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A SENSITIVE TEST OF QUANTUM ELECTRODYNAMICS.
THE 2S-2P ENERGY DIFFERENCE OF μ MESIC HYDROGEN

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A B S T R A C T

The 2S and 2P energy levels of μ mesic hydrogen are evaluated with an accuracy of the order of the width produced by the 2P-1S decay. It is shown that a measurement of the 2S-2P transition frequencies with this accuracy would give a precise determination of order α and order α^2 electron vacuum polarization effects, thus providing a very good test of q.e.d.

1. INTRODUCTION

The present work is a study of the 2S and 2P energy levels of the μ mesic hydrogen atom (μp). The motivation for doing it is that a kind of Lamb-Retherford experiment has apparently become possible, in which transitions are induced between the levels 2S and 2P by a submillimetre wave generator ¹⁾. It is therefore of some interest to know what information could be extracted from such an experiment.

The principle of the experiment should be the following. The 2S level is metastable and is appreciably populated when μ^- mesons are stopped in gaseous hydrogen. Inducing transitions between the 2S and 2P levels has then as a consequence an enhancement of the counting rate of the 2P-1S (1.9 keV) γ rays. The 2S-2P resonance frequency can therefore be detected by a resonant enhancement of the 1.9 keV γ rays counting rate.

The first point to notice is that the 2P states have a width Γ , due to dipole transitions to the 1S level

$$\Gamma = \frac{1}{\tau} \approx 1.2 \cdot 10^{11} \text{ sec}^{-1} . \quad (1)$$

Γ will also be the width of the 2S-2P lines. Let I be the intensity of the exciting light, and let ω be its frequency, with a spread narrower than Γ . The signal is then ²⁾

$$R = \sum_i \frac{4\pi^2\alpha}{h} g r_{\text{Bohr}}^2 I f_i \frac{\Gamma/2\pi}{(\omega - \omega_i)^2 + \Gamma^2/4} \quad (2)$$

The index i refers to the different lines in which the transition is split. The f_i 's are intensity factors of the order of unity (see Table II).

Under the best conditions (resonance), the transition rate is

$$R_i = \frac{4\pi^2\alpha}{h} g r_{\text{Bohr}}^2 I f_i \frac{2}{\pi \Gamma} \approx 1.4 \cdot 10^8 f_i I \text{ sec}^{-1} \quad (2')$$

if I is measured in units 10^6 W/cm^2 . On the basis of this number and of a realistic estimate of the population of the $2S$ level, one cannot expect a detailed determination of the resonance shape. So Γ is a reasonable order of magnitude of the precision within which the resonance frequencies can be determined. We shall therefore consider only effects which are relevant with respect to that intrinsic precision.

Were it not for the existence of the electrons, the treatment of the μp system would be exactly the same as that of ordinary hydrogen: one has only to replace the mass of the electron by the mass of the muon in all formulae.

The relevant effect of electrons is a contribution to vacuum polarization, producing a modification of the photon propagator in a range of the order of the electron Compton wavelength $\lambda_e = \hbar/m_e c$. Since λ_e is just of the order of the Bohr radius ($\sim \hbar/(\alpha m_\mu c)$), the μp system is a very sensitive probe of this effect.

Indeed, we shall find ³⁾ that the main contribution to the $2S-2P$ energy splitting comes from graph a) of Fig. 2, and that also graphs b) of Fig. 2, representing electron vacuum polarization of order α^2 , give a well-detectable contribution.

In Section 2 we briefly review the treatment of ordinary hydrogen as given in Refs. ⁴⁾⁻⁷⁾, adapting it to the μp system within the desired accuracy.

In Section 3 we consider the effect of inserting electron bubbles. The only graph which gives a detectable contribution after insertion of electron loops is the one-photon exchange graph. The relevant contributions of electron vacuum polarization turn out to be the graphs in Fig. 2. Their effect is calculated by using a perturbation expansion of the Bethe-Salpeter equation ⁴⁾. Retardation effects are explicitly calculated and found to be relevant.

In Section 4, we summarize the results and answer the initial question on the information which can be extracted from a measurement of the $2S-2P$ transition frequencies.

2. EFFECTS OTHER THAN ELECTRON VACUUM POLARIZATION

We shall work in the frame of the Bethe-Salpeter equation ⁸⁾ (BSE), which is the correct way of describing the system. The kernel of the BSE for the μ^-p system is the sum of all irreducible graphs in which photons are exchanged between the proton and the muon.

In this Section we shall neglect the effects of electrons and adapt to μp the results of Refs. 4)-6), which refer to ordinary hydrogen. The effects of electrons will be treated in Section 3. The starting point of the theory in Ref. 4) is the observation that the static Coulomb interaction is the main interaction in the hydrogen atom. This applies equally well to μp . The reason is that the motion is essentially non-relativistic: the relative velocity on the first Bohr orbit is $\sim \alpha c$, and is independent of the mass of the orbiting particle. By working in the Coulomb gauge, one isolates the instantaneous Coulomb interaction [graph a), Fig. 1], from other minor terms of the kernel. In Ref. 4) a perturbation theory is developed for the BSE, in which the instantaneous Coulomb interaction is treated as unperturbed kernel, and the other graphs as a perturbation. Some detail of the method is recalled in the Appendix and in Section 3, where it is used to calculate the effect of electron vacuum polarization. In Refs. 4)-6), the theory is applied to calculate fine structure (fs), Lamb shift, and hfs of hydrogen. The results are as follows. fs and hfs are correctly given by the Breit equation ⁹⁾, i.e., by graph a) of Fig. 1, plus graph b), treated as instantaneous. Retardation effects in graph b) and graphs c), d), e) give contributions of relative order $\alpha m/M$, which are explicitly computed. Higher order corrections are of relative order $\alpha (m/M)^2$, α^2 or smaller ¹⁰⁾. So fs splitting is given by

$$E_{n,j} = \left\{ -\frac{1}{n^2} - \frac{\alpha^2}{n^3} \left(\frac{1}{k} - \frac{3}{4n} \right) \right\} R_y \quad (3)$$

where n is the principal quantum number, j is the angular momentum $|\vec{l} + \vec{s}|$ of the orbiting particle, and $K = j + \frac{1}{2}$. Here, and in what follows, all energies are given in units of $Ry = (\alpha^2 \mu c^2)/2$, in order to get simple formulae. $\mu = m_\mu / (1 + (m_\mu / M_p))$ is the reduced mass of the system.

