CM-P00058258

Ref.TH.1035-CERN

Archives

ON THE SIGNATURE AND PARITY OF INTERNAL TRAJECTORIES
IN THE GENERALIZED VENEZIANO MODEL FOR N PARTICLES

J.F.L. Hopkinson and Chan Hong - Mo
CERN -- Geneva

ABSTRACT

It is shown that in the N particle generalization of the Veneziano amplitude: 1) the parent trajectory has definite signature, while the daughters have in general mixed signature, and 2) the parent and first daughter trajectories have only normal parity poles, while the other daughters suffer in general from parity doubling.

1. A generalization of the Veneziano model to multi-particle interactions has been considered by several authors ¹⁾. In all these papers it was implicitly assumed that the internal trajectories have definite signatures and normal parities with respect to the external particles. The purpose of this note is merely to show that this is indeed the case for the parent trajectories. In general, however, the daughter trajectories will be degenerate in both signature and parity.

2. The amplitude for a reaction with N neutral external lines has the form

$$A_N = \sum B_N(1, 2, \dots, N) \quad (1)$$

where each term in the sum corresponds to a distinct cyclic ordering of the particles $1, \dots, N$. Ignoring for the present the possibility of ghost-killing factors and satellite terms, we have for B_N just the generalized beta function. For the particular ordering $(1, 2, 3, \dots, N)$, this is defined as

$$B_N = \int_0^1 du_{12} \dots du_{1,N-2} \frac{x_{12}-1}{u_{12}} \frac{x_{23}-1}{u_{23}} \dots \frac{x_{1,N-2}-1}{u_{1,N-2}} \frac{x_{N-2,N-1}-1}{u_{N-2,N-1}} \prod_{1 \leq \alpha < \beta \leq N-1} \frac{z_{\alpha\beta}}{w_{\alpha\beta}} \quad (2)$$

$$= \int_0^1 du_{12} \dots du_{1,N-2} b_N(u_{12}, \dots, u_{1,N-2})$$

where

$$x_{ij} = -\alpha(S_{i,i+1,\dots,j}) = -\alpha_{ij}^0 - \alpha' (p_i + p_{i+1} + \dots + p_j)^2 \quad (\equiv 0 \text{ if } i=j)$$

$$z_{ij} = x_{ij} - x_{i+1,j} - x_{i,j-1} + x_{i+1,j-1}$$

$$w_{ij} = 1 - v_{ij-1} = 1 - u_{1i} u_{1,i+1} \dots u_{1,j-1}$$

Remembering that $u_{1,j} \equiv u_{j+1,N}$, we can rearrange this expression to

$$B_N = \int_0^1 du_{12} \dots du_{1,M-1} b_{M+1}(u_{12}, \dots, u_{1,M-1}) \cdot du_{M+2,N} \dots du_{N-1,N} b_{N-M+1}(u_{M+2,N}, \dots, u_{N-1,N})$$

$$\cdot \frac{x_{1M}-1}{u_{1M}} (1-u_{1M})^{x_{M,M+1}-1} \prod_{\substack{\alpha < \beta \\ \alpha, \beta \in A_M^N}} \frac{z_{\alpha\beta}}{w_{\alpha\beta}} \quad (3)$$

where

$$A_M^N = \{ (\alpha, \beta) \mid 1 < \alpha \leq M < \beta < N, \quad \beta - \alpha > 1 \}$$

The residue of the $(J+1)^{\text{th}}$ pole in the $(1, M)$ channel $(-x_{1, M} = \alpha_{1, M} = J)$, corresponding to Fig. 1a, is

$$R_J = \int_0^1 du_{1,2} \dots du_{1,M-1} b_{M+1} \cdot du_{M+2,N} \dots du_{N-1,N} b_{N-M+1} \cdot G_J \quad (4)$$

where

$$G_J = \frac{1}{J!} \left[\frac{\partial^J}{\partial u_{1,M}^J} G \right]_{u_{1,M}=0} \quad (5)$$

$$G = (1 - u_{1,M})^{K_{M+1}-1} \prod_{A_M^N} w_{\alpha\beta}^{z_{\alpha\beta}}$$

