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CONSTRUCTION OF COVARIANT AND GAUGE INVARIANT T^* PRODUCTS

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A B S T R A C T

The problem of constructing sea-gull terms, which render time-ordered products of arbitrary operators Lorentz covariant, is investigated.

We determine the necessary and sufficient conditions that the commutators of operators must satisfy in order that it be possible to covariantize their time-ordered product. We give an explicit formula for the sea-gull in terms of the relevant Schwinger term. We show that our considerations, which use formal equal time commutators, are equally valid for commutators defined by use of the Bjorken-Johnson-Low theorem.

We then address ourselves to the question of whether sea-gulls and Schwinger terms cancel in the Ward-Takahashi identities which these covariant T^* products satisfy. We give necessary and sufficient conditions for this cancellation to take place. Specific examples of the general theory are given.

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1. INTRODUCTION

In applications of current algebra, it is frequently desired to construct covariant time ordered products, T^* products, for the currents. As is well known, the ordinary time ordered product, the T product, need not be covariant due to singularities of the current commutators on the light cone. Covariance is then effected by the addition of a sea-gull term. One also wishes to study the Ward-Takahashi identities which these T^* products satisfy. These identities, which in a complete theory may be derived from the gauge principles that are satisfied, are usually obtained with the help of Feynman's conjecture that the gradient terms in the commutators, the Schwinger terms, cancel against the divergences of sea-gulls¹⁾. Thus the derivation of these identities proceeds "naïvely": one ignores Schwinger terms and sea-gull terms altogether. However, it has been shown recently that in calculating the $\pi^0 \rightarrow 2\gamma$ decay rate in the ∇ model by use of PCAC, this desired cancellation does not occur, and Feynman's conjecture is not valid, due to this anomalous behaviour²⁾. It has also been observed that T^* products involving the energy momentum tensor will not in general satisfy Ward-Takahashi identities which are got by ignoring sea-gulls and Schwinger terms³⁾.

In this paper we address ourselves to the following problem. What are the conditions that commutators must satisfy so that

- 1) it is possible to construct a covariant T^* product;
- 2) how should the sea-gull, which covariantizes the T product, be constructed from the commutators; and
- 3) what additional constraints are placed on the sea-gull such that the Ward-Takahashi identities, obtained by ignoring Schwinger terms and sea-gull terms, are valid⁴⁾?

In Section 2 we solve the above problem for T products of two operators. Section 3 is devoted to specific examples of the general theory developed in the previous Section. In Section 4 the treatment is generalized to three operators, indicating how the techniques are to be extended to an arbitrary number of operators.

We shall restrict our investigation to the case that the commutators contain at most one derivative of a delta function. This restriction is only for notational economy - it will be quite evident how to generalize to an arbitrary number of delta function gradient terms. Also, when dealing with currents, we shall always consider the currents to be conserved, or to satisfy PCAC, though again the generalization is trivial.

2. CONSTRUCTION OF SEA-GULLS

A. The problem of Lorentz covariance

Consider two operators, A and B, whose commutator is known at equal times. It is desired to determine the sea-gull C, such that the T product $T(AB)$ can be covariantized. We have found that the easiest approach to this problem of Lorentz covariance makes use of an explicit, unit timelike vector n^μ , $n^2 = 1$, which permits one to express non-covariant objects like T products and equal time commutators in explicitly covariant, though frame dependent (n dependent) fashion. Thus we have

$$T^*(x, y) = T(x, y; n) + C(x, y; n), \quad (1)$$

$$T(x, y; n) = \theta([x-y] \cdot n) A(x) B(y) + \theta([y-x] \cdot n) B(y) A(x), \quad (2)$$

$$\begin{aligned} \delta([x-y] \cdot n) [A(x), B(y)] = & S(y; n) \delta^4(x-y) + \\ & + S^\alpha(y; n) P_{\alpha\beta} \partial^\beta \delta^4(x-y). \end{aligned} \quad (3)$$

In the above C is an n dependent sea-gull which, when summed with the n dependent T product, yields a covariant (n independent) T^* product. Equation (3) is a statement of the equal time commutator; a Schwinger term, $S^\nu(y;n)$, has been allowed for. $P_{\alpha\beta}$ is a spacelike projection operator, which assures that no time derivatives of the delta function are present.

$$P_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta. \quad (4)$$

By hypothesis, $T^*(x,y)$ is independent of n , hence insensitive to variations of n . However, since n^2 is constrained to unity, only variations on a spacelike direction are unrestricted. These remarks indicate that (1), (2) and (3) have the following consequence

$$\begin{aligned} P^{\mu\nu} \frac{\delta}{\delta n^\nu} T^*(x,y) &= P^{\mu\nu} (x-y)_\nu \delta([x-y] \cdot n) [A(x), B(y)] + \\ &+ P^{\mu\nu} \frac{\delta}{\delta n^\nu} C(x,y;n) = 0 \end{aligned} \quad (5)$$

$$P^{\mu\nu} \frac{\delta}{\delta n^\nu} C(x,y;n) = S^\nu(y;n) P_\nu{}^\mu \delta^4(x-y). \quad (6)$$

The above reflects the well-known fact that no sea-gull is necessary when Schwinger terms are absent, i.e., the T product is already covariant. The projection operator occurring in (6) can be removed if one allows for an additional term, proportional to n^ν to be present. However, since S^ν is defined by (3) only up to terms proportional to n^ν , we may absorb such terms in S^ν , and replace (6) by

$$C(x,y;n) = C(y;n) \delta^4(x-y). \quad (7a)$$

$$\frac{\delta}{\delta n^\nu} C(y; n) = S^\nu(y; n). \quad (7b)$$

Evidently it follows from (7b) that S_{AB} must be curlless.