In these units the width $\hbar \Gamma$ of Eq. (1) is given by

$$\hbar \Gamma = 0.6 \cdot 10^{-3} \alpha^2 Ry \quad (4)$$

According to Eq. (3) the levels $2S_{\frac{1}{2}}, 2P_{\frac{1}{2}}$ are degenerate and the level $2P_{\frac{3}{2}}$ is separated by an amount

$$\delta \equiv \Delta E_{\frac{3}{2}} - \Delta E_{\frac{1}{2}} = \frac{1}{16} \alpha^2 Ry = .0625 \alpha^2 Ry \quad (5)$$

The recoil corrections of Ref. 4) are negligible with respect to our precision Eq. (4).

hsf is described by the matrix elements between Schrödinger non-relativistic states, of the operator

$$hfs = -2\mu_0 \left\{ \frac{8\pi}{3} \delta(\vec{r}) \vec{z} \cdot \vec{\mu}_p + \frac{1}{r^3} \vec{l} \cdot \vec{\mu}_p - \frac{1}{r^3} \left[\vec{z} \cdot \vec{\mu}_p - \frac{3(\vec{z} \cdot \vec{r})(\vec{\mu}_p \cdot \vec{r})}{r^2} \right] \right\} \quad (6)$$

In Eq. (6), $\vec{\mu}_p$ is the magnetic moment of the proton, and μ_0 the gyromagnetic ratio of the muon, which can be taken equal to the Dirac value $e\hbar/(2m_\mu c)$ in the limits of our accuracy.

In a representation in which the states $^1S_{\frac{1}{2}}, ^3S_{\frac{1}{2}}, ^1P_{\frac{1}{2}}, ^3P_{\frac{1}{2}}, ^3P_{\frac{3}{2}}, ^5P_{\frac{3}{2}}$ in the order are a basis, the operator hfs is represented by the matrix

$$hfs = \beta \begin{pmatrix} -3/4 & & & & & \\ & 1/4 & & & & \\ & & -1/4 & & & \\ & & & 1/2 & -\sqrt{2}/48 & \\ & & & -\sqrt{2}/48 & -1/2 & \\ & & & & & 1/20 \end{pmatrix} \quad (7)$$

where

$$\beta = \frac{2}{3} g_p \frac{\mu^2}{m_\mu M_p} \alpha^2 R_y = .1694 \alpha^2 R_y . \quad (8)$$

$g_p = 2.7927$ is the Landé factor of the proton.

The recoil corrections of Ref. 6) are completely negligible. We remark explicitly that β/δ , the ratio of hfs to fs, is proportional to μ/M_p as appears from Eqs. (5) and (8). This ratio is therefore ~ 200 times larger for μp than for ordinary hydrogen. This is the reason why we have calculated the non-diagonal elements between the states $^3P_{\frac{1}{2}}$ and $^3P_{\frac{3}{2}}$ in the matrix Eq. (7), which are neglected in the treatment of ep, where the fs splitting between these two states is large compared to β .

Lamb shift ^{5),11)} [graphs f) of Fig. 2] is for the S state

$$\begin{aligned} \Delta E_{2S}^{LS} &= \frac{\alpha}{3\pi} \left(\frac{\mu}{m_\mu} \right)^3 \left[2 \ln \frac{1}{\alpha} - \ln \frac{K_0(2,0)}{R_y} + \frac{19}{30} \right] \alpha^2 R_y \\ &= 4.9 \cdot 10^{-3} \alpha^2 R_y \end{aligned} \quad (9)$$

The shift of the 2P states is $\sim 2\%$ of the shift of the 2S state, and therefore outside the limits of our accuracy. Corrections due to the finite size of the proton are given by

$$\left. \begin{aligned} \Delta E_{2S}^{FS} &= \frac{2\pi}{3} e^2 r_p^2 |u(0)|^2 = \frac{1}{6} \frac{r_p^2}{r_{Bohr}^2} R_y \\ \Delta E_{2P}^{FS} &= 0 \end{aligned} \right\} \quad (10)$$

r_p^2 is defined by the equation

$$G_E(k^2) \sim 1 - \frac{1}{6} r_p^2 k^2 ,$$

$G_E(k^2)$ being the electric form factor of the proton.

Equations (10) are correct up to terms of relative order

$$\frac{r_p^2}{r_{Bohr}^2} \sim \left(\frac{1}{300} \right)^2$$

coming from the approximation of taking only the second momentum of the charge distribution, and of relative order $\sim \alpha^2 \ln \alpha$ coming from the approximation of taking

$$\int \frac{d^3 p}{(2\pi)^3} \varphi(\vec{p}) = u(0)$$

with u the Schrödinger wave function. Assuming $r_p^2 = 0.80 \pm 0.02$ fermi, Ref. ¹²⁾, as it comes from a fit of low momentum transfer ep scattering data, we get

$$\Delta E_{25}^{FS} = (.0258 \pm .0006) \alpha^2 R_4 \quad (11)$$

To the desired accuracy we have considered all the effects, but the vacuum polarization due to the electrons.

