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WEAK SELF-MASSSES AND CABIBBO ANGLE. DISCUSSION OF HIGHER ORDERS

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A B S T R A C T

Our previous argument, relating the value of the Cabibbo angle to the symmetry breaking of chiral $SU_2 \times SU_2$ and allowing for a calculation of the angle is extended to higher orders in weak interactions. It is shown that the derivation goes through at any order in a quark model. In a more general frame, in terms of commutators of currents and of currents and divergences, the derivation is extended to fourth order.

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1. INTRODUCTION

At the last High Energy Conference in Vienna we presented an argument relating the value of the Cabibbo angle to the symmetry breaking of chiral $SU_3 \times SU_2$ and allowing for a satisfactory calculation of the angle ^{1),2)}. The argument was based on the requisite that there should be no quadratic divergent contributions to a general second order matrix element for emission and reabsorption of a W boson, besides a possible SU_3 singlet term. It was also reported in Ref. 2) that the argument goes through also at fourth order. In this note we shall discuss the higher order contributions. We shall show that in a quark model the argument applies at any order. We shall also present the fourth order derivation for the more general case, employing Bjorken's technique ³⁾ to relate the coefficients of the divergent terms to equal time commutators.

In Section 2, we shall summarize our previous work on the subject and discuss our assumptions. In Section 3, we shall treat the quark model at any order. In Section 4, a general procedure will be applied to fourth order. Section 5 will contain a summary of formal properties of the algebra of charges and divergences.

2. SUMMARY OF OUR PREVIOUS WORK

We shall first repeat the general view-point underlying our calculation ⁴⁾. For simplicity we shall first use the language of a naïve quark model. (Because of the assumed form of the symmetry breaking Hamiltonian, this model contains some elements of generality.)

Starting from p , n , and λ quarks, with masses m_p , m_n , and m_λ , the highest divergent terms at any order (i.e., terms proportional to powers of $G\Lambda^2$, where G is the weak coupling constant and Λ is a cut-off) amount to a term ⁵⁾

$$\delta m_p \bar{p}p + \delta m_n \bar{n}n + \delta m_\lambda \bar{\lambda}\lambda + \delta m (\bar{n}\lambda + \bar{\lambda}n) + \delta m_5 (\bar{n}\gamma_5\lambda - \bar{\lambda}\gamma_5n) \quad (1)$$

where the coefficients δm_p , δm_n , δm_λ , δm and δm_5 are divergent integrals. Renormalization theory would suggest that one introduces counter terms to such infinities and interprets m_p , m_n , and m_λ as the "physical" quark masses, assigned from the outside and unpredictable from the renormalized theory. We shall adopt this view-point, except for the request we make that the theory shall be able to predict the weak contributions to the differences among the quark masses, that is, in particular, that there should be no divergent terms contributing to $(m_p + \delta m_p) - (m_n + \delta m_n)$ and to $(m_n + \delta m_n) - (m_\lambda + \delta m_\lambda)$. This is equivalent to requiring that :

$$\delta m_p = \delta m_n = \delta m_\lambda \quad (2)$$

For consistency, the requisite should apply not only to the highest divergent terms at each order, but also to possible lower divergent terms contributing to such mass shifts or to additional quantities that we expect the theory should predict. This might turn out to be impossible - the interpretation of our argument should then be modified ⁶⁾.

At lowest order (second order in W exchange), Eq. (2) gives ²⁾

$$m_p = m_n \cos^2 \theta, \quad m_n \cos^2 \theta = m_\lambda \sin^2 \theta \quad (3)$$

or, equivalently,

$$\tan^2 \theta = \frac{m_n}{m_\lambda}, \quad \frac{1}{m_p} = \frac{1}{m_n} + \frac{1}{m_\lambda} \quad (4)$$

We shall show in this note that these results hold at any order.

The interpretation of Eqs. (4) has already been discussed ²⁾. The first equation gives θ in terms of two quark masses; the second one relates among themselves the three masses (i.e., the three a priori independent parameters that break $SU_3 \times SU_3$). Significantly, when $m_p - m_n$ vanishes (as one may expect in the absence of electromagnetism) one has simultaneously $m_p/m_\lambda (=m_n/m_\lambda) \rightarrow 0$ (i.e., exact $SU_2 \times SU_2$ and $\theta \rightarrow 0$). Indeed, one can argue directly that when $\theta = 0$ the weak currents are divergenceless in the limit of $SU_2 \times SU_2$, as they clearly coincide with the (negative chirality) currents of $SU_2 \times SU_2$. But the quadratic divergent terms in the second order self-masses actually arise from the current divergences.

In a more general framework than the above quark model picture, one assumes a symmetry breaking Hamiltonian density of the form :

$$H' = \epsilon_0 u_c + \epsilon_8 u_8 + \epsilon_3 u_3 \quad (5)$$

In the quark model H' is nothing but the sum of the quark mass terms, allowing the identification :

$$\begin{aligned} -m_p &= \sqrt{2} \epsilon_0 + \epsilon_8 + \sqrt{3} \epsilon_3 \\ -m_n &= \sqrt{2} \epsilon_0 + \epsilon_8 - \sqrt{3} \epsilon_3 \\ -m_\lambda &= \sqrt{2} \epsilon_0 - 2 \epsilon_8 \end{aligned} \quad (6)$$

It will be suggestive to retain this identification also in the general case.