$$\frac{\delta}{\delta n^\beta} S^\alpha(y; n) = \frac{\delta}{\delta n^\alpha} S^\beta(y; n) \quad (8)$$

This is our first result: a T product can be covariantized if and only if the Schwinger term is curlless in the sense (8). When that condition is satisfied, the sea-gull is given by the line integral

$$C(y; n) = \int dn'_\alpha S^\alpha(y; n'). \quad (9)$$

Alternatively one may view (7) as a formula for the Schwinger term, given the sea-gull. Since the T^* product is n independent, this also means that the Schwinger term is determined from the T product by the relation

$$\begin{aligned} P^{\mu\nu} \frac{\delta}{\delta n^\nu} T(x, y; n) &= P^{\mu\nu} (x-y)_\nu \delta([x-y] \cdot n) [A(x), B(y)] = \\ &= -S^\nu(y, n) P_\nu{}^\mu \delta^\mu(x-y). \end{aligned} \quad (10)$$

There exists another connection between T products and commutators as given by the Bjorken-Johnson-Low⁵⁾ theorem, which relates the high energy behaviour of T products to equal time commutators. In our notation the theorem can be stated as follows

$$T(q; n) \equiv \int d^4x e^{iq \cdot x} T(x, 0; n), \quad (11)$$

$$S(q, n) \equiv \int d^4x e^{iq \cdot x} \delta(x \cdot n) [A(x), B(0)], \quad (12)$$

$$\lim_{q \cdot n \rightarrow \infty} (q \cdot n) T(q; n) = i S(q; n). \quad (13)$$

When this result is combined with our formula (10), appropriately Fourier transformed, we find the unexpected relation

$$\lim_{q \cdot n \rightarrow \infty} q \cdot n P^{\mu\nu} \frac{\delta}{\delta q^\nu} T(q; n) = - P^{\mu\nu} \frac{\delta}{\delta n^\nu} T(q; n). \quad (14)$$

That this equation is indeed true can be directly ascertained from (11) with the help of the integral representation of the step function.

Therefore our considerations apply with equal validity to commutators defined via the Bjorken-Johnson-Low theorem. Of course (9) does not determine the sea-gull uniquely; any covariant, n independent, expression may be added to (9). We shall learn below how to fix the form of C further.

What is the meaning of the curl condition (8)? It can be shown that (8) is a consequence of the requirement that the equal time commutator, hence the Schwinger term, is a restriction to equal times of a Lorentz covariant, n independent, unequal time commutator. This demonstration is performed in the Appendix. Thus (8) is always satisfied for commutators arising in a Lorentz covariant theory.

B. Further constraints arising from conservation and/or PCAC

To continue our analysis, we now take the operators A and B to be currents, A^μ and B^ν . All formulas derived above still hold; but the notation is now modified to exhibit the additional two indices μ, ν . We take the currents to be conserved, or to

satisfy PCAC in the specific sense given below. It turns out that conservation and PCAC, combined with Lorentz invariance, lead to further restrictions. To exhibit these, we return to (2), (3) and (7), which now read

$$\begin{aligned} T^{*\mu\nu}(x,y) &= \theta([x-y] \cdot n) A^\mu(x) B^\nu(y) + \\ &+ \theta([y-x] \cdot n) B^\nu(y) A^\mu(x) + C^{\mu\nu}(y;n) \delta^4(x-y) + \tilde{C}^{\mu\nu}(x,y). \end{aligned} \quad (15)$$

where we have allowed for the possibility of adding an n independent sea-gull $\tilde{C}^{\mu\nu}$; by necessity it is local in $x-y$.

$$\begin{aligned} \delta([x-y] \cdot n) [A^\mu(x), B^\nu(y)] &= S^{\mu\nu}(y;n) \delta^4(x-y) + \\ &+ S^{\mu\nu/\alpha}(y;n) P_{\alpha\beta} \partial^\beta \delta^4(x-y), \end{aligned} \quad (16)$$

$$S^{\mu\nu/\alpha}(y;n) = \frac{\delta}{\delta n_\alpha} C^{\mu\nu}(y;n). \quad (17)$$

Diverging (15) with $\partial/\partial X_\mu$, we find

$$\begin{aligned} \frac{\partial}{\partial x^\mu} T^{*\mu\nu}(x,y) &= \theta([x-y] \cdot n) \partial_\mu A^\mu(x) B^\nu(y) + \\ &+ \theta([y-x] \cdot n) B^\nu(y) \partial_\mu A^\mu(x) + C^{\mu\nu}(y;n) \partial_\mu \delta^4(x-y) + \\ &+ n_\mu S^{\mu\nu}(y;n) \delta^4(x-y) + n_\mu S^{\mu\nu/\alpha}(y;n) P_{\alpha\beta} \partial^\beta \delta^4(x-y) \\ &+ \frac{\partial}{\partial x^\mu} \tilde{C}^{\mu\nu}(x,y). \end{aligned} \quad (18)$$

For conserved currents, the T product on the right-hand side of (18) is absent. The remaining terms must be covariant, since T^* possesses this property. Thus we find immediately that

$$\eta_\mu S^{\mu\nu}(y; n) = I^\nu(y), \quad (19)$$

$$C^{\beta\nu}(y; n) + \eta_\mu S^{\mu\nu/\alpha}(y; n) P_\alpha^\beta = I^{\beta\nu}(y). \quad (20)$$