Insertion of electron bubbles in graphs c)-f) of Fig. 1 would give a small correction to effects which are per se negligible. So we have only to consider the insertion of electron loops in the one-photon exchange graphs. The lowest order terms are shown in Fig. 2.

In the next Section we shall calculate the energy shift due to the graphs of Fig. 2. Higher order vacuum polarization graphs will be shown to be negligible.

3. ELECTRON VACUUM POLARIZATION

As stated at the end of Section 2, we have to consider the insertion of electron loops only in the one-photon exchange graph.

This amounts to modifying by electron vacuum polarization the field produced by one of the particles on the other, as if it were generated by an external source.

In the Coulomb gauge ¹³⁾ we have for the static field

$$-\frac{4\pi\alpha}{|\vec{k}|^2} \longrightarrow -\frac{4\pi\alpha}{|\vec{k}|^2} \{1 - \Pi(k^2)\} \quad (12)$$

and for the transverse field

$$\begin{aligned} & \frac{4\pi\alpha}{k^2} \left\{ \vec{\alpha}_p \cdot \vec{\alpha}_\mu - \frac{(\vec{k} \cdot \vec{\alpha}_p)(\vec{k} \cdot \vec{\alpha}_\mu)}{|\vec{k}|^2} \right\} \longrightarrow \\ & \frac{4\pi\alpha}{k^2} \left\{ \vec{\alpha}_p \cdot \vec{\alpha}_\mu - \frac{(\vec{k} \cdot \vec{\alpha}_p)(\vec{k} \cdot \vec{\alpha}_\mu)}{|\vec{k}|^2} \right\} \cdot \{1 - \Pi(k^2)\} \end{aligned} \quad (13)$$

$\Pi(k^2)$ is the so-called vacuum polarization kernel ¹⁴⁾.

In what follows, we shall make wide use of the spectral representation

$$\frac{\Pi(k^2)}{k^2} = -i \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{m^2} \Delta_F(k^2, m^2) dm^2 \quad (14)$$

where

$$\rho(m^2) = \frac{1}{\pi} \text{Im } \Pi(m^2)$$

We first treat the perturbation term $(4\pi\alpha/|\vec{k}|^2)\Pi(k^2)$ of Eq. (12), using the formula (A.9).

We notice that the vacuum polarization kernel is at least of the order α^2 [graph a) in Fig. 2]. So the perturbation formula (A.9) is correct up to terms of relative order α^2 and it makes sense to calculate by Eq. (A.9) vacuum polarization effects up to order α^3 [graphs b) in Fig. 2]. Fortunately, higher order effects will be found to be beyond the precision Eq. (4).

The level shift for the perturbation in Eq. (12) is

$$\Delta E = 4\pi\alpha \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \bar{\Psi}^c(p, p) \gamma_0^{(\omega)} \gamma_0^{(\omega')} \frac{\Pi(k^2)}{|\vec{k}|^2} \Psi_c(p+k, p) \quad (15)$$

By using the spectral representation Eq. (14), and the expression (A.6) for Ψ^c , the integrations on the fourth components in Eq. (15) are easily performed. The functions to be integrated are products of Feynman denominators. The result is :

$$\Delta E = 4\pi\alpha \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \chi_{E++}^*(\vec{p}) \chi_{E++}(\vec{p}+\vec{k}) \times$$

$$\left\{ \frac{\pi(-|\vec{k}|^2)}{|\vec{k}|^2} + \frac{1}{2} [E'(\vec{p}) + E'(-\vec{p}) - E] \frac{1}{|\vec{k}|^2} \int_{4m_e^2}^{\infty} \frac{\rho(m^2) dm^2}{(m^2+k^2)^{3/2}} \right\} \quad (16)$$

Only the terms with positive energy components ($\chi_{E++}(\vec{p})$) have been retained. Terms with negative energy components are of relative order $\alpha^3 \ln^2 \alpha$ or smaller, as can be proved by using Eqs. (A.7) and the fact that ^{(14), (15)}

$$\pi(k^2) \underset{|\vec{k}|^2 \gg 4m_e^2}{\sim} \ln \frac{|\vec{k}|}{4m_e^2} \quad (17)$$

In the positive energy terms contributions of relative order α^2 have been neglected. The first term in the right-hand side of Eq. (16) is the instantaneous potential, the second term is a small correction coming from retardation. By virtue of Eq. (17) the first term is not more singular at the origin than the Coulomb potential, and we know that Schrödinger non-relativistic wave functions give the correct Coulomb energy, apart from first corrections which are of relative order α^2/n . Neglecting corrections of $O(\alpha^2)$ we can therefore replace the wave functions χ in Eq. (16) by non-relativistic wave functions, at least for the first term. We shall do the same for the second one, which is small, even if its behaviour at the origin is slightly worse.

In the $\vec{r}$ representation, Eq. (16) can be written

$$\Delta E = 4\pi\alpha \int d^3r |\varphi_{ne}(\vec{r})|^2 \left\{ V(r) + \frac{\alpha}{r} W'(r) \right\} \quad (18)$$

where

$$V(r) = \int \frac{d^3k}{(2\pi)^3} \frac{\pi(-|\vec{k}|^2)}{|\vec{k}|^2} e^{i\vec{k} \cdot \vec{r}} \quad (19)$$

and

$$W'(r) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\vec{r}}}{|\vec{k}|^2} \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{(m^2+k^2)^{3/2}} dm^2 \quad (20)$$

The factor α/r in the second term of Eq. (18) comes from the factor $[\bar{E}^{(\mu)}(\vec{p}) + E^{(p)}(-\vec{p}) - E]$ in Eq. (16) by use of the equation of motion (A.4).

