CM-P00057826

Archives

Ref. TH. 911

PION-PION PHASE SHIFT INFORMATION FROM THE REACTION $\bar{p}p \rightarrow 3\pi$

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A B S T R A C T

The relativistic forms of the Faddeev equations are studied in connection with any general decay process into three pions. The pion-pion interaction is parametrized through off-shell form factors for each partial wave to the relevant phase shifts. The equations are solved numerically for the case of $\bar{p}p \rightarrow 3\pi$, where the annihilation is at rest, and compared with experimental data for the Dalitz plot for the $\pi^+ \pi^- \pi^0$ final state. Phase shift solutions for the P and D waves are fed in which simply correspond to the ρ and f^0 mesons and the best fits from a wide-ranging set of solutions of the S wave phase shifts are found. We find evidence in favour of an S wave resonance close to the ρ mass with a corresponding width of 480 MeV.

I. - INTRODUCTION

In spite of the numerous attempts to study low-energy pion-pion interactions, our knowledge concerning the phase shifts which parametrize these interactions, in particular the S wave interaction, remains rather uncertain. Over the last few years a host of solutions for the S wave phase shift δ_0^0 (we use the notation δ_{2J}^{2I}) have been put forward with varying levels of confidence and although many of these contradict one another it is possible to discern a trend amongst the more recent and reliable estimates. These latest solutions favour δ_0^0 reaching 90° between 700 and 850 MeV but to be increasing very slowly at the resonance position ^{1),2)}. This is in sharp contrast to the earlier predictions of a low mass object (~ 400 MeV) with a narrow width (< 100 MeV).

Phenomenological calculations of $\pi\pi$ phase shifts can be classified into mainly three types :

- i) : analysis of peripheral production of pions in πN collisions ;
- ii) : dispersion relation analyses of πN scattering ;
- iii) : analysis of $K \ell_4$ decays.

Each of these methods has certain defects - none is very "clean" in the sense that in each case strong assumptions, the exact nature of which varies according to the authors, have to be made concerning interactions other than the one between pions. The difficulty encountered by the analyses ¹⁾⁻¹⁷⁾ of the reaction $\pi N \rightarrow \pi\pi N$ in terms of peripheral production is the domination of the final state by the pion-nucleon interaction and also the freedom of choice for the largely unknown off-shell form factor. Information about S wave pion-pion scattering from dispersion relation analysis of backward pion-nucleon scattering has become more precise as the corresponding information on πN phase shifts has improved and expanded. Early calculations ¹⁸⁾ required δ_0^0 to rise rapidly to about 30° and then to slowly fall off while more recent and detailed calculations ¹⁹⁾ found solutions consistent with quite a wide class of behaviour for δ_0^0 . The

latest solutions ²⁰⁾ for the S and D wave phase show resonant behaviour in the range 600-700 MeV. $K_{\ell 4}$ decays have recently ²¹⁾⁻²³⁾ become a focus of study of $\pi\pi$ interaction. This promises to be an important source of information in the region below the mass of the K meson in spite of the number of parameters involved.

In this paper we present yet another way of determining phase shifts which we believe to have certain advantages compared with those mentioned above. The method involves a detailed analysis of the structure of the Dalitz plot corresponding to the production of three pions from proton-antiproton annihilations at rest. The outstanding feature of this reaction is that the only allowed final state interaction is the one we wish to determine and there is no contamination by pion-nucleon interaction. Furthermore since the $\bar{p}p$ system is so massive we can hope to study the $\pi\pi$ interaction from threshold all the way up to 1740 MeV.