In the above, I^ν and $I^{\beta\nu}$ are Lorentz covariant, n independent. For PCAC currents we again obtain the same result, if we interpret PCAC to mean that $\partial_\mu A^\mu$ is a sufficiently gentle operator, so that it does not give rise to Schwinger terms when commuted with B^ν . In that case the T product in the right-hand side of (14) is separately covariant, and the remaining terms must again satisfy (19) and (20). Similar results are obtained by diverging with $\partial/\partial y^\nu$:

$$\eta_\nu S^{\mu\nu}(y; n) - \partial_{y^\nu} C^{\mu\nu}(y; n) = J^\mu(y), \quad (21)$$

$$C^{\mu\beta}(y; n) + \eta_\nu S^{\mu\nu/\alpha}(y; n) P_\alpha^\beta = J^{\mu\beta}(y). \quad (22)$$

Here J^μ and $J^{\mu\beta}$ are again covariant, n independent quantities, not necessarily equal to I^μ and $I^{\mu\beta}$.

The solution to (19) is

$$S^{\mu\nu}(y; n) = n^\mu I^\nu(y) + P^\mu_\alpha S^{\alpha\nu}(y; n), \quad (23)$$

while (20) and (22) require

$$n_\alpha C^{\alpha\beta}(y; n) = n_\alpha I^{\alpha\beta}(y), \quad (24a)$$

$$\eta_\beta C^{\alpha\beta}(y; n) = \eta_\beta J^{\alpha\beta}(y). \quad (24b)$$

Note that (24) shows that

$$n_\alpha n_\beta (I^{\alpha\beta}(y) - J^{\alpha\beta}(y)) = 0 \quad (25)$$

which means that if $I^{\alpha\beta} \neq J^{\alpha\beta}$, then their difference must be a Lorentz covariant tensor antisymmetric in α and β . Also (20) and (22) can be written in the form

$$n_\mu S^{\mu\nu/\alpha}(y;n) P_\alpha^\beta = I^{\beta\nu}(y) - C^{\beta\nu}(y;n), \quad (26a)$$

$$n_\nu S^{\mu\nu/\alpha}(y;n) P_\alpha^\beta = J^{\mu\beta}(y) - C^{\mu\beta}(y;n), \quad (26b)$$

which by virtue of (24) gives

$$\begin{aligned} n_\mu n_\nu S^{\mu\nu/\alpha}(y;n) P_\alpha^\beta &= n_\nu (I^{\beta\nu}(y) - J^{\beta\nu}(y)) = \\ &= n_\mu (J^{\mu\beta}(y) - I^{\mu\beta}(y)). \end{aligned} \quad (26c)$$

It is seen from (26c) that a Schwinger term in the 00 commutator is present if and only if $I^{\beta\nu} \neq J^{\beta\nu}$. If such an object is indeed present, i.e., $S^{00/i} \neq 0$, it must be the $0i$ component of an anti-symmetric Lorentz tensor. This result will assume great relevance to the problem of cancellation of Schwinger terms with divergences of sea-gulls, as will be discussed below.

Equations (23) and (24) place subtle restrictions on the Lorentz properties of sea-gulls, commutators and Schwinger terms. [An example is (26c).] These restrictions have been derived by us previously using a slightly different formalism³⁾. Here we will merely quote some of the results; they can be easily derived in the present formalism by manipulating (17), (23) and (24). The restrictions are:

- 1) once the term proportional to the delta function in the $[\bar{A}^0 B^0]$ commutator, i.e., S^{00} has been determined, then the delta function portion of the $[\bar{A}^0 B^i]$ and the $[\bar{B}^0 A^i]$ commutator is also known;
- 2) if the Schwinger term in the $[\bar{A}^0, B^i]$ and in the $[\bar{A}^i, B^0]$ commutators vanishes, i.e., $S^{0i/j} = S^{i0/j} = 0$; then all the gradient terms in the commutator are absent. In particular it is impossible to have a gradient term only in the $[\bar{A}^i, B^j]$ commutator; and it is therefore possible to determine $S^{ij/k}$ from the other components of $S^{\mu\nu/\alpha}$;
- 3) when the $[\bar{A}^0, B^0]$ Schwinger term is absent a symmetry property for the $[\bar{A}^0, B^i]$ Schwinger term is satisfied. To exhibit the symmetry, we introduce the subscripts A, B on $S^{\mu\nu/\alpha}$:

$$S_{AB}^{0i/j} = S_{BA}^{0j/i} \quad (27)$$

C. Cancellation of Schwinger terms with divergences of sea-gulls

We now have exploited all the constraints that Lorentz covariance and conservation or PCAC impose on the sea-gulls and Schwinger terms. Next, we enquire what further constraints are present when it is insisted that, in the derivation of Ward-Takahashi identities, a cancellation occurs between the divergence of the sea-gull and the Schwinger term, i.e., that Feynman's conjecture holds. Evidently, this requirement asserts that there is no gradient of delta function in the Ward-Takahashi identities

$$\begin{aligned} \frac{\partial}{\partial x^\mu} T^{*\mu\nu}(x, y) &= T(\partial_\mu A^\mu(x) B^\nu(y)) + I^\nu(y) \delta^4(x-y) \\ &+ I^{\beta\nu}(y) \partial_\beta \delta^4(x-y) + \frac{\partial}{\partial x^\mu} \tilde{C}^{\mu\nu}(x, y). \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y^\nu} T^{*\mu\nu}(x,y) &= T(A^\mu(x) \partial_\nu B^\nu(y)) - \mathcal{J}^\mu(x) \delta^4(x-y) + \\ &- \mathcal{J}^{\mu\beta}(y) \partial_\beta \delta^4(x-y) + \frac{\partial}{\partial y^\nu} \tilde{C}^{\mu\nu}(x,y). \end{aligned} \quad (29)$$