Put $G^{(1)} = (\partial/\partial u_{1,M})G = FG$, so $G^{(2)} = (F^2 + F^{(1)})G$, etc. Now $G|_{u_{1,M}=0}$ is equal to unity, and F and each of its derivatives with respect to $u_{1,M}$ is a first degree polynomial in the dual variables. Therefore, clearly,

$$G^{(J)}|_{u_{1,M}=0} = \left[\frac{\partial^J}{\partial u_{1,M}^J} G \right]_{u_{1,M}=0}$$

is a polynomial in the dual variables of degree J , and the highest power comes only from the term $F^J G|_{u_{1,M}=0} = F_0^J(J)$. Putting $F_0(1) = f_1 + f_0$, where f_1 is the linear and f_0 the constant part of F_0 with respect to the dual variables. Thus the leading power of G_J is f_1^J , and from (5)

$$f_i = - \sum_{\substack{1 < \alpha \leq M \\ < \beta < N}} v_{\alpha, M+1} v_{M+1, \beta+1} z_{\alpha\beta} \quad (6)$$

where the definitions of v and z have been generalized in the obvious way: $v_{i,i} = u_{1,i}$; $v_{i,i-1} = 1$; $z_{i,i+1} = x_{i,i+1}$. The contribution of the parent trajectory to the residue is then

$$R_J^{(P)} = \int_0^1 du_{1,2} \dots du_{1,M-1} b_{M+1} \cdot du_{M+2,N} \dots du_{N-1,N} b_{N-M+1} \cdot (f_1)^J \quad (7)$$

3. Alternative forms to (2) can be obtained for B_N by choosing other sets of independent integration variables instead of the set $u_{1,j}$ ($2 \leq j \leq N-2$) which corresponds to Fig. 2a. Such transformations have been considered in general by Koba and Nielsen ¹⁾. We have found, in particular, that for discussing the signature of internal trajectories it is convenient to consider also the representation of B_N in terms of the set : $u_{1,j}$ ($2 \leq j \leq M$), $u_{M+1,j}$ ($M+2 \leq j \leq N-1$) corresponding to Fig. 2b.

The expression for B_N in terms of these variables has been given by Koba and Nielsen, but let us outline the derivation here, in our notation. We can write the new variables in terms of the old

$$u_{M+1,\beta} = \frac{w_{M+1,\beta} w_{M,\beta+1}}{w_{M,\beta} w_{M+1,\beta+1}} \quad (M+1 < \beta < N), \quad (u_{M+1,N} = u_{1,N}) \quad (8)$$

Inverting these relations, we get

$$u_{1,\beta} = \frac{(1 - \eta_{\beta+1,N-1})(1 - \eta_{\beta,N})}{(1 - \eta_{\beta,N-1})(1 - \eta_{\beta+1,N})} \quad (M+1 < \beta < N) \quad (9)$$

where $\eta_{\alpha,\beta} = u_{M+1,\alpha} u_{M+1,\alpha+1} \dots u_{M+1,\beta}$ ($M < \alpha \leq \beta < N$);
 $\eta_{\beta+1,\beta} = 1$, $\eta_{M+1,\beta} = 0$. Defining a new set $\bar{w}_{\alpha,\beta}$ by

$$\begin{aligned} \text{a) } 1 < \alpha \leq \beta-1 \leq M, \quad w_{\alpha,\beta} &= \bar{w}_{\alpha,\beta}, \quad \bar{w}_{\alpha,\beta} = 1 - v_{\alpha,\beta-1} \\ \text{b) } 1 < \alpha \leq M < \beta-1 < N, \quad w_{\alpha,\beta} &= \bar{w}_{\alpha,\beta} (1 - \eta_{\beta,N})^{-1}, \quad \bar{w}_{\alpha,\beta} = 1 - v_{\alpha,M} - \eta_{\beta,N} + v_{\alpha,M} \eta_{\beta,N-1} \\ \text{c) } M < \alpha \leq \beta-1 < N, \quad w_{\alpha,\beta} &= \bar{w}_{\alpha,\beta} \frac{(1 - u_{1,M})(\eta_{\beta,N-1})}{(1 - \eta_{\beta,N})(1 - \eta_{\alpha,N-1})}, \quad \bar{w}_{\alpha,\beta} = 1 - \eta_{\alpha,\beta-1} \end{aligned} \quad (10)$$

all the dependent variables can be obtained from

$$u_{\alpha,\beta} = \frac{\bar{w}_{\alpha,\beta} \bar{w}_{\alpha-1,\beta+1}}{\bar{w}_{\alpha-1,\beta} \bar{w}_{\alpha,\beta+1}} \quad (\alpha \neq 1, M+1) \quad (11)$$