In the general second order matrix element,

$$M^{(2)} = \frac{(-ig_W)^2}{2} \frac{i}{(2\pi)^4} \int d^4 q \frac{1}{q^2 - M_W^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2} \right) \times$$

$$\times \int d^4 x e^{-iqx} \langle \alpha | T \{ j_\mu(x) j_\nu^\dagger(0) + j_\nu^\dagger(x) j_\mu(0) \} | \beta \rangle - i \langle \alpha | C(0) | \beta \rangle + \quad (7)$$

- vacuum amplitude

where j_μ is the source of the charged W (of mass M_W and coupling g_W), C is the counter term, and $|\alpha\rangle$ and $|\beta\rangle$ are states of strong interacting particles, one can isolate the quadratic divergent term in the form $\langle \alpha | m_2 | \beta \rangle = \langle 0 | m_2 | 0 \rangle$, where ^{1), 2)}

$$m_2 = \frac{i}{2(2\pi)^4} \left(\frac{g_W \Lambda}{M_W} \right)^2 \{ [Q, D^\dagger(0)] + [Q^\dagger, D(0)] \} - i C(0) \quad (8)$$

The assumptions are :

- i) that Schwinger terms are c numbers;
- ii) that the Bjorken limit can be applied to the Fourier transform of $T(D(x)D^\dagger(0))$.

For a Cabibbo current and for the assumed H^1 , m_2 is of the form :

$$m_2 = b_0 u_0 + b_3 u_3 + b_8 u_8 + b_6 u_6 + b_7 v_7 \quad (9)$$

(The commutator algebra of charges and divergences has interesting features which are presented in the last Section of this note.) Our requirement of absence of quadratic divergences in $\Delta S=0$ octet amplitudes corresponds to imposing the two equations $b_3=0$ and $b_8=0$, or, explicitly :

$$b_3 = \frac{1}{\sqrt{3}} (\sqrt{2} \epsilon_0 + \epsilon_8) \sin^2 \theta + \epsilon_3 (\cos^2 \theta + 1) = 0 \quad (10)$$

$$b_8 = \frac{1}{3} (\sqrt{2} \epsilon_0 + \epsilon_8) (2 - 3 \sin^2 \theta) + 2 \epsilon_8 \sin^2 \theta + \frac{1}{\sqrt{3}} \sin^2 \theta \epsilon_3 = 0 \quad (11)$$

These equations are, of course, identical to Eqs. (3), through the substitution of Eqs. (6). The counter term C , in Eq. (8), will account for the diverging coefficients b_0 , b_6 and b_7' . However, u_6 and v_7 are easily seen to be proportional to the divergences of the vector and axial currents, respectively, with unitary index 7, and, as such, they will not contribute between states of equal four momentum ⁷⁾. This remark, very useful in second order calculation, does not justify the elimination of counter terms transforming as u_6 and v_7 , which become necessary at higher order ⁸⁾.

Besides the quadratic divergence, the matrix element $M^{(2)}$ in Eq. (7) also contains a logarithmic divergent term, m_0 , which can again be extracted by Bjorken's method

$$m_0 = \frac{ig_w^2}{2(2\pi)^4} \int \frac{d^4 q}{q^4} \int d^4 x e^{iqx} \delta(x_0) \left\{ \frac{1}{M_w^2} [\dot{D}(x), \dot{D}^\dagger(0)] + \frac{1}{M_w^2} [\dot{D}(x)^\dagger, D(0)] + \right. \\ \left. + \left[j_K(0), \frac{\partial}{\partial x_0} j_K^\dagger(x) - \frac{\partial}{\partial x_K} j_0^\dagger(x) \right] + \left[j_K^\dagger(0), \frac{\partial}{\partial x_0} j_K(x) - \frac{\partial}{\partial x_K} j_0(x) \right] \right\} \quad (12)$$

If j_μ is identified with the (conserved) electromagnetic current, the commutator in Eq. (12) is the one which is known to be responsible for the logarithmic divergences in calculations of electromagnetic mass differences ⁹⁾. We expect that a mechanism will be found ¹⁰⁾ for making electromagnetic mass differences finite and, because of the close similarity of the relevant commutators, one may expect that the same mechanism will also eliminate the unwanted logarithmic divergences from W exchange. In any case, the logarithmic divergences will bring additional requirements on the theory, not "directly" related to those from the quadratic divergences. Indeed, the commutators in Eq. (8) depend only on that part of the Hamiltonian which breaks $SU_3 \times SU_3$, whereas the commutators in Eq. (12) depend on the entire Hamiltonian.

The general procedure to extract the divergent terms, using Bjorken's technique, can be extended to the fourth order matrix element

$$\begin{aligned}
 M^{(4)} = & \frac{(-ig_w)^4}{8} \left[\frac{i}{(2\pi)^4} \right]^2 \int d^4q d^4k \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_w^2} \right) \frac{1}{q^2 - M_w^2} \left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{M_w^2} \right) \frac{1}{k^2 - M_w^2} \times \\
 & \times \int d^4x d^4y d^4z e^{-iq(x-y) - ikz} \langle \alpha | T \{ [j_\mu(x) j_\nu^\dagger(y) + j_\mu^\dagger(x) j_\nu(y)] [j_\rho(z) j_\sigma^\dagger(0) + j_\rho^\dagger(z) j_\sigma(0)] \} | \beta \rangle + \quad (13) \\
 & + \frac{ig_w^2}{2} \frac{i}{(2\pi)^4} \int d^4q \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_w^2} \right) \frac{1}{q^2 - M_w^2} \int d^4x d^4y e^{-iq(x-y)} \langle \alpha | T \{ [j_\mu(x) j_\nu^\dagger(y) + j_\mu^\dagger(x) j_\nu(y)] C(0) \} | \beta \rangle + \\
 & - \frac{1}{2} \int d^4x \langle \alpha | T(C(x) C(0)) | \beta \rangle = \text{vacuum amplitude},
 \end{aligned}$$

where $C(x)$ is the counter term (a superposition of u_0 , u_6 , and v_7). One finds that the same conditions, [Eqs. (10) and (11), or, equivalently, Eqs. (3), plus the conditions for the logarithmic term, Eq. (12)] which guarantee the absence of unwanted divergences at second order, are sufficient to guarantee the absence of the corresponding divergences, at fourth order. We have not yet proved, however, that all the conditions used, in particular those relevant for the second order logarithmic divergences, are in fact necessary to guarantee the absence of the unwanted divergences¹¹⁾. We are at present investigating this point, as well as the extension to higher orders of the Bjorken procedure.