If we are to have the sea-gulls cancel the Schwinger term in the Ward identity $\tilde{C}^{\mu\nu}$ must be chosen so that

$$\tilde{C}^{\mu\nu}(x,y) = -I^{\mu\nu}(x) \delta^4(x-y) + F^{\mu\nu}(x,y), \quad (30a)$$

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu}(x,y) = 0. \quad (30b)$$

Similarly the second Ward identity requires

$$\tilde{C}^{\mu\nu}(x,y) = -\mathcal{J}^{\mu\nu}(x) \delta^4(x-y) + F'^{\mu\nu}(x,y), \quad (31a)$$

$$\frac{\partial}{\partial y^\nu} F'^{\mu\nu}(x,y) = 0. \quad (31b)$$

It is therefore always possible to satisfy one of the two Ward identities. This is because $\tilde{C}^{\mu\nu}$ can be chosen to be equal to $-I^{\mu\nu}(x) \delta^4(x-y)$, thereby satisfying (30) or equal to $-\mathcal{J}^{\mu\nu}(x) \delta^4(x-y)$, thereby satisfying (31). However, it is not always possible to satisfy both identities. This can be achieved if and only if $X^{\mu\nu} \equiv I^{\mu\nu} - \mathcal{J}^{\mu\nu} = -X^{\nu\mu}$ satisfies:

$$\partial_\mu X^{\mu\nu} = \partial_\nu X^{\mu\nu} = 0. \quad (32)$$

This is because if $X^{\mu\nu}$ satisfies (32) then it can be written as $X^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha X_\beta$, and we then choose $\tilde{C}^{\mu\nu}$ to be

$$\begin{aligned} \tilde{C}^{\mu\nu}(x,y) &= -I^{\mu\nu}(x) \delta^4(x-y) - \varepsilon^{\mu\nu\alpha\beta} X_\alpha(y) \partial_\beta \delta^4(x-y) \\ &= -\mathcal{J}^{\mu\nu}(y) \delta^4(x-y) - \varepsilon^{\mu\nu\alpha\beta} X_\alpha(x) \partial_\beta \delta^4(x-y), \end{aligned} \quad (33)$$

satisfying both of the above requirements.

This condition is also necessary, since if there exist functions F and F' which satisfy Eqs. (30) and (31), then $X^{\mu\nu}(x)$ must satisfy $\partial_\mu^x \partial_\nu^y \{ X^{\mu\nu}(x) \delta^4(x-y) \} = 0$, and therefore Eq. (32).

It is seen from (26c) that this severely restricts the form of the $[\bar{A}^0, B^0]$ Schwinger term, which must be of the form

$$\delta(x_0 - y_0) [A^0(x), B^0(y)] \sim \epsilon_{ijk} \partial^j X^k(y) \partial^i \delta(x-y). \quad (34)$$

It is amusing to note that such a Schwinger term arises in the local commutators of an electromagnetic current which contains an anomalous magnetic moment ($J^\nu = \partial_\mu \bar{\psi} \sigma^{\mu\nu} \psi$).

Thus our conclusion is that although it is always possible to choose the sea-gull so that in one Ward-Takahashi identity divergences of sea-gulls and Schwinger terms cancel, it need not be the case that this cancellation will occur in both identities. The necessary and sufficient condition for the Feynman conjecture to be valid for both identities is that the Schwinger term in $[\bar{A}^0, B^0]$ be either absent or of the form given in (34).

Since the non-cancellation in both identities, as well as the Schwinger terms in the $[\bar{A}^0, B^0]$ commutator, involve the quantity $X^{\mu\nu}$, an object which is antisymmetric in μ and ν as shown above, we may expect the occurrence of these unexpected terms whenever antisymmetric tensors arise naturally in the theory.

Finally, we must emphasize that although it is always possible to satisfy one Ward-Takahashi identity, within the context of the present investigation, there may exist further requirements on the T^* product which would prevent one from adding the sea-gull which affects Feynman's conjecture. Examples of such further restrictions will be exhibited in the next Section.

3. EXAMPLES

A. We present three examples in application of the above theory. Consider first the minimal system of commutators, valid in scalar electrodynamics

$$\begin{aligned} [\mathcal{D}^0(x,t), \mathcal{D}^0(y,t)] &= 0 \\ [\mathcal{D}^0(x,t), \mathcal{D}^i(y,t)] &= S(y) \partial^i \delta(x-y) \\ [\mathcal{D}^i(x,t), \mathcal{D}^k(y,t)] &= 0 \end{aligned} \quad (35a)$$

In our covariant notation, this may be rewritten as

$$\delta([x-y] \cdot n) [\mathcal{D}^\mu(x), \mathcal{D}^\nu(y)] = S(y) (n^\mu g^{\nu\alpha} + n^\nu g^{\mu\alpha}) P_{\alpha\beta} \partial^\beta \delta^4(x-y). \quad (35b)$$