It is tedious but not difficult to show that the Jacobian is :

$$\Delta_{M,N} = \frac{\partial(u_{M+1,M+2}, u_{M+1,M+3}, \dots, u_{M+1,N-1})}{\partial(u_{1,M+1}, u_{1,M+2}, \dots, u_{1,N-2})} = (-)^{N-M} \frac{\bar{w}_{M+2,M+3} \dots \bar{w}_{N-1,N}}{w_{M,M+1} \dots w_{N-2,N-1}} \quad (12)$$

and B_N can therefore be written as :

$$B_N = \int_0^1 du_{1,2} \dots du_{1,M-1} b_{M+1}(u_{1,2}, \dots, u_{1,M-1}) \cdot du_{M+1,M+2} \dots du_{M+1,N-1} b_{N-M+1}(u_{M+1,M+2}, \dots, u_{M+1,N-1})$$

$$du_{1,M} u_{1,M}^{x_{1,M}-1} (1-u_{1,M})^{x_{M,N-1}-1} \cdot \prod_{1 \leq \alpha < M} u_{1,N}^{2x_N} \prod_{\substack{1 \leq \alpha < M \\ < \beta-1 < N-1}} \bar{w}_{\alpha\beta}^{2x_\beta} \quad (13)$$

4. Consider the internal trajectory in $B_N(1,2,\dots,N)$ corresponding to Fig. 1a. In the sum (1) for the actual amplitude, there will be another term corresponding to Fig. 1b, with the external lines reflected on one side of the diagram, namely $B_N(1,2,\dots,M,N,\dots,M+1)$. To prove that the parent trajectory under consideration has definite signature, it is sufficient to prove that the residue $\tilde{R}_J$ of the $(J+1)^{th}$ pole in $B_N(1,2,\dots,M,N,\dots,M+1)$ has the form

$$\tilde{R}_J = (-)^J R_J^{(P)} + r_J \quad (14)$$

where r_J is a polynomial of degree $(J-1)$ in the dual variables.

To do this we write $B_N(1,2,\dots,M,N,N-1,\dots,M+1)$ in the representation (13). We introduce the new variables $\tilde{x}_{i,j}$:

$$\tilde{x}_{ij} = -\alpha(S_{i,i+1,\dots,M,N,N-1,j}) = -\tilde{x}_{ij}^e - \alpha'(p_i + \dots + p_M + p_{N-1} + p_j)^2 \quad (15)$$

when $i \leq M < j$ (otherwise $\tilde{x}_{ij} = x_{ij}$). From the trivial identity :

$$S_{ij} + \tilde{S}_{ijM} = S_{i,i-1} + S_{iM} + S_{M+1,j} + S_{j+1,N} - S_{iM} \quad (i \leq M < j < N) \quad (16)$$

it follows that

$$\tilde{X}_{ij} = -x_{ij-1} + g_1(i,j) \quad \tilde{Z}_{ij} = z_{ij} + g_2(i,j) \quad (17)$$

where g_1 and g_2 are independent of the dual variables. Performing the relabelling in (13), and taking the residue at the pole $x_{1,M} = -J$ gives

$$\tilde{R}_J = \int_0^1 du_{1,2} \dots du_{1,M-1} b_{M+1} \dots du_{M+2,N} \dots du_{N-1,N} b_{N-M+1} \dots \tilde{G}_J \quad (18)$$

where

$$\begin{aligned} \tilde{G}_J &= \frac{1}{J!} \left[\frac{\partial^J}{\partial u_{1,M}^J} \tilde{G} \right]_{u_{1,M}=0} \\ \tilde{G} &= (1-u_{1,M})^{\tilde{X}_{M,M+2}-1} \prod_{1 \leq \alpha < M} \tilde{W}_{\alpha,N}^{\tilde{Z}_{\alpha,N}} \prod_{\substack{1 \leq \alpha < M \\ \beta-1 < N-1}} \tilde{W}_{\alpha,\beta}^{\tilde{Z}_{\alpha,\beta}} \end{aligned} \quad (19)$$

As before, put $\tilde{G}^{(1)} = (\partial/\partial u_{1,M})\tilde{G} = \tilde{F}\tilde{G}$, etc., Similarly, the leading power of $\tilde{G}_J$ in the dual variables is $\tilde{F}_1^J$ where

$$\tilde{f}_1 = - \left\{ \tilde{X}_{M,M+2} + \sum_{1 \leq \alpha < M} \tilde{Z}_{\alpha,N} V_{\alpha,M-1} + \sum_{\substack{1 \leq \alpha < M \\ \beta-1 < N-1}} (V_{\alpha,M-1} + V_{M+1,\beta-1} + V_{\alpha,M-1} V_{M+1,\beta-1}) \tilde{Z}_{\alpha,\beta} \right\} \quad (20)$$

since under the labelling change $\alpha \rightarrow N+M-\alpha+1$ if $\alpha > M$ and