By using the spectral representation Eq. (14), Eq. (19) becomes

$$V(r) = - \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{m^2} dm^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\vec{r}}}{k^2+m^2} = - \int_{4m_e^2}^{\infty} \frac{dm^2 \rho(m^2)}{m^2} \frac{e^{-mr}}{4\pi r} \quad (21)$$

The functions $\varphi_{nl}(\vec{r})$ being the product of an exponential, times a polynomial, if one uses Eq. (21) the integration over $\vec{r}$ in Eq. (18) is elementary.

For the contribution of $V(r)$ to the 2S-2P energy difference, we get

$$\Delta E^I = \Delta E_{2S}^I - \Delta E_{2P}^I = -2 Ry \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{(m+1)^4} dm^2$$

or, changing variables from m^2 to x , defined by $(4m_e^2)/x = m^2$:

$$\Delta E^I = -2 Ry \int_0^1 dx \frac{a^2}{(x+a)^4} \rho\left(\frac{a^2}{x}\right) \quad (22)$$

where

$$a = \frac{2m_e}{\alpha\mu} = 1.4748$$

is the value of $2m_e$ in our atomic units. To order α (14), (15) [graph a) of Fig. 2] :

$$\wp\left(\frac{\alpha^2}{x}\right) = \frac{\alpha}{3\pi} \left(1 + \frac{x}{2}\right) \sqrt{1-x} \quad (23)$$

To order α^2 [sum of graphs b) in Fig. 2] :

$$\wp\left(\frac{\alpha^2}{x}\right) = \frac{\alpha^2}{3\pi^2} F(\sqrt{1-x}) \quad (24)$$

where ¹⁵⁾

$$\begin{aligned} F(z) = & z \left[-\frac{19}{24} + \frac{55}{72} z^2 - \frac{1}{3} z^4 - \frac{1}{3} (3-z^2) \ln \frac{64z^4}{(1-z^2)^3} \right] \\ & + \ln \frac{1+z}{1-z} \left[\frac{33}{16} + \frac{23}{8} z^2 - \frac{23}{16} z^4 + \frac{1}{6} z^6 + \left(\frac{3}{2} + z^2 - \frac{1}{2} z^4 \right) \ln \frac{(1+z)^3}{8z^2} \right] \\ & + \left(\frac{3}{2} + z^2 - \frac{z^4}{2} \right) \left[4 \operatorname{Li}_2 \left(\frac{1-z}{1+z} \right) + 2 \operatorname{Li}_2 \left(\frac{z-1}{1+z} \right) \right] \end{aligned} \quad (24')$$

Inserting expressions (23) and (24) into Eq. (22), we get, by numerical integration

$$\Delta E_{\alpha}^I = -1.5225 \alpha^2 Ry \quad (25)$$

$$\Delta E_{\alpha^2}^I = -0.0112 \alpha^2 Ry \quad (26)$$

The energy splitting (26) is ~ 20 times larger than the experimental resolution, Eq. (4).

The value of ΔE_{α}^I in Eq. (25) agrees with that obtained by using the analytical expression for the same quantity, given in Ref. ¹⁶⁾. We have recalculated it for checking the method ¹⁷⁾ in view of the calculation of the fourth order contribution, Eq. (26). We notice that the perturbation $V(r)$ is spherically symmetric and also spin independent within the required accuracy. So it can only mix states with

different quantum number n , and therefore, by the result of Eq. (25) it affects the wave functions by terms $O(\alpha^2)$. Therefore the analysis of Section 2 is not affected by the presence of electron vacuum polarization.

We now evaluate the retardation correction, second term in the right-hand side of Eq. (18), for the 2S-2P energy difference. We first Fourier transform

$$\frac{1}{r} \left[|\varphi_{2S}(r)|^2 - \frac{1}{3} \sum_{m=-1}^1 |\varphi_{2P,m}(r)|^2 \right]$$

getting (in atomic units)

$$\frac{1}{2} \left\{ \frac{1}{1+k^2} - \frac{3}{(1+k^2)^2} + \frac{2}{(1+k^2)^3} \right\}$$

and then we perform the integration in Eq. (18) in momentum space. We get

$$\Delta E^R = \Delta E_{2S}^R - \Delta E_{2P}^R = 4\pi\alpha Ry \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} \left[\frac{1}{1+k^2} - \frac{3}{(1+k^2)^2} + \frac{2}{(1+k^2)^3} \right] \times \right. \\ \left. \frac{1}{k^2} \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{(m^2+k^2)^{3/2}} dm^2 \right\}$$

or, interchanging the order of integrations

$$\Delta E^R = \frac{\alpha}{\pi} Ry \int_{4m_e^2}^{\infty} dm^2 \rho(m^2) \int_0^{\infty} dk \left[\frac{1}{1+k^2} - \frac{3}{(1+k^2)^2} + \frac{2}{(1+k^2)^3} \right] \frac{1}{(m^2+k^2)^{3/2}}$$

After integrating over k , by changing variable from m^2 to a^2/x^2 , we finally get

$$\Delta E_{\alpha}^R = \frac{2a^2}{3\pi^2} \int_0^1 dx \frac{\sqrt{1-x^2}}{(a^2-x^2)^3} (1+\frac{x^2}{2}) \left\{ (x^4 + \frac{5}{2}a^2x^2 + \frac{a^4}{4}) \frac{1}{(a^2-x^2)^{1/2}} \frac{\sin^{-1} \sqrt{1-x^2}}{a^2} \right. \\ \left. - 2x^3 - \frac{7}{4}a^2x \right\} a^2 Ry$$

Numerical integration gives

$$\Delta E_{\alpha}^R = .0103 \alpha^2 R_y \quad (27)$$

This is of the order of 0.6% of ΔE_{α}^I , Eq. (25). The analogous correction for the fourth order term is negligible. We have yet to evaluate the contribution of the transverse part of Eq. (13). We recall (see Section 2) that the unperturbed term in Eq. (13) [graph b) of Fig. 1] gives m/M corrections to fs and the hfs coupling when treated as instantaneous; the corresponding recoil corrections are negligible. So the effect of vacuum polarization is expected to be of order α times m/m fs or α times hfs.