The Schwinger term $S^{\mu\nu/\alpha}(y,n)$ is given by

$$S^{\mu\nu/\alpha}(y;n) = S(y) [n^\mu g^{\nu\alpha} + n^\nu g^{\mu\alpha}] \quad (36)$$

and is seen to satisfy the curl condition (8). A sea-gull which covariantizes $T(j^\mu(x)j^\nu(y))$ is given by $S(x)n^\mu n^\nu \delta^4(x-y)$. However, this choice for $C^{\mu\nu}$ will not satisfy Feynman's conjecture in either index. We therefore choose the sea-gull to be

$$C^{\mu\nu}(x,y;n) = S(x) (n^\mu n^\nu - g^{\mu\nu}) \delta^4(x-y). \quad (37)$$

Therefore

$$T^*\{\mathcal{D}^\mu(x)\mathcal{D}^\nu(y)\} = T\{\mathcal{D}^\mu(x)\mathcal{D}^\nu(y)\} + S(x) (n^\mu n^\nu - g^{\mu\nu}) \delta^4(x-y) \quad (38)$$

is covariant, and satisfies the Ward-Takahashi identity

$$\frac{\partial}{\partial x^\mu} T^{*\mu\nu}(x,y) = 0. \quad (39)$$

B. As a second example, we consider the anomalous commutators which have been discovered in connection with the axial-vector, vector, vector triangle graph ^{6),7)}.

The one-photon matrix elements of the $[J^\mu, J_5^\nu]$ commutator can be represented by the following

$$\begin{aligned} [\mathcal{J}^0(x,t), \mathcal{J}_5^0(y,t)] &= 2c \tilde{F}_{ij}^0 \partial_j \delta(x-y), \\ [\mathcal{J}^0(x,t), \mathcal{J}_5^i(y,t)] &= c \tilde{F}_{ij}^0 \partial_j \delta(x-y), \\ [\mathcal{J}^i(x,t), \mathcal{J}_5^0(y,t)] &= -c \partial_j \tilde{F}^{ij}(x) \delta(x-y) + \\ &\quad + c \tilde{F}^{ij}(y) \partial_j \delta(x-y). \end{aligned}$$

(40a)

Here $c = ie/8\pi^2$ and $\tilde{F}^{\mu\nu}$ is the antisymmetric, conserved electromagnetic dual tensor. This is rewritten in the covariant notation as

$$\begin{aligned} \delta([x-y] \cdot n) [\mathcal{J}^\mu(x), \mathcal{J}_5^\nu(y)] &= -c n^\nu P_{\alpha\beta} \partial^\alpha \tilde{F}^{\mu\beta}(x) \delta^\nu(x-y) + \\ &\quad + \bar{S}^{\mu\nu}(x; n) \delta^\nu(x-y) + \\ &\quad + c [\tilde{F}^{\mu\gamma}(x) (g^\alpha_\gamma n^\nu + g^{\alpha\nu} n_\gamma) + \tilde{F}^{\nu\gamma}(y) (g^\alpha_\gamma n^\mu + g^{\alpha\mu} n_\gamma)] P_{\alpha\beta} \partial^\beta \delta^\nu(x-y). \end{aligned}$$

(40b)

In (40b) $\bar{S}^{\mu\nu}$ allows for a possible delta function commutator in the space-space components, which is not determined by (30a) ($n_\mu \bar{S}^{\mu\nu} = 0 = \bar{S}^{\mu\nu} n_\nu$). No unknown gradient terms are present, since they could contribute only to the space-space commutator which is impossible; see above.

Before we apply our analysis to the above commutator, we remark that these commutators arise from a diagram which cannot maintain both vector and axial current conservation⁸⁾. (It is assumed that all masses have been set to zero so that the axial current should be conserved.) However, working only to order e as we are, we may neglect this fact. It follows from (40b) that

$$\begin{aligned} S^{\mu\nu}(x;n) &= -c n^\nu P_{\alpha\beta} \partial^\alpha \tilde{F}^{\mu\beta}(x) + \bar{S}^{\mu\nu}(x;n) \\ &= c n^\nu n_\alpha n_\beta \partial^\alpha \tilde{F}^{\mu\beta}(x) + \bar{S}^{\mu\nu}(x;n) \end{aligned} \quad (41a)$$

$$\begin{aligned} S^{\mu\nu/d}(y;n) &= c \tilde{F}^{\mu r}(y) (g^\alpha_r n^\nu + g^{\alpha\nu} n_r) + \\ &\quad + c \tilde{F}^{\nu r}(y) (g^\alpha_r n^\mu + g^{\alpha\mu} n_r). \end{aligned} \quad (41b)$$

Note that the Schwinger term satisfies the curl condition. Therefore, a sea-gull which covariantizes the T product can be given:

$$C^{\mu\nu}(y;n) = c \tilde{F}^{\mu r}(y) n_r n^\nu + c \tilde{F}^{\nu r}(y) n_r n^\mu \quad (42)$$

Equation (19) is satisfied

$$I^\nu(y) = n_\mu S^{\mu\nu}(y;n) = n_\mu \bar{S}^{\mu\nu}(y;n) = 0; \quad (43a)$$

similarly, (21) also holds

$$\begin{aligned} J^\mu(y) &= n_\nu S^{\mu\nu}(y;n) - \partial_\mu C^{\mu\nu}(y;n) = \\ &= c n_\alpha n_\beta \partial^\alpha \tilde{F}^{\mu\beta}(y) - c n_r n^\nu \partial_\nu \tilde{F}^{\mu r}(y) = 0. \end{aligned} \quad (43b)$$

Equations (24) imply that

$$\begin{aligned} I^{\mu\nu}(y) &= -c \tilde{F}^{\mu\nu}(y), \\ J^{\mu\nu}(y) &= c \tilde{F}^{\mu\nu}(y). \end{aligned} \quad (44)$$