3. HIGHER ORDERS IN A QUARK MODEL

We shall first discuss in detail the fourth order calculation; the generalization to all orders will then be evident. The possible kinds of graphs are reported in Fig. 1

Fig. 1

Fourth order self-mass terms in the quark model. Graphs of type (b) are not allowed because of charge conservation.

We have omitted vacuum polarization graphs which are obviously unessential to our argument.

We notice immediately that graphs of type (b) are not allowed from charge conservation (we are assuming only charged W mesons). Let us discuss briefly the graphs of the kind (a). When the external particle is a proton-like quark, p , the intermediate fermions are, according to the graph, n or λ , then p , then again n or λ . The quadratic divergent term will be proportional to

$$m_n (\cos^4 \theta + 2 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) = m_p$$

Similarly, when the external particle is n , the intermediate fermions are p , n or λ , and p , and one gets

$$m_n (\cos^4 \theta + \cos^2 \theta \sin^2 \theta) = m_n \cos^2 \theta$$

When λ is the external particle, the intermediate fermions are again p , n or λ , and p , and the quadratic divergence must be proportional to :

$$m_\lambda (\sin^2 \theta \cos^2 \theta + \sin^4 \theta) = m_\lambda \sin^2 \theta$$

In conclusion, these graphs contribute proportionally to m_p , $m_n \cos^2 \theta$, and $m_\lambda \sin^2 \theta$, to the quadratic divergent terms in the self-masses of p , n , and λ respectively. This was exactly the result obtained at second order. The condition that the quadratic divergent terms do not contribute to mass differences at second order implies therefore that also the quadratic divergences from graphs of type (a) contribute only a common mass renormalization to p , n , and λ . Thus we are left with evaluating the graphs of type (c).

To deal with the three quarks and their possible mixing, we shall employ a matrix formalism in the space of p , n , and λ . Let us define a matrix $\Sigma(K)$, where K is the quark four-momentum, describing the (quadratic divergent) contributions from the second order graphs of Fig. 2 :

Fig. 2

Second order self - mass graphs
in the quark model

The matrix $\Sigma(K)$ has elements Σ_{pp} , Σ_{nn} , $\Sigma_{n\lambda}$, $\Sigma_{\lambda n}$, and $\Sigma_{\lambda\lambda}$, the remaining ones being obviously zero. We know that $\Sigma(K)$ also includes contributions which either cancel out (unitary singlet terms) or vanish (terms proportional to divergences) when one calculates the quark mass differences on the mass shell. These terms are proportional to what remains of the commutator ^{1), 2)} sum, :

$$[Q, D^\dagger(0)] + [Q^\dagger, D(0)] \quad (14)$$

[see Eq. (8)] after imposing the conditions which assure the disappearance of quadratic divergences in the mass differences [vanishing of the coefficients multiplying u_3 and u_8 in the expansion of Eq. (14), see Refs. 1) and 2)]. The relevant expression is given in Eqs. (41)-(43) in the last Section of the present paper, or more explicitly, in terms of quark masses, in Eq. (46).

Our attitude is that terms which do not contribute to the mass differences between particles belonging to the same SU_3 multiplet can be subtracted by suitable counter terms, whose addition, in fact, does not restrict the predictive power of the theory. The convergence conditions, on the other hand, will make the weak mass differences finite and thus predictable by the theory. In this sense our programme is more ambitious than a straight renormalization procedure where counter terms would be introduced also in correspondence to the divergent mass differences, thus making in fact such mass differences quite arbitrary. In the quark model we are using here, one can imagine adding counter terms of the kind $\bar{p}p, \bar{n}n, \bar{\lambda}\lambda, \bar{n}\lambda + \bar{\lambda}n$ and $\bar{n}\gamma_5\lambda - \bar{\lambda}\gamma_5n$ to the Lagrangian, corresponding to a sum of pieces that behave like u_0, u_3, u_8, u_6 and v_7 . What we require in addition from a straight renormalization procedure is that no terms proportional to u_3 and u_8 have to be added, thus making it possible for the theory to predict mass differences. [We are referring only to quadratic divergences and assuming the existence of some mechanism which acts similarly for the logarithmic divergences (or a different explanation for that case).] Instead of the matrix $\Sigma(K)$ describing the graphs in Fig. 2, we must thus consider a matrix $\Sigma(K)-C$, where C is a mass counter term proportional to the expression in Eq. (14), or, explicitly, to the expression in Eq. (23). The graphs of the type (c) correspond to the lowest iteration of the graphs of Fig. 2 and the corresponding amplitudes can be extracted from the propagator expansion

$$\begin{aligned}
 S &= \frac{1}{K-M} + \frac{1}{K-M} (\Sigma-C) \frac{1}{K-M} + \frac{1}{K-M} (\Sigma-C) \frac{1}{K-M} (\Sigma-C) \frac{1}{K-M} + \dots \\
 &= \frac{1}{K-M} \frac{1}{1 - (\Sigma-C) \frac{1}{K-M}}
 \end{aligned}
 \tag{15}$$

