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INFINITE SERIES REPRESENTATION OF THE N - POINT FUNCTION

IN THE GENERALIZED VENEZIANO MODEL

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A B S T R A C T

Explicit formulas for the N - point function in the generalized Veneziano model as an infinite sum over lower order functions are presented. The form of the amplitude exhibits clearly its physical interpretation as is believed to be useful for theoretical as well as practical applications.

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The Veneziano amplitude ¹⁾ for the idealized case of spin zero external particles has been generalized to five-particle processes by Bardakçi and Ruegg ²⁾ and to six and more particles by Chan ³⁾ and by Chan and Tsou ^{4),5)}. For $N \leq 7$ it has been explicitly shown that the amplitude possesses the following properties: (1) crossing symmetry, (2) analyticity, (3) correct residues, (4) Regge behaviour, (5) duality.

In this letter we present explicit formulas for the N -point function B_N as infinite sums over functions of lower order and we discuss briefly some of the properties of B_N . We have verified that our expression for B_N agrees with the integral form developed independently by Chan and Tsou ⁵⁾ on the basis of an earlier conjecture ³⁾, and that it satisfies the requirements (1)-(5) above. Also it has the nice property of showing explicitly most of the poles in the amplitude.

1. THE N - POINT FUNCTION

We only consider here the idealized case when all external particles are scalars. The first pole on each trajectory is assumed to have zero spin. We further assume real linear trajectories with a universal slope.

For the permutation of the N particles corresponding to Fig. 1 we define the variables ^{*)} x_{ij} in terms of the trajectories in the corresponding channel: $x_{ij} = -\alpha_{i,i+1,\dots,j}(s_{i,i+1,\dots,j})$, $x_{1N} = x_{ii} = 0$. In the graph of Fig. 1 the internal lines represent the variables x_{1i} and the two-body variables between adjacent particles are $x_{i,i+1}$.

*) The advantage of this choice of variables has been pointed out by Tsou Sheung Tsun.

We now define B_N by the recurrence formula

$$B_N(x) = \sum_{\substack{k_{i,N-1} \\ = 0}}^{\infty} \left\{ \prod_{i=2}^{N-3} (-)^{k_{i,N-1}} \binom{z_{i,N-1}}{k_{i,N-1}} \right\} B_4(x_{N-2,N-1}, x_{N-1,N} + \sum_{i=2}^{N-3} k_{i,N-1}) B_{N-1}(x') \quad (1)$$

where

$$B_4(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

is the Veneziano amplitude,

$$z_{ij} = x_{ij} - x_{i+1,j} - x_{ij-1} + x_{i+1,j-1} \quad (2)$$

and the variables x'_{ij} are defined by

$x'_{ij} =$	$j < N-2$	$j = N-2$	$j = N-1$
$i = 1$	$x_{ij} + \sum_{\alpha=2}^j k_{\alpha,N-1}$	$x_{ij} + \sum_{\alpha=2}^{j-1} k_{\alpha,N-1}$	-
$i > 1$	x_{ij}	x_{ij}	$x_{i,j+1}$

(3)

This general formula is not very transparent, so let us consider immediately the particular cases $N=5$ and $N=6$. The five-point function is

$$B_5(x) = \sum_{k=0}^{\infty} (-)^k \binom{z_{24}}{k} B_4(x_{34}, x_{45} + k) B_4(x_{12} + k, x_{23}) \quad (4)$$

and B_6 may be written

$$B_6(x) = \sum_{\substack{k_{25} k_{35} \\ = 0}}^{\infty} (-)^{k_{25} + k_{35}} \binom{z_{25}}{k_{25}} \binom{z_{35}}{k_{35}} B_4(x_{45}, x_{56} + k_{25} + k_{35}) \quad (5)$$

$$B_5(x_{12} + k_{25}, x_{23}, x_{34}, x_{46} + k_{25} + k_{35}, x_{24})$$

The definition of the variables x' looks complicated in general, but can be explained quite simply. For example, the variables of B_5 in Eq. (5) are just the variables of the five-point function with particles 5 and 6 considered as "particle" 5, and with summation parameters added to the variables corresponding to internal lines. The way the $k_{\alpha\beta}$ are added to the internal line variables can be inferred from Eq. (3) and Fig. 2.

Introducing the standard integral representation for B_4 , it is easily seen that for $N=5, 6, 7$, our expression for B_N agrees with the already known expressions ²⁾⁻⁴⁾. In the following we will attempt to justify our definition for arbitrary N .