The two final states resulting from $\bar{p}p$ collisions are (i) $\bar{p}p \rightarrow 3\pi^0$ and (ii) $\bar{p}p \rightarrow \pi^+ \pi^- \pi^0$. The first reaction is particularly clean since no P wave $\pi\pi$ interaction is allowed and so the removal of the ρ meson allows the S wave interaction to be seen at its clearest. The systematic way to proceed would be clearly to first analyze reaction (i) in order to determine the S and D wave phase shifts and then to feed these into reaction (ii) in order to determine the P wave behaviour. However, although very good experimental data have recently been published for the second reaction ²⁴⁾ which allow a detailed analysis of the Dalitz plot there are not, at the moment, published data for reaction (i). It is hoped that data will become available before long ²⁵⁾ for the $3\pi^0$ final state. The procedure we have adopted is to analyze reaction (ii) by feeding in phase shift solutions for the P and D wave interactions which simply describe the ρ and f^0 mesons and get the S wave phase shifts as output. Thus it is quite likely that our P wave solution underestimates the low-energy tail of the ρ .

We study the general decay process $A \rightarrow 3\pi$ using the Faddeev equations²⁶⁾ and then solve these equations numerically. Since the energy of the $\bar{p}p$ system is large compared with the pion mass, it is necessary to use the relativistic form of the Faddeev equations, the detailed forms of which have been developed for the case of spin zero particles by Freedman, Lovelace and Namyslowski²⁷⁾. Fortunately much valuable work has already been carried out on the numerical solving of the non-relativistic Faddeev equations by Hetherington and Schick²⁸⁾ and Barbour and Schult²⁹⁾ and we have simply extended these methods to the relativistic case. The calculation is also interesting in providing evidence on the importance, or otherwise, of multiple scattering effects in final state interactions - in particular the so-called triangular singularity contributions.

In Section 2, we go through the rather complicated notation and description of kinematics necessary for three-pion systems. Section 3 describes the relativistic Faddeev equations while the method of solving these equations for the process $A \rightarrow 3\pi$, where A is a system with well-defined quantum numbers, is described in Section 4. In the final Section, we study the process $\bar{p}p \rightarrow \pi^+\pi^-\pi^0$ applying the techniques from the preceding Sections and present our conclusions concerning the S wave interaction. Our main conclusion is consistent with the views mentioned at the beginning of this Section - namely that δ^0_0 resonates near the mass of the ρ meson but the width of the resonance is large. Our best solution corresponds to an S wave resonance of mass 780 MeV and width 480 MeV.

2. - KINEMATICS

Consider the process $A \rightarrow 3\pi$. The total four-momentum $\hat{P}$ (all four-momenta denoted with hats) is

$$\hat{P} = \hat{k}_1 + \hat{k}_2 + \hat{k}_3 \quad (1)$$

where $\hat{k}_\alpha$ is the momentum of pion α . The mass of A is W , so that in the centre-of-mass frame

$$\hat{P} = (W, \underline{0})$$

We define, with $\alpha\beta\gamma$ cyclic

$$\hat{p}_\gamma = \frac{1}{2}(\hat{k}_\alpha - \hat{k}_\beta) \quad (2)$$

as the relative momentum of the $(\alpha\beta)$ subsystem and

$$\hat{q}_\gamma = \frac{1}{3}[2\hat{k}_\gamma - (\hat{k}_\alpha + \hat{k}_\beta)] \quad (3)$$

as the momentum of pion γ relative to the $(\alpha\beta)$ subsystem, so that in the centre-of-mass frame the three momenta are

$$\underline{q}_\alpha = \underline{k}_\alpha \quad (4)$$

We shall denote magnitudes of three momenta simply by k_α , p_α , q_α , etc., and on-shell quantities with a bar -

$$\overline{q}_\alpha = \sqrt{1 + q_\alpha^2} \quad (5)$$

$$\overline{p}_\alpha = 0 \quad (6)$$

where we have taken the pion-mass as unity.

w_γ is defined as the energy of the $(\alpha\beta)$ subsystem in the $(\alpha\beta)$ rest frame so that

$$q_\alpha = \frac{1}{2W} \left[\left\{ W^2 - (\bar{w}_\alpha + 1)^2 \right\} \left\{ W^2 - (\bar{w}_\alpha - 1)^2 \right\} \right]^{\frac{1}{2}} \quad (7)$$

and

$$p_\alpha = \frac{1}{2} \left[\bar{w}_\alpha^2 - 4 \right]^{\frac{1}{2}} \quad (8)$$

The most convenient angular momentum description is in terms of the helicity states for three particles developed by Wick³⁰⁾.