With the choice (32) for the entire sea-gull, neither Ward-Takahashi is satisfied and additional covariant terms must be added to the sea-gull. To evaluate $\tilde{C}^{\mu\nu}(x,y)$, we consult (33) and find

$$\tilde{C}^{\mu\nu}(x,y) = c \tilde{F}^{\mu\nu}(y) \delta^4(x-y) - \varepsilon^{\mu\nu\alpha\beta} X_\alpha(y) \partial_\beta \delta^4(x-y), \quad (45a)$$

$$\tilde{C}^{\mu\nu}(x,y) = -c \tilde{F}^{\mu\nu}(y) \delta^4(x-y) - \varepsilon^{\mu\nu\alpha\beta} X_\alpha(x) \partial_\beta \delta^4(x-y). \quad (45b)$$

For the two equations (45) to be consistent, we must have

$$\begin{aligned} 2c \tilde{F}^{\mu\nu}(y) \delta^4(x-y) &= \varepsilon^{\mu\nu\alpha\beta} [X_\alpha(y) - X_\alpha(x)] \partial_\beta \delta^4(x-y) = \\ &= \varepsilon^{\mu\nu\alpha\beta} \partial_\beta X_\alpha(y) \delta^4(x-y) \\ &= -4c \varepsilon^{\mu\nu\alpha\beta} \partial_\beta A_\alpha(y) \delta^4(x-y). \end{aligned} \quad (46)$$

We have expressed $\tilde{F}^{\mu\nu}$ in terms of the vector potential A_α . Evidently, we must have $-X^\alpha = cA^\alpha$ and both Ward-Takahashi identities are satisfied with the following choice of the sea-gull

$$\begin{aligned} c (\tilde{F}^{\mu\nu}(y) n_\nu n^\nu + \tilde{F}^{\nu\alpha}(y) n_\nu n^\alpha + \tilde{F}^{\mu\nu}(y)) \delta^4(x-y) + \\ + c \varepsilon^{\mu\nu\alpha\beta} A_\alpha(y) \partial_\beta \delta^4(x-y). \end{aligned} \quad (47)$$

We now show that this gradient term violates another requirement that the T^* product must satisfy. Note that we wish to take matrix elements of $T^*\{j^\mu(x)j_5^\nu(y)\}$ between vacuum and a one photon state. The explicit dependence of the sea-gull on the vector potential, would lead to a matrix element which would not be gauge invariant with respect to that photon. Therefore, the gradient term that we were forced to introduce into the sea-gull to satisfy both Ward-Takahashi identities is unacceptable, and the sea-gull must be chosen to be of the form

$$c(\tilde{F}^{\mu r}(y)\eta_r\eta^\nu + \tilde{F}^{\nu r}(y)\eta_r\eta^\mu + \epsilon\tilde{F}^{\mu\nu}(y)\delta^4(x-y)). \quad (48)$$

Here ϵ is an arbitrary parameter which is determined only when it is decided which Ward-Takahashi identity is to be satisfied. The choice $\epsilon = 1$ effects cancellation of Schwinger terms and sea-gulls in the μ identity, while $\epsilon = -1$ performs this service for the ν identity. In any case, it is impossible to satisfy the requirement for both identities. If the former choice is made, then

$$\frac{\partial}{\partial y^\nu} T^*\{j^\mu(x)j_5^\nu(y)\} = 2c\tilde{F}^{\mu r}(y)\partial_r\delta^4(x-y). \quad (49)$$

C. As a third example we consider the T^* product of a conserved current J^λ and the stress energy tensor $\theta^{\mu\nu}$

$$T^*\{J^\lambda(x)\theta^{\mu\nu}(y)\}.$$

The sea-gull, which covariantizes the respective T product now carries three indices, $C^{\lambda\mu\nu}$, and it must be symmetric in $\mu\nu$ since $\theta^{\mu\nu}$ possesses this symmetry. To establish whether or not the commutators involve Schwinger terms which permit the composition of Feynman's conjecture on all three Ward-Takahashi identities, we must specify the nature of the commutator $[J^\lambda, \theta^{\mu\nu}]$.

This object is in general model dependent. However, certain components are model independent. These are (for conserved currents) ⁹⁾

$$[\mathcal{T}^0(x,t), \theta^{00}(y,t)] = -i \mathcal{T}^i(y) \partial_i \delta(x-y), \quad (50a)$$

$$[\mathcal{T}^0(x,t), \theta^{0i}(y,t)] = i \mathcal{T}^0(y) \partial^i \delta(x-y), \quad (50b)$$

$$[\mathcal{T}^i(x,t), \theta^{00}(y,t)] = i \partial_0 \mathcal{T}^i(x,t) \delta(x-y) + i \mathcal{T}_0(x,t) \partial^i \delta(x-y). \quad (50c)$$

Thus we see that $S^{00\nu/i} \neq 0$, $S^{0\mu 0} \neq 0$ and $S^{\lambda 00/i} \neq 0$. It is therefore impossible to satisfy Feynman's conjecture in all three Ward-Takahashi identities, and some of these identities contain gradient terms. Unlike the gradient terms, discussed in subsection B above, the present occurrence is in no way anomalous. It merely reflects the nature of the gauge group which is satisfied for gravitons (which couple to the stress tensor).