2. CROSSING SYMMETRY AND ANALYTICITY

The amplitude is invariant under cyclic and anticyclic permutations of the external particles. This is shown by obtaining an integral representation for B_N and proving that the integral has the right symmetry properties. We first observe that Eq. (1) may be considered as a splitting of B_N into B_4 and B_{N-1} around the vertex $N-2$. By repeated application of this splitting around the vertices $N-3, N-4$ etc., down to 3, we obtain

$$B_N(x) = \sum_{k_{\alpha\beta}} \left\{ \prod_{i=2}^{N-2} \left\{ \prod_{A_i} (-)^{k_{\alpha\beta}} \binom{z_{\alpha\beta}}{k_{\alpha\beta}} \right\} B_4(x_{i,i+1}, x_{i,i} + \sum_{A_i} k_{\alpha\beta}) \right\} \quad (6)$$

where A_i is the index set $2 \leq \alpha \leq i$, $i+1 \leq \beta \leq N-1$, $\beta - \alpha \geq 2$. If we introduce the usual integral representation of B_4 , each sum is a binomial series, and it may be shown that the resulting integral is exactly the expression of Chan and Tsou. Their integral has been shown to have the correct symmetry properties ⁵⁾.

This agreement also implies that our expression has the right analyticity properties. These can also be proved from Eq. (1) by induction. Assume that B_{N-1} has the right pole structure, that is, poles only occur in those combinations that can be associated with the graphs having $N-1$ external lines. It is clear from the cyclic symmetry and from Eq. (1) that this implies the same properties for B_N , and since this is true for B_4 it follows for all N .

3. RESIDUES

To work out the residue of a pole in one of the variables, we distinguish two cases. First, suppose there is a pole in $x_{N-1,N}$ which by cyclic symmetry may represent any of the $\alpha_{i,i+1}$. Putting $x_{N-1,N} = -J$ ($J = \text{non-negative integer}$) and using Eq. (1), the pole only occurs in the terms where

$$K \equiv \sum_{i=2}^{N-3} k_{i,N-1} \leq J$$

The residue of the pole is then seen to be

$$R_{N,N-1}^{(J)} = (-)^J \sum_{K=0}^J \left\{ \prod_{i=2}^{N-3} \binom{z_{i,N-1}}{k_{i,N-1}} \right\} \binom{x_{N-2,N-1} - 1}{J-K} B_{N-1}(\bar{x}) \quad (7)$$

It is seen that the polynomial multiplying B_{N-1} is of maximum degree J , as it should be.

Now consider the case of a pole in x_{ij} ($3 \leq j \leq N-2$), which corresponds to an internal line in Fig. 2. It is advantageous to recombine the B_4 in Eq. (6) into a B_{j+1} and a B_{N-j+2} function

$$B_N(x) = \sum_{k_{\alpha\beta}} \left\{ \prod_{\substack{\alpha=2, j-1 \\ \beta=j+1, N-1}} (-)^{k_{\alpha\beta}} \binom{z_{\alpha\beta}}{k_{\alpha\beta}} \right\} B_{j+1}(x') B_{N-j+2}(x'') \quad (8)$$

The variables x' can be obtained by considering $(j+1)$ as the cluster $(j+1, j+2, \dots, N)$ and by adding to x_{1i} all $k_{\alpha\beta}$ where $\alpha \leq i$. Similarly, to obtain x'' , add to x_{1i} all $k_{\alpha\beta}$ where $\beta \geq i+1$ and let $(1, 2, \dots, j-1) \rightarrow 1$, $j \rightarrow 2$, $j+1 \rightarrow 3$, etc. We then split the B_{N-j+2} in Eq. (8) about the vertex $j+1$, and put $x_{ij} = -J$. The pole is now only in the middle B_4 function, and the residue turns out to be

$$R_{ij}^{(J)} = (-)^J \sum_{K=0}^J \left\{ \prod_{\alpha} \binom{z_{\alpha\beta}}{k_{\alpha\beta}} \right\} \binom{x_{jj+1} - 1}{J-K} B_{j+1}(x') B_{N-j+1}(x'') \quad (9)$$

The polynomial multiplying $B_{j+1} \cdot B_{N-j+1}$ is clearly a function only of those variables which do not belong entirely to one cluster and of maximum degree J .