One therefore has

$$S^{-1} = K - M - \Sigma + C$$

and the poles (renormalized masses) will be given by the equation

$$\det[S^{-1}] = \det(K - M - \Sigma + C) = 0 \tag{16}$$

The matrix elements of Σ (quadratic divergent terms) are easily obtained from the graph in Fig. 2. One finds

$$\Sigma_{pp} = f_2 \not{x} \frac{1+i\gamma_5}{2} (\cos^2 \theta + \sin^2 \theta) = f_2 \not{x} \frac{1+i\gamma_5}{2}; \quad \Sigma_{nn} = f_2 \not{x} \frac{1+i\gamma_5}{2} \cos^2 \theta$$

$$\Sigma_{\lambda\lambda} = f_2 \not{x} \frac{1+i\gamma_5}{2} \sin^2 \theta; \quad \Sigma_{n\lambda} = f_2 \not{x} \frac{1+i\gamma_5}{2} \cos \theta \sin \theta = \Sigma_{\lambda n}$$

where f_2 is proportional to g_W^2/m_W^2 and to a quadratically divergent integral. The matrix elements of C are proportional to the matrix elements of the expression in Eq. (46), $\sqrt{2} E'_0 + E'_8$, which we write compactly as $\bar{\psi} E \psi$, where ψ is the quark triplet. The matrix C is then $C = \frac{1}{2} f_2 E$. It will be convenient, in order to have a closed form solution of the determinantal equation, to substitute $\frac{1}{2} f_2 \rightarrow -(\sqrt{1-f_2} - 1)$, corresponding to an addition of terms $O(f^2)$, terms which are indeed left arbitrary in C as long as we limit our calculation of the determinant up to $O(f_2^2)$ [we are iterating $O(f_2)$ terms]. We thus take $C = (1 - \sqrt{1-f_2})E$ and write Eq. (16) as : [after inserting the conditions $m_p = m_n \cos^2 \theta = m_\lambda \sin^2 \theta$]

$$C = \begin{bmatrix} \not{x} - m_p - \Sigma_{pp} + (1 - \sqrt{1-f_2}) m_p; & 0 & 0 \\ 0 & \not{x} - m_n - \Sigma_{nn} + (1 - \sqrt{1-f_2}) m_n \cos^2 \theta; & -\Sigma_{n\lambda} + (1 - \sqrt{1-f_2}) \frac{1}{2} \not{x} \times \sin \theta \cos \theta [(m_\lambda + m_n) - i(m_\lambda - m_n) \gamma_5] \\ 0 & -\Sigma_{n\lambda} + (1 - \sqrt{1-f_2}) \frac{1}{2} \sin \theta \cos \theta [(m_\lambda + m_n) - i(m_\lambda - m_n) \gamma_5]; & \not{x} - m_\lambda - \Sigma_{\lambda\lambda} + (1 - \sqrt{1-f_2}) m_\lambda \sin^2 \theta \end{bmatrix} \quad (17)$$

Equation (17) is an algebraic equation in K^2 . It is most conveniently solved by choosing a representation with γ_5 diagonal and going to

the particle rest frame. One can check immediately that $K^2 = m_p^2$ is the solution of

$$\det [K - m_p - \Sigma_{pp} + (1 - \sqrt{1 - f_2}) m_p] = 0$$

and, with a little more algebra, that $K^2 = m_n^2$ and $K^2 = m_\lambda^2$ are both solutions of

$$\det M(\lambda, n) = 0$$

where $M(\lambda, n)$ is the minor formed by the last two rows and columns in the matrix of Eq. (17) (in its compact form). This completes our proof at fourth order.

It is clear that the proof, in fact, applies to any order. At any finite order $2n+2$ we shall consider separately the "proper" graphs (such as those of Figs. 1a and 1b) and the "improper" ones (such as that of Fig. 1c). The term "improper" is actually applied here in a slightly more general context than in problems with one single fermion, since, for instance in a graph like Fig. 1c, the situation where the external particle is n and the virtual neutral particle is λ , does not correspond to a vanishing denominator (as it does when the internal λ is replaced by n). By a simple extension of the fourth order result, one immediately sees that proper graphs give contributions proportional to m_p , $m_n \cos^2 \theta$, and $m_\lambda \sin^2 \theta$ to p , n and λ respectively. Therefore these graphs will just contribute, because of the relation $m_p = m_n \cos^2 \theta = m_\lambda \sin^2 \theta$, an equal renormalization to p , n , and λ , which can be accounted for by a counter term proportional to u_0 .