To make the present example more specific, we consider a special (trivial) case of the $[\mathcal{T}^\lambda, \theta^{\mu\nu}]$ commutator algebra, which holds for example in a massless, scalar theory with $\mathcal{T}_\mu = \partial_\mu \varphi$. For the commutator we take

$$\begin{aligned} \delta(x-y) \cdot n [\mathcal{T}^\lambda(x), \theta^{\mu\nu}(y)] = & i \{ \mathcal{T}^\mu(y) (n^\nu g^{\lambda\alpha} + n^\lambda g^{\nu\alpha}) + \\ & + \mathcal{T}^\nu(y) (n^\mu g^{\lambda\alpha} + n^\lambda g^{\mu\alpha}) \\ & - g^{\mu\nu} (g^{\lambda\alpha} \mathcal{T}^\alpha(y) n_\alpha + n^\lambda \mathcal{T}^\alpha(y)) \} P_{\alpha\beta} \partial^\beta \delta^\nu(x-y). \end{aligned} \quad (51)$$

The Schwinger term satisfies the curl condition and a covariantizing sea-gull can be chosen to be

$$C^{\lambda\mu\nu}(x,y;n) = -i \{ \bar{\psi}^\mu(y) P^{\nu\lambda} + \bar{\psi}^\nu(y) P^{\mu\lambda} - g^{\mu\nu} \bar{\psi}^\lambda(y) P_\alpha^\lambda \} \delta^4(x-y). \quad (52)$$

With this choice of sea-gull, we have

$$n_\lambda C^{\lambda\mu\nu} = 0 \quad (53)$$

and the Ward-Takahashi identity in λ satisfies the Feynman conjecture

$$\frac{\partial}{\partial x^\lambda} T^* \{ \bar{\psi}^\lambda(x) \theta^{\mu\nu}(y) \} = 0. \quad (54)$$

However, now

$$n_\mu C^{\lambda\mu\nu}(x,y;n) = -i \{ \bar{\psi}^\lambda(x) n_\alpha g^{\nu\lambda} - n^\nu \bar{\psi}^\lambda(x) \} \delta^4(x-y), \quad (55a)$$

$$n_\nu C^{\lambda\mu\nu}(x,y;n) = -i \{ \bar{\psi}^\lambda(x) n_\alpha g^{\mu\lambda} - n^\mu \bar{\psi}^\lambda(x) \} \delta^4(x-y). \quad (55b)$$

Thus the Ward-Takahashi identity for μ and ν contains gradient terms. One could relinquish (54) but cancel Schwinger terms and divergences of sea-gulls for the μ identity by choosing a different sea-gull, by adding to (52) $i \{ J^\mu(x) g^{\nu\lambda} - g^{\mu\nu} J^\lambda(x) \}$. However, this choice of the sea-gull is not symmetric in μ and ν and thus is not acceptable. Therefore we must allow the μ and ν Ward-Takahashi identities to contain the extra terms.

This is the case in general when one is dealing with the energy momentum tensor. Due to the fact that for any operator A , which is not a Lorentz scalar, both the commutators $[A, \theta_{0\nu}]$ and $[A, \theta_{\mu 0}]$ have Schwinger terms, one cannot satisfy the naïve Ward identities for $\theta_{\mu\nu}$ by adding a sea-gull symmetric in μ and ν .

4. EXTENSION TO MANY OPERATORS

It is not difficult to extend our method of covariantization to T products of many operators. We shall illustrate the procedure for the T^* product of three operators J , K , L . We define the sea-gull by

$$T^*\{J(x)K(y)L(z)\} = T\{J(x)K(y)L(z)\} + C(x,y,z;n). \quad (56)$$

If we differentiate with respect to n , we derive as the analogue of (7)

$$\begin{aligned} \frac{\delta}{\delta n^\mu} C(x,y,z;n) = & \delta(x-y) T\{S_\mu^{JK}(x;n)L(z)\} + \\ & + \delta(x-z) T\{S_\mu^{LJ}(z;n)K(y)\} + \delta(y-z) T\{S_\mu^{KL}(y;n)J(x)\}, \end{aligned} \quad (57)$$

where S_μ^{JK} is the Schwinger term in the commutator of J and K . This equation is integrable if the right-hand side of (57) is curlless. The curl of (57) involves two terms. First we have a combination of T products of the curl of the two-operator Schwinger term, e.g.,

$$\frac{\delta}{\delta n^\nu} S_\mu^{JK} - \frac{\delta}{\delta n^\mu} S_\nu^{JK}.$$

These terms vanish as shown previously. The remaining term is

$$\begin{aligned} \delta(x-y)\delta(x-z)\{ & S_{\mu\nu}^{JK,L}(x;n) + S_{\mu\nu}^{LJ,K}(x;n) + S_{\mu\nu}^{KL,J}(x;n) - \\ & - S_{\nu\mu}^{JK,L}(x;n) - S_{\nu\mu}^{LJ,K}(x;n) - S_{\nu\mu}^{KL,J}(x;n)\}, \end{aligned} \quad (58)$$

where $S_{\mu\nu}^{JK,L}$ is the Schwinger term in the commutator of S_μ^{JK} with L ; i.e.,

$$\delta([x-z],n) [S_\mu^{JK}(x;n), L(z)] = \dots + S_{\mu\nu}^{JK,L}(x) p^\nu \partial_\alpha \delta^4(x-z). \quad (59)$$

One can easily show that (48) vanishes identically if the Jacobi identity holds for the operators J, K and L.