4. REGGE BEHAVIOUR

We now consider particles 1 and N as incoming. Here we only show Reggeization for $N=5$ and $N=6$; the generalization to arbitrary N should be fairly obvious. Only the leading term in the Regge limit is considered.

We consider first B_5 as given by Eq. (4) and introduce the variable $\lambda_3 = x_{23}x_{34}/x_{24}$ which must be kept fixed in the Regge limit ⁶⁾. Writing the B_4 functions in terms of Γ functions, introducing the integral representation for $\Gamma(x_{12}+k)$ and $\Gamma(x_{45}+k)$, and picking out the leading term from the binomial coefficient we get, in the Regge limit,

$$B_5(x) \sim \int_0^\infty \int_0^\infty du dv u^{x_{12}-1} v^{x_{34}-1} e^{-u-v} \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{uv}{\lambda_3}\right)^k x_{23}^{-x_{12}} x_{34}^{-x_{34}}$$

This gives

$$B_5(x) \sim x_{23}^{-x_{12}} x_{34}^{-x_{34}} G_5(x_{12}, x_{34}; \lambda_3) \quad (10)$$

$$G_5(x, y; \lambda) = \int_0^\infty \int_0^\infty du dv u^{x-1} v^{y-1} e^{-u-v-\frac{uv}{\lambda}} \quad (11)$$

as obtained by Bardakçi and Ruegg²⁾.

Now consider B_6 . We substitute the asymptotic expression (10) for B_5 into Eq. (5) for B_6 , express the B_4 in Eq. (5) in terms of Γ functions and introduce the integral representation for $\Gamma(x_{56}+k_{25}+k_{35})$. It will be seen that the k dependence from the leading term of the binomial coefficients exactly cancels the k dependence of the energy variables from the B_5 function in the triple Regge limit $x_{25} \sim (\lambda_3 \lambda_4)^{-1} x_{23} x_{34} x_{45}$ ⁶⁾. The sums then give exponentials as above, and we get

$$B_6(x) \sim x_{23}^{-x_{12}} x_{34}^{-x_{13}} x_{45}^{-x_{14}} G_6(x_{12}, x_{13}, x_{14}; \lambda_3, \lambda_4) \quad (12)$$

$$G_6(x, y, z; \lambda_1, \lambda_2) = \int_0^\infty \int_0^\infty \int_0^\infty du dv dw u^{x-1} v^{y-1} w^{z-1} e^{-u-v-w-\frac{uv}{\lambda_1}-\frac{vw}{\lambda_2}-\frac{uvw}{\lambda_1\lambda_2}} \quad (13)$$

which is Chan's result³⁾.

From the structure of Eq. (1) it is clear that Reggeization may be shown for all N by induction. It also follows that B_N Reggeizes correctly when only some of the x_{ii+1} tend to infinity.

The residue function G_N may be inferred from Eqs. (11) and (13). It is easily seen that G_N factorizes ^{*}), and that the vertex function for an internal vertex is related to a Whittaker function. For instance

$$\zeta_6 = \Gamma(x_{12}) V(x_{12}, x_{13}, \lambda_2) \Gamma(x_{13}) V(x_{13}, x_{14}, \lambda_4) \Gamma(x_{14})$$

$$V(x, y; \lambda) = \frac{1}{\Gamma(y)} \int_0^\infty du u^{y-1} e^{-u} \left(1 + \frac{u}{\lambda}\right)^{-x}$$

A further discussion will be given elsewhere.

6. CONCLUSION

One question left aside so far is the problem of convergence of the series. For B_5 the series converges for $x_{24} > 0$, but in the general case the region of convergence has not yet been worked out. It can be proved that a region of convergence always exists and that it is independent of the x_{1i} .

We would like to point out some advantages of the present formula for B_N . First, it exhibits clearly the physical interpretation of the amplitude and the universal dynamical assumption underlying it. Second, the series will be useful for actual calculation, to which the integral form is not easily applicable. Finally, it seems that our expression may prove useful in construction of approximate expressions which probably will be needed for $N \geq 7$ because of the rapid increase of complexity for large N .

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^{*}) This was first pointed out by K. Bardakçi.

FIG.1

FIG.2

R E F E R E N C E S

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FIGURE CAPTIONS

Figure 1 : The N - particle graph used in the definition of B_N . The black dots indicate the vertices defining the splitting in Eqs. (1) and (8) respectively.

Figure 2 : Splitting of B_N into B_4 and B_{N-1} for the cases $N=5, 6, 7$ showing the addition of the summation parameters to the variables representing the internal lines.