$$\eta_{\alpha,\beta} \rightarrow v_{N+M-\alpha, N+M-\beta}.$$

Now we relate the $\{\tilde{x}\}$ to the $\{x\}$ using (17), and keep only the terms linear in the dual variables :

$$-\tilde{X}_{M,M+2} \simeq X_{M,M+1} \quad (21-a)$$

The coefficient of $v_{\alpha,M-1}$ ($1 < \alpha < M$) is

$$-\tilde{Z}_{\alpha,N} - \sum_{M+1 < \beta < N} \tilde{Z}_{\alpha,\beta} \simeq Z_{M,\beta} \quad (21-b)$$

and the coefficient of $v_{M+1,\beta-1}$ ($M+1 < \beta < N$) is

$$-\sum_{1 \leq \alpha < M} \tilde{z}_{\alpha\beta} \simeq z_{M,\beta} \quad (21-c)$$

and the coefficient of $v_{\alpha, M-1} v_{M+1, \beta}$ ($1 < \alpha < M, M+1 < \beta < N$) is

$$\tilde{z}_{\alpha\beta} \simeq z_{\alpha\beta} \quad (21-d)$$

Therefore

$$\tilde{f}_i \simeq \sum_{\substack{1 < \alpha < M \\ < \beta < N}} v_{\alpha, M-1} v_{M+1, \beta-1} z_{\alpha\beta} = -f_i \quad (22)$$

so that (14) is satisfied. This establishes the signature of the parent trajectory.

It can easily be seen by working out the terms in R_J and $\tilde{R}_J$ of lower degree in the dual variables that the two contributions from the daughter trajectories are not so simply related. This means that the daughter poles have in general mixed signatures.

5. A method of incorporating isospin into the generalized Veneziano model has been suggested ²⁾, which apart from retaining all the desirable properties of the neutral meson model treated above, excludes exotic resonances in all channels. For external mesons with isospin $I=1$ and $G=-$ for example, this involves simply the replacement

$$B_N(1, 2, \dots, N) \longrightarrow \frac{1}{2} \text{Tr}(\tau_{a_1} \tau_{a_2} \dots \tau_{a_N}) B_N(1, 2, \dots, N) \quad (23)$$

for each term in the sum (1), where a_i is the isospin of the external particle i . The pole in the term corresponding to Fig. 1a will occur in both isospin states, with the residue,

$$\begin{aligned} & \frac{1}{2} \text{Tr}(\tau_{a_1} \tau_{a_2} \dots \tau_{a_N}) \cdot \frac{1}{2} \text{Tr}(\tau_{a_{M+1}} \dots \tau_{a_N}) R_J \\ & + \sum_x \frac{1}{2} \text{Tr}(\tau_{a_1} \dots \tau_{a_N} \tau_x) \cdot \frac{1}{2} \text{Tr}(\tau_x \tau_{a_{M+1}} \dots \tau_{a_N}) R_J \end{aligned} \quad (24)$$

where the first term corresponds to an $I=0$ exchange and the second term to an $I=1$ exchange. Similarly, for the term corresponding to Fig. 1b, we have the residue

$$\begin{aligned} & \frac{1}{2} \text{Tr}(\tau_{a_1} \tau_{a_2} \dots \tau_{a_M}) \cdot \frac{1}{2} \text{Tr}(\tau_{a_N} \tau_{a_{N-1}} \dots \tau_{a_{M+1}}) \tilde{R}_J \\ & + \sum_X \frac{1}{2} \text{Tr}(\tau_{a_1} \dots \tau_{a_M} \tau_X) \cdot \frac{1}{2} \text{Tr}(\tau_X \tau_N \dots \tau_{a_{M+1}}) \tilde{R}_J \end{aligned} \quad (25)$$

From the identity

$$\text{Tr}(\tau_X, \tau_{X_L} \dots \tau_{X_R}) = (-)^L \text{Tr}(\tau_{X_L} \dots \tau_{X_R} \tau_X) \quad (26)$$

and the relation between R_J and $\tilde{R}_J$ deduced above (14), we can readily see that for the amplitude A_N

- 1) parent trajectories with the same G parity, but different isospins have different signatures $(\rho - f_0; \omega - A_2)$;
- 2) parent trajectories with the same isospin but different G parities have opposite signatures $(\rho - A_2; \omega - f_0)$;
- 3) daughter trajectories have in general mixed signatures and isospins.

6. The leading power in the dual variables of the residue R_J is, (6),

$$F_0 = \left\{ - \sum_{\substack{1 \leq \alpha \leq M \\ < \beta < N}} \rho_\alpha \sigma_\beta z_{\alpha\beta} \right\}^J \quad (27)$$

where $\rho_\alpha = v_{\alpha, M-1}$, $\sigma_\beta = v_{M+1, \beta-1}$.