The most delicate part of the argument is the one concerning the "improper" graphs, for which additional counter terms proportional to u_6 and v_7 will be needed. The argument, however, is almost the same as for fourth order. We again define a matrix Σ which contains all "proper" graphs [terms diverging like powers of $(G\Lambda^2)$ up to order $2n$:

$$\Sigma = \Sigma_2 + \Sigma_4 + \dots + \Sigma_{2n}$$

The matrix elements of Σ will be of the form

$$\Sigma_{PP} = F_n \times \frac{1+\gamma_s}{2}$$

$$\Sigma_{nn} = F_n \times \frac{1+\gamma_s}{2} \cos^2 \theta$$

$$\Sigma_{\lambda\lambda} = F_n \times \frac{1+\gamma_s}{2} \sin^2 \theta$$

$$\Sigma_{n\lambda} = \Sigma_{\lambda n} = F_n \times \frac{1+\gamma_s}{2} \sin \theta \cos \theta$$

where $F_n = f_2 + f_4 + \dots + f_{2n}$. We now choose the counter term as :

$$C_n = (1 - \sqrt{1 - F_n}) E = \left(\frac{1}{2} f_2 + \frac{1}{2} f_4 + \frac{1}{8} f_2^2 + \dots \right) E$$

We recall that the solutions of Eq. (17) were $K^2 = m_n^2$ and $K^2 = m_\lambda^2$, independent of the value of the parameter f_2 . Clearly, $K^2 = m_n^2$ and $K^2 = m_\lambda^2$ will therefore be the propagator poles at order $2n+2$. The counter term C_n is thus sufficient to renormalize the "improper" diagrams. The full counter term will contain an additional piece proportional to u_0 to renormalize the "proper" graphs, as shown before.

We have thus shown that at any finite order, the highest divergences are renormalized by a counter term transforming as a definite superposition of components u_0, u_6 , and v_7 of the same $(3, \bar{3})$ and $(\bar{3}, 3)$ tensor to which the symmetry breaking Hamiltonian belongs, provided the conditions $m_p = m_n \cos^2 \theta = m_\lambda \sin^2 \theta$ hold. The latter conditions are identical with the second order conditions for absence of quadratic divergences in the mass differences inside an SU_3 multiplet. The value of θ follows directly from the above equation, using some recent analyses of strong interaction data ¹²⁾.

4. GENERAL FOURTH ORDER CALCULATION

The interaction Hamiltonian is assumed to be of the form :

$$g_w (\int_{\mu} w_{\mu} + \int_{\mu}^{\dagger} w_{\mu}^{\dagger}) - C \quad (18)$$

where j_{μ} is the source of the intermediate boson field w_{μ} and is taken to be of the form

$$j_{\mu} = \frac{1}{\sqrt{2}} (j_{\mu}^{(1)} + i j_{\mu}^{(2)}) \cos \theta + \frac{1}{\sqrt{2}} (j_{\mu}^{(4)} + i j_{\mu}^{(5)}) \sin \theta \quad (19)$$

with $j_{\mu}^{(i)} = v_{\mu}^{(i)} + A_{\mu}^{(i)}$ ($i=0,1,\dots,8$); C is the counter term (a linear superposition of u_0 , u_6 , and v_7). The highest divergent part (quadratic divergence) in the second order matrix element of Eq. (7) can be separated and expressed in terms of the operator m_2 in Eq. (8). In Fig. 3 the reduction by use of Ward identities and by assuming the validity of the Bjorken limit is carried out graphically.

Fig. 3

Graphical reduction of the Fourier transform of the T product of two currents

The highest divergence arises from the $q_\mu q_\nu$ term. Acting with q_ν , by partial integration, gives rise to two terms : in the first term a derivative $\partial/\partial x_\nu$ acts inside the T product; in the second term $\partial/\partial x_\nu$ acts on the Θ function leading to an equal time commutator. In the latter term, the dependence inside the integral $\int d^4x$ in Eq. (7) gets lost (we are assuming c number Schwinger terms) and by symmetric integration, the integral $\int d^4q$ gives zero. Acting with q_μ on the surviving term gives a less divergent term (T product of two divergences) and a term proportional to a commutator of a charge and a divergence. This graphical method is useful when dealing with higher orders.

The counter term will be determined from Eq. (8) :

$$C(z) = \frac{1}{2(2\pi)^4} \left(\frac{g_w \Lambda}{M_w} \right)^2 \{ [Q, D^+(z)] + [Q^+, D(z)] \} \quad (20)$$

and it has indeed components only along u_0, u_6 , and v_7 , by virtue of our physical assumption of non-divergent mass shift differences, realized by the two equations (10) and (11).

The reduction is very lengthy and cumbersome for the fourth order matrix element of Eq. (13). We shall not reproduce such reduction, in view of its complexity, but use our graphical formalism to visualize the final result. Starting from the graph of Fig. 4, by

Fig. 4

Graphical representation of the double Fourier transform of the T product of four currents

successive contractions with q_μ , q_ν , K_ζ , and K_σ , partial integrations and elimination of less divergent terms, one finally arrives at the graphs of the kind shown in Fig. 5 :

Fig. 5

Graphs resulting from the reduction of the graph of Fig. 4

The assumptions necessary to carry out the reduction are related to the use of partial integration and to the assumption that the Bjorken limit applies, so as to be able to eliminate less divergent integrals at various steps in the course of the procedure. Again Schwinger terms are assumed to be c numbers, and symmetric integration is applied to eliminate odd integrands in the virtual momenta. The quartic (highest divergence) term in $M^{(4)}$, Eq. (13), is finally obtained as :

$$\langle \alpha | m_4 | \beta \rangle = \langle c | m_4 | c \rangle$$

where m_4 is the sum of the following terms (multiplied by some coefficients) :

$$\left(\frac{g_w \Lambda}{M_w}\right)^4 \{ [Q, D^\dagger(0)] + [Q^\dagger, D(0)] \} \quad (21)$$

$$\left(\frac{g_w \Lambda}{M_w}\right)^4 \{ [Q, [Q^\dagger, c(0)]] + [Q^\dagger, [Q, c(0)]] \} \quad (22)$$