One can therefore integrate (49). When use is made of the fact that the objects S^{JK} are themselves divergences, an integration by parts can be performed and we finally get

$$C(x, y, z; n) = T\{C^{JK}(x, n) L(z)\} \delta(x-y) + T\{C^{LJ}(z, n) K(y)\} \delta(x-z) + \\ + T\{C^{KL}(y, n) J(x)\} \delta(y-z) + \delta(x-y) \delta(x-z) \left\{ \int_0^n dn' \mu [C_{\mu}^{JK, L}(x; n') + \right. \\ \left. + C_{\mu}^{LJ, K}(x; n') + C_{\mu}^{KL, J}(x; n') \right\}, \quad (60)$$

where C^{JK} is the sea-gull in the T^* product of J and K, and $C_{\mu}^{JK, L}$ the Schwinger term in the commutator of $C^{JK}(x; n)$ with $L(z)$. This can also be written as

$$C(x, y, z; n) = T^*\{C^{JK}(x; n) L(z)\} \delta(x-y) + T^*\{C^{LJ}(z; n) K(y)\} \delta(x-z) + \\ + T^*\{C^{KL}(y; n) J(x)\} \delta(y-z). \quad (61)$$

In the same manner one can covariantize an arbitrary product of operators.

A P P E N D I X

We show here that the curl condition (8) is always true when the Schwinger term arises in an equal time commutator which is a restriction to equal times of a Lorentz covariant, unequal time commutator. We begin with the defining equation (3) (the subscripts A and B are suppressed)

$$\begin{aligned} \delta([x-y] \cdot n) [A(x), B(y)] &= S(y; n) \delta^4(x-y) + \\ &+ S^\alpha(y; n) P_{\alpha\beta} \partial^\beta \delta^4(x-y). \end{aligned} \quad (A.1)$$

In this expression n is varied in a spacelike direction. The variation of the left-hand side is

$$\begin{aligned} P^{\mu\nu}(x-y), \delta'([x-y] \cdot n) [A(x), B(y)] &= \\ = P^{\mu\nu}(x-y), n^\omega \left[\frac{\delta}{\delta x^\omega} \left(\delta([x-y] \cdot n) [A(x), B(y)] \right) - \delta([x-y] \cdot n) [A_\omega(x), B(y)] \right] \\ = P^{\mu\nu}(x-y), n^\omega \left[S(y; n) \partial_\omega \delta^4(x-y) + S^\alpha(y; n) P_{\alpha\beta} \partial^\beta \partial_\omega \delta^4(x-y) \right. \\ \left. - \delta([x-y] \cdot n) [A_\omega(x), B(y)] \right]. \end{aligned} \quad (A.2)$$

Here $A_w \equiv \partial_w A$. The commutator of A_w with B is taken to be, by definition

$$\begin{aligned} \delta([x-y] \cdot n) [A_\omega(x), B(y)] &= G_\omega(y; n) \delta^4(x-y) + \\ &+ G_{\omega/\alpha}(y; n) P^{\alpha\beta} \partial_\beta \delta^4(x-y) + \\ &+ G_{\omega/\alpha\beta}(y; n) P^{\alpha\gamma} P^{\beta\delta} \partial_\gamma \partial_\delta \delta^4(x-y) + \dots \end{aligned}$$

(A.3)

The dots indicate possible derivatives, higher than the second.

Note the $G_{\omega/\alpha\beta}(y; n)$ can be chosen to be symmetric in α and β . Thus (A.2) becomes

$$\begin{aligned} &- P^{\mu\alpha} n^\omega S^\alpha(y; n) \partial_\omega \delta^4(x-y) + P^{\mu\alpha} n^\omega G_{\omega/\alpha}(y; n) \delta^4(x-y) + \\ &+ 2 P^{\mu\alpha} P^{\beta\delta} n^\omega G_{\omega/\alpha\beta}(y; n) \partial_\delta \delta^4(x-y) + \dots \end{aligned}$$

(A.4)

Now the dots indicate gradients of delta functions higher than the first. The variation of the right-hand side of (A.1) is

$$\begin{aligned} &P^{\mu\nu} \frac{\delta}{\delta n^\nu} S(y; n) \delta^4(x-y) + P_{\alpha\beta} P^{\mu\nu} \frac{\delta}{\delta n^\nu} S^\alpha(y; n) \partial^\beta \delta^4(x-y) \\ &- P^{\mu\beta} S^\alpha(y; n) n_\alpha \partial_\beta \delta^4(x-y) - P^{\mu\alpha} S_\alpha(y; n) n_\beta \partial^\beta \delta^4(x-y) \end{aligned}$$

(A.5)

Equating (A.4) with (A.5) requires

$$P^{\mu\alpha} \frac{\delta}{\delta n^\alpha} S(y; n) = P^{\mu\alpha} n^\omega G_{\omega/\alpha}(y; n),$$

(A.6a)

$$P_{\alpha\beta} P^{\mu\nu} \frac{\delta}{\delta h^\nu} S^\alpha(y;n) = n_\alpha S^\alpha(y;n) P^\mu_\beta + \\ + 2 P^{\mu\alpha} p^{\delta\beta} n^\omega G_{\omega\alpha\delta}(y;n). \quad (\text{A.6b})$$

Cancelling the projection operators yields finally

$$\frac{\delta}{\delta h^\alpha} S(y;n) = n^\omega G_{\omega\alpha}(y;n), \quad (\text{A.7a})$$

$$\frac{\delta}{\delta h^\nu} S^\alpha(y;n) = n_\beta S^\beta(y;n) g^\alpha_\nu + 2 n^\omega G_{\omega\nu}{}^\alpha(y;n). \quad (\text{A.7b})$$

The second equation shows that

$$\frac{\delta}{\delta h^\nu} S^\alpha(y;n)$$

is symmetric in ν and α , which was to be proven.

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