By inserting the perturbation Eq. (13) into Eq. (A.9), we get, by the same technique as used in deriving Eq. (16),

$$\Delta E = -4\pi\alpha \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \left[\chi_{E++}^+(\vec{p}) \left\{ \vec{\alpha}_{\mu i} \cdot \vec{\alpha}_{\mu f} - \frac{(\vec{\alpha}_{\mu i} \cdot \vec{k})(\vec{\alpha}_{\mu f} \cdot \vec{k})}{|\vec{k}|^2} \right\} \chi_{E++}^-(\vec{p}+\vec{k}) \right. \\ \left. V'(\vec{p}, \vec{k}) \right] \quad (28)$$

where $V'(\vec{p}, \vec{k})$ is given by the equation

$$V'(\vec{p}, \vec{k}) = \int_{\sqrt{4m_e^2 + |\vec{k}|^2}}^{\infty} dp_0 \frac{p_0(p_0^2 - |\vec{k}|^2)}{p_0^2 - |\vec{k}|^2} \left\{ \frac{1}{p_0^2 + E^{(\mu)}(\vec{p}) + E^{(\mu)}(\vec{p}+\vec{k}) - E} + \frac{1}{p_0^2 + E^{(\mu)}(\vec{p}+\vec{k}) + E^{(\mu)}(\vec{p}) - E} \right\} \quad (29)$$

For all important values of $\vec{p}$ and $\vec{k}$ in the integral (28) [see Eqs. (A.8)-(A.8')]]

$$E^{(\mu)}(\vec{p}) + E^{(\mu)}(\vec{p}+\vec{k}) - E \approx E^{(\mu)}(\vec{p}+\vec{k}) + E^{(\mu)}(\vec{p}) - E \approx \frac{\alpha^2 \mu}{n^2}$$

the average binding energy. On the other hand, $p^0 \gg \sqrt{4m_e^2 + |\vec{k}|^2} \sim \alpha\mu$. So we can develop the right-hand side of Eq. (29) in power series of the energies in the denominators, getting

$$V'(\vec{p}, \vec{k}) = \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{m^2} \frac{1}{m^2 + k^2} dm^2 +$$

$$2 \left[E^{(\mu)}(\vec{p}) + E^{(p)}(\vec{p}) - E \right] \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{m^2} \frac{dm^2}{(m^2 + k^2)^{3/2}} + O(\alpha^2) \quad (30)$$

The first term in Eq. (30) is the instantaneous part, the second one is a correction due to retardation.

By using the equations of motion (A.4), the second term in Eq. (30) is equivalent in the $\vec{r}$ representation to a potential

$$V_R(r) = -\frac{\alpha}{2\pi^2} \int_{4m_e^2}^{\infty} \frac{\rho(m^2)}{m^2} \frac{K_0(mr)}{r} dm^2$$

where $K_0(z)$ is the modified Bessel function of order zero. Compared with the instantaneous potential, Eq. (21), $V_R(r)$ has, apart from logarithms, the same behaviour at the origin and at infinity, and a factor α in front. So it gives effects of the order α times the instantaneous part, and therefore - as we shall see - it is negligible. For the instantaneous part we get, by Eqs. (28) and (30) and by using the non-relativistic expression for the matrix elements of the $\vec{\alpha}$ matrices¹⁸⁾,

$$\Delta E_{n,j} = -2\pi \left(\frac{\mu^2}{m_\mu m_p} \right) \alpha^2 R_y \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \varphi_{n,j}^*(\vec{p}) \times$$

$$\left\{ -\frac{2}{3} |\vec{k}|^2 \vec{\sigma}_\mu \cdot \vec{\sigma}_p + 4 \left[p^2 - \frac{(\vec{p} \cdot \vec{k})^2}{|\vec{k}|^2} \right] + 2i (\vec{k} \times \vec{p}) (\vec{\sigma}_\mu + \vec{\sigma}_p) \right. \quad (31)$$

$$\left. + [(\vec{k} \cdot \vec{\sigma}_\mu)(\vec{k} \cdot \vec{\sigma}_p) - \frac{1}{3} |\vec{k}|^2 \vec{\sigma}_\mu \cdot \vec{\sigma}_p] \right\} \frac{\pi(-|\vec{k}|^2)}{|\vec{k}|^2} \varphi_{n,j}(\vec{p} + \vec{k})$$