$$\left(\frac{g_w \Lambda}{M_w}\right)^4 \int d^4 x e^{-iqx} \delta(x_0) \sum_{k=1}^3 \left[j_k^{(3)}(x), \frac{\partial}{\partial x_0} j_k^{(3)}(0) - \frac{\partial}{\partial x_k} j_0^{(3)}(0) \right] \quad (23)$$

$$\begin{aligned}
 & \frac{1}{2} \frac{1}{4(2\pi)^4} \left(\frac{g_w \Lambda}{M_w} \right)^4 \int d^4x T \left\{ \left[[Q, D^\dagger(x)] + [Q^\dagger, D(x)] \right] \left\{ [Q, D^\dagger(0)] + [Q^\dagger, D(0)] \right\} \right\} + \\
 & - \frac{i}{2(2\pi)^4} \left(\frac{g_w \Lambda}{M_w} \right)^2 \int d^4x T \left\{ \left[[Q, D^\dagger(x)] + [Q^\dagger, D(x)] \right] C(0) \right\} + \quad (24) \\
 & - \frac{1}{2} \int d^4x T \left\{ C(x) C(0) \right\}
 \end{aligned}$$

In Eq. (23) $j_\mu^{(3)}(x)$ stands for the commutator $[Q, j_\mu^\dagger(x)]$ (it is the third-component in the rotated frame). The term in Eq. (23) contains the commutators :

$$\left[j_\kappa^{(3)}(x), \frac{\partial}{\partial x_\mu} j_\kappa^{(3)}(0) - \frac{\partial}{\partial x_\mu} j_0^{(3)}(0) \right] \delta(x_0)$$

In field algebra ¹³⁾ these commutators are q numbers. Models have, however, been proposed ¹⁰⁾ to obtain finite second order electromagnetic mass differences, where such commutators are c numbers. We also recall ^{1),2)} that the condition that such commutators are c numbers is also required to guarantee the absence of logarithmic divergences in the calculation of weak second order mass shifts. Under such a condition the commutator in Eq. (23) does not contribute to our fourth order quadratic divergent mass shifts. It is immediately seen that the sum in Eq. (24) does similarly give vanishing contribution under the condition in Eq. (20) for the counter term. Finally, Eq. (20) also shows that the terms in Eqs. (21) and (22) have the same structure of the first term on the right-hand side of Eq. (8). Therefore, when the conditions of Eqs. (10) and (11) are satisfied, such terms do not contain contributions proportional to u_3 and u_8 . Our proof is thus completed.

It is very interesting to compare our general proof, here at fourth order, with the corresponding proof for the free quark model (see Section 3). In particular, we recall that no conditions bearing on commutators of the type occurring in Eq. (23) were used in dealing with the quark model. The point is the following. Such commutators

can be formally evaluated in a free quark model. They fail to exhibit a covariant form. Following Bjorken's reasoning [see Ref. 3], one will re-establish covariance through Schwinger terms. The explicit evaluation shows that the contributions from such terms are proportional to

$$[[\sum_i \epsilon_i \lambda_i, \lambda'_3], \lambda'_3] + [\{\sum_i \epsilon_i \lambda_i, \lambda'_3\}, \lambda'_3]$$

where $\sum_i \epsilon_i \lambda_i$ is the quark mass term and λ'_3 is obtained from λ_3 by a Cabibbo rotation (see Section 5). Therefore, in a free quark model, even if such terms appear with non-vanishing coefficients, all the conditions will anyway reduce to those in Eqs. (10) and (11).

5. COMMUTATOR ALGEBRA OF CHIRAL CHARGES AND CHIRAL DIVERGENCES

In this last Section we shall summarize the formal properties of the commutator algebra of chiral charges and divergences. These properties will be of importance in a later work.

We call Q_i^V and Q_i^A ($i=0,1,\dots,8$) the vector and axial generators of chiral $U_3 \times U_3$; $J_{i\mu}^V(x)$ and $J_{i\mu}^A(x)$ the associated currents; and $D_i^V = J_{i\mu}^V / \partial x_\mu$ and $D_i^A = J_{i\mu}^A / \partial x_\mu$ their divergences.

We shall also introduce ¹⁴⁾ the scalar densities $u_i(x)$ together with pseudoscalar densities $v_i(x)$, which form together a $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU_3 \times SU_3$. They satisfy the commutation relations

$$[Q_i^V, u_j] = i f_{ijk} u_k \quad [Q_i^V, v_j] = i f_{ijk} v_k \quad (25)$$

$$[Q_i^A, u_j] = -i d_{ijk} v_k \quad [Q_i^A, v_j] = i d_{ijk} u_k \quad (25')$$

We shall also define

$$Q_i = Q_i^V + Q_i^A \quad ; \quad J_{i\mu} = J_{i\mu}^V + J_{i\mu}^A \quad ; \quad D_i = D_i^V + D_i^A$$

The charges Q_i generate an SU_3 subgroup of $SU_3 \times SU_3$. Suppose we add to an $SU_3 \times SU_3$ symmetric Hamiltonian a symmetry breaking term :

$$\mathcal{H}' = \sum_k \epsilon_k u_k(x) \quad (26)$$

where the ϵ_k 's are real parameters (of course, ϵ_k can differ from zero only for $k = 0, 3, 8$). From (26) it follows that

$$D_i^V = \sum_k \epsilon_k f_{kij} u_j \quad D_i^A = \sum_k \epsilon_k d_{kij} v_j \quad (27)$$

Besides

$$D_i = D_i^V + D_i^A = \sum_l \epsilon_l (f_{l,j} u_j + d_{l,j} v_j) \quad (28)$$

it is convenient to introduce

$$E_i \equiv -i\sqrt{\frac{2}{3}} [Q_0, D_i] = \sum_l \epsilon_l (-f_{l,j} v_j + d_{l,j} u_j) \quad (28')$$