The $z_{\alpha\beta}$ can be written in terms of the external momenta as,

$$\begin{aligned} z_{\alpha\beta} &= -(\alpha_{\alpha\beta}^0 - \alpha_{\alpha+1, \beta}^0 - \alpha_{\alpha, \beta-1}^0 + \alpha_{\alpha+1, \beta-1}^0) - (2\alpha') \rho_\alpha \cdot \rho_\beta \\ &\simeq -(2\alpha') \rho_\alpha \cdot \rho_\beta \end{aligned} \quad (28)$$

So the leading power of F_0 is

$$(2\alpha')^J \left\{ \sum_{\substack{1 \leq \alpha \leq M \\ < \beta < N}} \rho_\alpha \sigma_\beta (\rho_\alpha \cdot \rho_\beta) \right\}^J \quad (29)$$

Expanding this bracket, the general term is

$$p_{\alpha_1} p_{\alpha_2} \dots p_{\alpha_J} \cdot \sigma_{\beta_1} \sigma_{\beta_2} \dots \sigma_{\beta_J} \quad (30)$$

where $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_J \leq M$, $M \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_J \leq N$,
with coefficient

$$\sum_{P_i P_j} (p_{\alpha_{i_1}} p_{\beta_{j_1}}) (p_{\alpha_{i_2}} p_{\beta_{j_2}}) \dots (p_{\alpha_{i_J}} p_{\beta_{j_J}}) \quad (31)$$

which is clearly the contraction $T_1 \cdot T_2$ of

$$T_1^{(\mu_J)} = \sum_{P_i} p_{\alpha_{i_1}}^{\mu_1} p_{\alpha_{i_2}}^{\mu_2} \dots p_{\alpha_{i_J}}^{\mu_J} \quad (32)$$

$$T_2^{(\mu_J)} = \sum_{P_j} p_{\beta_{j_1}}^{\mu_1} p_{\beta_{j_2}}^{\mu_2} \dots p_{\beta_{j_J}}^{\mu_J}$$

where $P_i = (i_1, i_2, \dots, i_J)$ is any permutation of the indices $(1, 2, \dots, J)$, and the (μ_J) are the Lorentz indices on the external four-momenta p . These tensors T_1 and T_2 are clearly fully symmetric Lorentz tensors, which as is well known contain only normal parity representations of the rotation group.

Thus the residue of B_N at any Regge pole is a polynomial of degree J in the dual variables whose leading term can be written as a fully symmetric tensor in the external momenta, so the spin J (and $J-1$) components have only the normal parity (i.e., $(-)^J$). The next term in the residue, degree $J-1$, can be written as a tensor in the momenta, but will in general be of mixed symmetry. However, a mixed symmetry Lorentz tensor (rank $J-1$) contains spin $J-1$ with normal parity and lower spins with both parities. Thus, in the amplitude (1) :

- i) a parent trajectory and its first daughter both have normal parity with respect to the external particles (note that N is even if the external lines have negative intrinsic parity);
- ii) the second and lower daughters have in general mixed parities.

We notice also that if the trajectories do not have a universal slope, then (28) is not true and the first daughter will have mixed parity like the other daughters.

ACKNOWLEDGEMENTS

We wish to gratefully acknowledge useful conversations with
Drs. Ruhl, J.C. Taylor and E. Plahte.

R E F E R E N C E S

- 1) The five-point function was introduced by :
K. Bardakçi and H. Ruegg, Phys.Letters 28B, 342 (1968);
M. Virasoro, Phys.Rev.Letters 22, 37 (1969);
and the higher functions by :
Chan Hong-Mo, Phys.Letters 28B, 425 (1969);
Chan Hong-Mo and Tsou Sheung Tsun, Phys.Letters 28B, 485 (1969);
C.J. Goebel and B. Sakita, Phys.Rev.Letters 22, 257 (1969);
Z. Koba and H.B. Nielson, Niels Bohr Institute preprint, January
(1969) - to be published in Nuclear Phys.;
Z. Koba and H.B. Nielsen, Niels Bohr Institute preprint, February
(1969) - to be published in Nuclear Phys.;
J.F.L. Hopkinson and E. Plahte, Phys.Letters 28B, 489 (1969).
- 2) J.E. Paton and Chan Hong-Mo, CERN preprint TH.994 (1969) - to be
published in Nuclear Phys.

Fig. 1a.

Fig. 1b.

Fig. 2a.

Fig. 2b.