The shift ΔE_{nj} in Eq. (31) is of formal order $(\alpha^2 \mu^2)/(m_\mu m_p)$ times the shift given by Eq. (16). It is easily recognized that this is the actual order of magnitude for all the terms in the right-hand side of Eq. (31) but the first one which is highly singular at the origin and contributes to the hfs splitting of the S states ¹⁹⁾. From this term we get, correcting for the anomalous magnetic moment of the proton, a modification of the hfs of the 2S level

$$\Delta hfs_{2S} = + \beta \vec{s}_\mu \cdot \vec{s}_p \ 8\pi \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \varphi_{2S}^*(\vec{p}) \pi(-|\vec{k}|^2) \varphi_{2S}(\vec{p}+\vec{k}) \quad (32)$$

where β is defined by Eq. (8). Or

$$\Delta hfs_{2S} = - \epsilon \ hfs_{2S}$$

where

$$- \epsilon = \frac{4}{\pi} \int_0^\infty dk \left\{ \frac{2}{(k^2+1)^2} - \frac{7}{(k^2+1)^3} + \frac{6}{(k^2+1)^4} \right\} k^2 \pi(-k^2) \quad (33)$$

For evaluating the integral in Eq. (23) we cannot use the spectral representation Eq. (14) and perform first the integral over k , then the integral over m^2 . If we proceed in this way, we get a divergent result. Due to the singular behaviour of the perturbation, the order of integrations cannot be exchanged. So we have to use the explicit expression for $\pi(-|\vec{k}|^2)$, namely to order α ¹⁴⁾

$$\pi(-k^2) = -\frac{\alpha}{3\pi} F(\delta) \quad \delta = \sqrt{1 + \frac{4m_e^2}{k^2}} \quad (34)$$

$$F(\delta) = \frac{8}{3} - \delta^2 + \frac{1}{2} (\delta^2 - 3\delta) \ln \frac{\delta+1}{\delta-1}$$

Inserting Eq. (34) into Eq. (33) and changing variables from k to x , defined by $k = \sqrt{(1-x^2)/x^2}$ we get

$$\epsilon = \frac{4\alpha}{3\pi^2} \int_0^1 \sqrt{1-x^2} (2 - 7x^2 + 6x^4) F(\delta) dx$$

$$\delta = \sqrt{1 + \frac{\alpha^2 x^2}{1-x^2}}$$

or by numerical integration

$$\epsilon = 4.2 \cdot 10^{-3} \quad (35)$$

which means $\epsilon\beta = 0.71 \cdot 10^{-3} \alpha^2 \text{Ry}$, i.e., at the limits of the precision Eq. (4).

4. SUMMARY OF THE RESULTS AND CONCLUSIONS

The results are summarized in Tables I and II. There

$$\begin{aligned} -\gamma &= \Delta E_{\alpha}^I && [\text{Eq. (25)}] \\ &+ \Delta E_{\alpha^2}^I && [\text{Eq. (26)}] \\ &+ \Delta E_{\alpha}^R && [\text{Eq. (27)}] \\ &+ \Delta E_{2s}^{(LS)} && [\text{Eq. (9)}] \\ &+ \Delta E_{2s}^{(FS)} && [\text{Eq. (11)}] \end{aligned}$$

Numerically

$$-\gamma = (-1.4927 \pm .0006) \alpha^2 \text{Ry} \quad (36)$$

The uncertainty in γ comes from the error in the radius of the proton. β, δ, ϵ are defined by Eqs. (8), (5) and (35). The conversion

factors to usual units are

$$\alpha^2 R_y = .13461 \text{ ev}$$

$$\frac{\alpha^2 R_y}{\hbar} = 2.0454 \cdot 10^{14} \text{ sec}^{-1}$$

On the basis of our numbers we can conclude that the results of the experiment sketched in Section 1 would be the following:

a) A very good measurement of order α electron vacuum polarization [graph a) in Fig. 2]. To our knowledge the existence of such an effect has already been tested :

- i) in heavy μ atoms X-ray spectra. Here, however, a quantitative determination is made difficult by nuclear effects;
- ii) by its small contribution to hydrogen Lamb shift. In this case the experimental precision is some per cent of this contribution ²⁰⁾;
- iii) by the fourth order $g-2$ of the muon, to which it contributes the dominant term (Fig. 3). Here the present experimental accuracy ²¹⁾ is of the order of some per cent of this term. However, the present accuracy is at the limits where strong interaction corrections to the photon propagator may become relevant ²²⁾.

In $\mu^- p$ no such problem arises. The reason is that, while in Fig. 3 the momentum transferred by the photon is of the order of the muon mass m_μ , in μp the momentum transfer is of the order of the Bohr momentum ($\alpha m_\mu \sim 1 \text{ MeV}$), which is far apart from the two-pion cut ²³⁾. On the other hand, in $\mu^- p$ the price one has to pay to strong interactions for getting such a situation, is the finite size correction to the Coulomb field Eq. (11) which requires independent experimental information.

b) A clear detection and a measurement to some per cent accuracy of order α^2 electron vacuum polarization [graphs b) of Fig. 2].

To our knowledge such an effect has never been observed. In the $g-2$ of the muon it enters in the sixth order term ²⁴⁾ which is expected to be obscured by strong interaction effects.

- c) The precision of the experiment being $\sim 10^{-3}$, no better value can be obtained for α or m_μ .

In conclusion, a measurement of the 2S-2P energy differences in the μp atom would provide a very good test of pure q.e.d.

As a final remark, we observe that the situation is not qualitatively different for the μ -He atom, and maybe also for other light μ atoms. Our calculations could be easily adapted to them, if they were more convenient for experimental reasons.

The author is indebted to J.S. Bell for useful criticism and discussions. He also wishes to thank E. Zavattini for raising his interest in the problem.