It can be verified that the commutators of E_i and D_i , with charges Q_i , are

$$[Q_i, D_j] = i f_{ijk} D_k + i d_{ijk} E_k \quad (29)$$

$$[Q_i, E_j] = -i d_{ijk} D_k + i f_{ijk} E_k \quad (29')$$

Equations (29) and (29') define the behaviour of the densities D_i , and E_i under the SU_3 subgroup whose generators are the Q_i (they belong to a reducible $3 \oplus 3 \oplus 3 \oplus \bar{3} \oplus \bar{3} \oplus \bar{3}$ representation). We shall now transform into a new SU_3 frame obtained from the original one through a Cabibbo rotation. We define :

$$Q'_k = e^{i\theta Q_7^V} Q_k e^{-i\theta Q_7^V} = \sum_j A_{kj} Q_j \quad (30)$$

and, correspondingly,

$$D'_k = \sum_j A_{kj} D_j \quad E'_k = \sum_j A_{kj} E_j \quad (31)$$

Equations (29) and (29') also hold for the rotated charges Q'_k and densities D'_k and E'_k . Furthermore, in terms of the rotated

$$u'_k = A_{kj} u_j \quad v'_k = A_{kj} v_j \quad (32)$$

$$D'_i = \sum_e \epsilon'_e (f_{eij} u'_j + d_{eij} v'_j) \quad (33)$$

$$E'_i = \sum_e \epsilon'_e (-f_{eij} v'_j + d_{eij} u'_j) \quad (33')$$

where

$$\epsilon'_e = \sum_m A_{em} \epsilon_m \quad (34)$$

For the description of weak interactions, we are interested in the SU_2 subgroup, whose generators are Q'_1, Q'_2, Q'_3 , or, in a spherical basis, Q'_+, Q'_- , and Q'_0 . The relevant commutators can be obtained from (29) and (29') taken in the rotated frame. One immediately finds

$$[Q'_+, D'_+] = [Q'_-, D'_-] = 0 \quad [Q'_3, D'_3] = \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8) \quad (35)$$

$$[Q'_\pm, D'_\mp] = \pm D'_3 + \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8) \quad (36)$$

$$[Q'_\pm, D'_3] = \mp D'_\pm \quad [Q'_3, D'_\pm] = \pm D'_\pm \quad (37)$$

$$[Q'_\pm, \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8)] = D'_\pm \quad [Q'_3, \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8)] = D'_3 \quad (38)$$

From Eqs. (35)-(38) one sees that

$$\begin{bmatrix} D'_+ \\ -D'_3 + \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8) \end{bmatrix}, \quad \begin{bmatrix} D'_3 + \frac{i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8) \\ D'_- \end{bmatrix} \quad (39)$$

transform like spinors with respect to the SU_2 subgroup of generators Q'_+ , Q'_- , and Q'_0 . From Eq. (36), one obtains for the commutator sum occurring in Eq. (8) :

$$[Q'_+, D'_-] + [Q'_-, D'_+] = \frac{2i}{\sqrt{3}} (\sqrt{2} E'_0 + E'_8) \quad (40)$$

The requirement of absence of quadratic divergence at lowest order is expressed by the condition that in the expression for $\sqrt{2}E'_0 + E'_8$ [as given by Eq. (33')] , the coefficients in front of u_3 and u_8 vanish. One then obtains :

$$\sqrt{2}E'_0 + E'_8 = a u_0 + \frac{\sqrt{3}}{2} \sin 2\theta (b D_7^V + \frac{1}{b} D_7^A) \quad (41)$$

where

$$a = \frac{\sqrt{2}}{3} [\sqrt{2} \epsilon_0 + \epsilon_8 + \frac{\sqrt{3}}{2} \sin^2 \theta (\epsilon_3 - \sqrt{3} \epsilon_8)] \quad (42)$$

$$b = \frac{1}{\sqrt{3}} (2\sqrt{2} \epsilon_0 - \sqrt{3} \epsilon_3 - \epsilon_8) / (-\sqrt{3} \epsilon_8 + \epsilon_3) \quad (42')$$

The divergences D_7^V and D_7^A are obtained from Eq. (28) :

$$D_7^V = -\frac{1}{2} (\sqrt{3} \epsilon_8 - \epsilon_3) u_6 \quad D_7^A = \frac{1}{2\sqrt{3}} (2\sqrt{2} \epsilon_0 - \sqrt{3} \epsilon_3 - \epsilon_8) v_7 \quad (43)$$

In a quark model one may identify the symmetry breaking term $\mathcal{H}'$ of Eq. (26) with the sum of the mass terms for the quarks :

$$\mathcal{H}' = m_p \bar{p} p + m_n \bar{n} n + m_\lambda \bar{\lambda} \lambda$$

where p , n , and λ stand for the p , n , and λ quark. One obtains :

$$a = -\frac{\sqrt{3}}{2} m_p \quad b = (m_\lambda + m_n) / (m_\lambda - m_n) \quad (44)$$

$$D_7^V = \frac{i}{2\sqrt{3}} (m_n - m_\lambda) u_6 \quad D_7^A = - \frac{i}{2\sqrt{3}} (m_n + m_\lambda) v_7 \quad (45)$$

and

$$[2E'_0 + E'_8] = - \left[\frac{3}{2} m_p u_0 - \frac{1}{2} \sin\theta \cos\theta [(m_\lambda + m_n) u_6 + (m_\lambda - m_n) v_7] \right] \quad (46)$$