A P P E N D I X

In this Appendix we briefly recall the formalism of Ref. 4), which is used in Section 3. Let

$$\psi(p, P) = S_F^{(\mu)}(p + \alpha P) S_F^{(p)}(-p + (1 - \alpha)P) \int \frac{d^4 p'}{(2\pi)^4} G(p, p', P) \psi(p', P) \quad (A.1)$$

be the BSE for the μ^-p system. In Eq. (A.1) the motion of the centre-of-mass (c.m.) has been separated :

$$\begin{aligned} P = P_\Lambda & \text{ is the total four momentum } p_\Lambda^{(\mu)} + p_\Lambda^{(p)} ; \\ p = p_\Lambda & \text{ is the relative four momentum } (1 - \alpha)p_\Lambda^{(\mu)} - \alpha p_\Lambda^{(p)} ; \end{aligned}$$

α is an arbitrary parameter corresponding to the arbitrariness in the definition of the c.m. co-ordinates. A convenient choice is $\alpha = m_\mu / (m_p + m_\mu)$; $S_F(p)$ is the fermion propagator, which - for our purposes - [see end of Section 2] can be approximated by the free propagator

$$S_F(p) = \frac{i}{\not{p} - m + i\epsilon}$$

$G(p, p', P)$ is the kernel;

$$\psi(p, P) = \psi(p_1, p_2) = \int d^4 x_1 \int d^4 x_2 e^{i(p_1 x_1 + p_2 x_2)} \langle 0 | T(\psi^{(\mu)}(x_1) \psi^{(p)}(x_2)) | S \rangle$$

is the BS function for the state $|S\rangle$ of interest. Equation (A.1) is a rigorous consequence of field theory^{8b)}. If the complete kernel G is approximated by its instantaneous Coulomb part

$$i \frac{4\pi\alpha}{|\vec{k}|^2} \gamma_0^{(\mu)} \gamma_0^{(p)}$$

equation (A.1) can be transformed into a three-dimensional equation for the function

$$\varphi(\vec{p}, P) = \int \frac{dp_0}{2\pi} \psi(p, P) \quad (\text{A.2})$$

by performing the integration on dp'_0 in the right-hand side of Eq. (A.1) and integrating both sides on dp_0 .

The resulting equation in the c.m.s. $[P_\lambda = (E, 0, 0, 0)]$ is

$$[E - H^{(\mu)}(\vec{p}) - H^{(\nu)}(-\vec{p})] \varphi_E(\vec{p}) = [\Lambda_+^{(\mu)}(\vec{p}) \Lambda_+^{(\nu)}(-\vec{p}) - \Lambda_-^{(\mu)}(\vec{p}) \Lambda_-^{(\nu)}(-\vec{p})] \cdot \int \frac{d^3k}{(2\pi)^3} \frac{-4\pi\alpha}{|\vec{k}|^2} \varphi_E(\vec{p} + \vec{k}) \quad (\text{A.3})$$

where $H(\vec{p}) = \vec{\alpha} \cdot \vec{p} + \beta m$, and

$$\Lambda_{\pm}(\vec{p}) = \frac{E(\vec{p}) \pm H(\vec{p})}{2E(\vec{p})}$$

with $E(\vec{p}) = \sqrt{p^2 + m^2}$. If we define $\chi_E(\vec{p})$ by the equation

$$[E - H^{(\mu)}(\vec{p}) - H^{(\nu)}(-\vec{p})] \chi_E(\vec{p}) = \int \frac{d^3k}{(2\pi)^3} \frac{-4\pi\alpha}{|\vec{k}|^2} \varphi_E(\vec{p} + \vec{k}) \quad (\text{A.4})$$

then

$$\varphi_E(\vec{p}) = [\Lambda_+^{(\mu)}(\vec{p}) \Lambda_+^{(\nu)}(-\vec{p}) - \Lambda_-^{(\mu)}(\vec{p}) \Lambda_-^{(\nu)}(-\vec{p})] \chi_E(\vec{p}) \quad (\text{A.5})$$

and, in the c.m.s.,

$$\psi^c(p, P) = S_F^{(\mu)}(p + \alpha P) S_F^{(\nu)}(-p + (1-\alpha)P) [E - H^{(\mu)}(\vec{p}) - H^{(\nu)}(-\vec{p})] \chi_E(\vec{p}) \quad (\text{A.6})$$

By the index c we mean that $\psi^c(p, P)$ is a solution of the approximate Coulomb problem. In the non-relativistic limit ($p \ll m_\mu$), Eq. (A.3) just becomes the Schrödinger equation for the hydrogen atom. In that limit it can easily be proved by using Eqs. (A.3) - (A.5) that

$$\left. \begin{aligned} \Lambda_-^{(\mu)}(\vec{p}) \chi_E(\vec{p}) &\sim \alpha^3 \Lambda_+^{(\mu)}(\vec{p}) \chi_E(\vec{p}) \\ \Lambda_-^{(p)}(-\vec{p}) \chi_E(\vec{p}) &\sim \alpha^3 \frac{m_\mu}{m_p} \Lambda_+^{(p)}(\vec{p}) \chi_E(\vec{p}) \end{aligned} \right\} \quad (\text{A.7})$$

and that

$$\varphi_{E++}(\vec{p}) \simeq \chi_{E++}(\vec{p}) = \varphi_{ne}(\vec{p}) \left\{ 1 + O\left(\frac{p+\alpha\mu}{\mu}\right)^2 \right\} \quad (\text{A.8})$$

$\varphi_{ne}(\vec{p})$ is the Schrödinger non-relativistic wave function. We recall that

$$\varphi_{ne}(\vec{p}) \sim \frac{1}{p^{4+2}} \quad (\text{A.8'})$$

for $p \gg \alpha\mu$. The energy shift produced by a perturbation graph $G'(K)$ is ⁴⁾:

$$\Delta E = i \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \bar{\psi}^c(p, p) G'(K) \psi^c(p, p) \quad (\text{A.9})$$

where

$$\bar{\psi}^c(p, p) = \bar{\chi}_E(\vec{p}) [E - H^{(\mu)}(\vec{p}) - H^{(p)}(-\vec{p})] S_F^{(\mu)}(p + \alpha p) S_F^{(p)}(-p + (1-\alpha)p)$$

and the normalization is

$$\int \varphi_E^+(\vec{p}) \chi_E(\vec{p}) \frac{d^3 p}{(2\pi)^3} = 1$$

If $G'(K) = G'(|\vec{K}|)$, i.e., if it is instantaneous, Eq. (A.9) becomes

$$\Delta E = i \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \bar{\chi}_E(\vec{p}) S'(\vec{k}) \chi_E(\vec{p} + \vec{k}) \quad (\text{A.10})$$

which is the usual perturbation theory.

Equations (A.9) and (A.10) are correct apart from terms of second order in the perturbation.

TABLE I

States	Energy
$1S_{\frac{1}{2}}$	$-\gamma - \frac{3}{4}\beta (1-\epsilon)$
$3S_{\frac{1}{2}}$	$-\gamma + \frac{\beta}{4} (1-\epsilon)$
$1P_{\frac{1}{2}}$	$-\beta/4$
$(3P_{\frac{1}{2}})^1 = \cos \frac{\theta}{2} 3P_{\frac{1}{2}} - i \sin \frac{\theta}{2} 3P_{\frac{3}{2}}$	$\frac{\delta}{2} + \sqrt{(\frac{\beta}{12} - \frac{\delta}{2})^2 + (\frac{\beta}{24} - \frac{\delta}{2})^2} \sim \frac{\beta}{12}$
$(3P_{\frac{3}{2}})^1 = \cos \frac{\theta}{2} 3P_{\frac{3}{2}} - i \sin \frac{\theta}{2} 3P_{\frac{1}{2}}$	$\frac{\delta}{2} + \sqrt{(\frac{\beta}{12} - \frac{\delta}{2})^2 + (\frac{\beta}{24} - \frac{\delta}{2})^2} \sim \delta - \frac{\beta}{12}$
$5P_{\frac{3}{2}}$	$\frac{\beta}{20}$

Energy eigenstates and eigenvalues. The upper index in the labels for the states is $2F+1$, with F the total angular momentum $= \vec{j} + \text{nuclear spin } \vec{I}$. The mixing between $3P_{\frac{1}{2}}$ and $3P_{\frac{3}{2}}$ is due to off diagonal matrix elements of hfs [Eq. (7)]

$$\sin \theta = \frac{\beta}{2\sqrt{(\beta - 6\delta)^2 + \frac{\beta^2}{8}}} \approx \theta \approx .28$$

In the expansion of the square roots, terms $\sim \beta/200$ have been neglected.

TABLE II

Transition	Frequency	Intensity f
$^1S_{\frac{1}{2}} \rightarrow ({}^3P_{\frac{1}{2}})^1$	$\gamma + \frac{5}{6}\beta + \frac{3}{4}\beta\varepsilon$	$\frac{1}{3}(1+\sin^2\theta) \sim \frac{1}{3}$
$^1S_{\frac{1}{2}} \rightarrow ({}^3P_{\frac{3}{2}})^1$	$\gamma + \frac{3}{4}\beta\varepsilon + \delta + \frac{2}{3}\beta$	$\frac{2}{3} - \frac{1}{3}\sin^2\theta \sim \frac{1}{3}$
${}^3S_{\frac{1}{2}} \rightarrow {}^1P_{\frac{1}{2}}$	$\gamma + \frac{\beta\varepsilon}{4} - \frac{\beta}{2}$	$\frac{1}{3}$
${}^3S_{\frac{1}{2}} \rightarrow ({}^3P_{\frac{1}{2}})^1$	$\gamma + \frac{\beta\varepsilon}{4} - \frac{\beta}{6}$	$\frac{2}{3} - \frac{1}{3}\sin^2\theta \approx \frac{2}{3}$
${}^3S_{\frac{1}{2}} \rightarrow ({}^3P_{\frac{3}{2}})^1$	$\gamma + \frac{\beta\varepsilon}{4} + \delta - \frac{\beta}{3}$	$\frac{1}{3}(1+\sin^2\theta) \sim \frac{1}{3}$
${}^3S_{\frac{1}{2}} \rightarrow {}^5P_{\frac{3}{2}}$	$\gamma + \frac{\beta\varepsilon}{4} - \frac{\beta}{5}$	$\frac{5}{3}$

For the definitions of $\beta, \gamma, \delta, \varepsilon$ see Section 4. The intensity factor f is defined by Eq. (2).

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This value is in agreement with the usual dipole form

$$G_E(q^2) = \frac{1}{(1 + q^2/11)^2} \quad [q^2 \text{ in } (\frac{GeV}{c})^2]$$

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L. Lovitch has solved the Dirac equation in the presence of the Coulomb potential plus the static vacuum polarization potential of order α , getting a level splitting which differs from ours by some parts in 10^5 . This confirms that we are correct in using non-relativistic wave functions. We thank L. Lovitch for communicating his result to us.
- 18) See, e.g., Ref. 7), Section 16.
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FIGURE CAPTIONS

Figure 1 : We are in the Coulomb gauge. Dashed lines represent instantaneous Coulomb photons, wavy lines represent transverse photons, solid lines muons, double solid lines protons.

Figure 2 : Electron vacuum polarization graphs. By these graphs we indicate schematically the sum of Coulomb and transverse contributions.

Figure 3 : Effect of electron vacuum polarization of order in the $g-2$ of the muon.

FIG. 1

FIG. 2

FIG. 3