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- 1) - R. Gatto, G. Sartori and M. Tonin - "Convergence Conditions and Cabibbo Angle", University of Padua Preprint, (July 1968).
- 2) - R. Gatto, G. Sartori and M. Tonin - CERN Preprint Th. 942, (September 1968), to appear in Phys.Letters.
- 3) - J.D. Bjorken - Phys.Rev. 148, 1467 (1966).
- 4) - It often happens that the interpretation of a theoretical result gets modified or enlarged in the subsequent discussions. Nevertheless we feel that we should, also for practical purposes, stick to a well-defined theoretical thesis. Such is our starting assumption that the theory shall be able to predict the values of the weak mass differences within an SU_3 multiplet. We like, however, to be completely open-minded as to a different interpretation of our results, provided it brings some real progress.
- 5) - This result is shown by an elegant argument by T.D. Lee, for the case of a neutral vector boson : T.D. Lee, CERN Preprint Th. 940, (September 1968). It is derived in the present paper for a charged vector boson.
- 6) - A new approach to the problem of divergences is being developed by T.D. Lee. We are grateful to Professor Lee for pointing out to us that in a convergent theory our determination of the Cabibbo angle from the vanishing of the coefficients of the quadratic divergences might be interpreted as expressing the smallness ($\sim G$) of the weak mass differences, as one would expect from weak interactions. Different proposals on the problem of divergences in weak interactions are due to : M. Gell-Mann, M.L. Goldberger, N.M. Kroll and F.E. Low - Preprint (June 1968), and M. Veltman - Orsay Preprint, (August 1968).
- 7) - C. Bouchiat, J. Iliopoulos and J. Prentki - CERN Preprint Th. 908 (1968).

- 8) - N. Cabibbo and L. Maiani (Istituto Superiore di Sanità, Roma, Preprint, October 1968) have recently advanced a proposal of substituting to our Eqs. (2) above the stronger requisite $\delta m_p = \delta m_n = \delta m_\lambda = 0$ (thus making the introduction of a counter-term behaving like u_0 unnecessary) at the same time including in the mass-shift terms accounting for tadpole electromagnetic contributions. Their proposal amounts to substituting for the two relations of Eqs. (3) above the three equations

$$m_p + \alpha = 0 \qquad m_n \cos^2 \theta + \beta = 0 \qquad m_\lambda \sin^2 \theta + \gamma = 0$$

where α , β and γ should be tadpole parts of the electromagnetic mass shifts. We note : (i) unfortunately this proposal will remain untestable until α , β and γ are calculated ; (ii) the only testable part of the proposal (if it is supplemented by the argument that it may not be a bad approximation to take $\beta = \gamma$) reduces to the second of our relations in Eq. (3), obtained by eliminating β ; (iii) even when counter terms behaving like u_0 are avoided, one is still left with the necessity of counter terms behaving like u_6 and v_7 ; (iv) concerning the assumption $\beta = \gamma$ (neglect of SU_3 corrections in the electromagnetic mass shifts), if SU_3 corrections were everywhere neglected (i.e., $\beta = \gamma$ and $m_n = m_\lambda$), one would have the unacceptable result $\cos^2 \theta = \sin^2 \theta$. The correct result, i.e., the second of our Eqs. (3), comes about, in the interpretation by Cabibbo and Maiani, from neglecting SU_3 corrections in the electromagnetic mass shifts while keeping them in the weak mass-shifts. This excludes simplest models where electromagnetic mass shifts are proportional to the initial masses (recalling that electromagnetism added to a chiral invariant Hamiltonian preserves chiral invariance).

- 9) - See for instance : M.B. Halpern and G. Segré - Phys.Rev.Letters 20, 1527 (1968) ;
G.C. Wick and B. Zumino - Phys.Letters 25B, 479 (1967).

10) - See for instance : T.D. Lee - Phys.Rev. 168, 1714 (1968) ;
S. Ciccariello, M. Tonin and G. Sartori - Nuovo Cimento
55A, 847 (1968).

11) - The authors, in footnote ⁸⁾ attribute to us a result : "...the leading divergences at each order have their exact form determined by the detailed structure of the symmetry breaking." This result is certainly true at second order and is due to C. Bouchiat, J. Iliopoulos and J. Prentki, Ref. ⁷⁾. The result is also true at any order in the quark model. Unfortunately we have not yet succeeded in giving a more general proof of such a conjecture (plausible in some respects).

12) - In terms of the parameters ϵ_0 and ϵ_8 of Eqs. (6), one can write ^{1),2)}

$$\tan^2 \theta \approx \frac{1}{3} \vartheta$$

where

$$\vartheta = - \left(1 + \frac{f_1 \epsilon_0}{\epsilon_8} \right)$$

The ratio $\sqrt{2} \epsilon_0 / \epsilon_8$ can be obtained from the work by S. Glashow and S. Weinberg (Phys.Rev.Letters 20, 224, 1968) and one obtains the estimate $\theta = 0.29$. Using the value $\epsilon_8 / \epsilon_0 = -1.25$ from the recent analysis of M. Gell-Mann, R.J. Oakes and B. Renner (Caltech Preprint 68-165), one obtains instead $\theta = 0.21$. Rather than preferring one value or the other, one should consider that large errors may be involved in the determinations of the relevant ratio

$$\sqrt{2} \epsilon_0 / \epsilon_8.$$

13) - N.M. Kroll, T.D. Lee and B. Zumino - Phys.Rev. 157, 1376 (1967);
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14) - M. Gell-Mann - Physics 1, 63 (1964).