Our definitions of p_α , q_α , W and $\bar{w}_\alpha$ are exactly those of Wick and we proceed to define our angles accordingly. The direction of $\underline{p}_\alpha$ is given by angles θ_α , ϕ_α and the direction of $\underline{q}_\alpha$ by Θ_α , Φ_α . Each of the sets of variables ($\alpha = 1, 2, 3$)

$$\bar{w}_\alpha \cos \Theta_\alpha \quad \Phi_\alpha \quad \cos \theta_\alpha \quad \phi_\alpha$$

constitutes a set of five independent variables which completely determine the configuration of a three-particle state.

We also need to define the angle χ_α which is the angle between $\underline{q}_\beta$ and $\underline{q}_\gamma$. The expressions for the angles in terms of the q_α can be found in Ref. 27).

We now list the various quantum numbers used :

- J = total angular momentum of A ;
- M = z component of J ;
- I = isospin of A ;
- j_γ = total angular momentum of the $(\alpha\beta)$ pion-pion subsystem ;
- m_γ = z component of j_γ ;
- T_γ = isospin of the $(\alpha\beta)$ subsystem ;
- L_γ = orbital angular momentum of pion γ relative to the $(\alpha\beta)$ subsystem ;
- t_γ = third component of isospin for pion γ .

Since J_Y , T_Y , L_Y are related to each other and to the parity of A , we shall often combine all three and label the particular $\pi\pi$ channel by N_Y .

We can then write the expression for the rate $A \rightarrow 3\pi$ as

$$d\left(\frac{1}{\tau}\right) = \frac{1}{2\pi W(2J+1)} \sum_M \int |\langle A | S_M^{JI} | 3\pi \rangle|^2 \delta^4(\hat{P} - \hat{k}_1 - \hat{k}_2 - \hat{k}_3) \frac{d^3k_1}{k_1^0} \frac{d^3k_2}{k_2^0} \frac{d^3k_3}{k_3^0} \quad (9)$$

It is convenient to write

$$\delta^4(\hat{P} - \sum \hat{k}_\alpha) \frac{d^3k_1}{k_1^0} \frac{d^3k_2}{k_2^0} \frac{d^3k_3}{k_3^0} = \frac{p_1 q_1}{W} d(\cos\Theta_1) d\phi_1 d\bar{w}_1 d(\cos\Theta) d\bar{\Phi} \quad (10)$$

If we assume the amplitude $\langle A | S_M^{JI} | 3\pi \rangle$ is independent of ϕ_1 , $\cos\Theta_1$ and $\bar{\Phi}_1$ then we obtain the right-hand side of equation (10) as

$$\left(\frac{\pi}{W}\right)^2 d\bar{w}_1^2 d\bar{w}_2^2 \quad (11)$$

In this case the density of the Dalitz plot can then be expressed as

$$\frac{d\left(\frac{1}{\tau}\right)}{d\bar{w}_1^2 d\bar{w}_2^2} = \frac{1}{(4\pi)^4 W^3 (2J+1)} \sum_M \left| \langle A | S_M^{JI}(\bar{w}_1^2, \bar{w}_2^2, \bar{w}_3^2) | 3\pi \rangle \right|^2 \quad (12)$$

3. - THE RELATIVISTIC FADDEEV EQUATIONS

In this Section we describe the basic model which we use to study a decay process into three pions. The full details of the theory are described by Freedman et al. ²⁷⁾ and we refer the reader to that paper for all proofs.

A) - Two-particle states

Consider the scattering of pions α, β . The scattering amplitude $T_Y(w_Y)$ satisfies the Bethe-Salpeter equation

$$T_Y(w_Y) = I_Y + T_Y(w_Y) G_Y^0(w_Y) I_Y \quad (13)$$

where I_Y is the sum of all irreducible graphs. G_Y^0 is the free two-particle Green's function given by

$$G_Y^0(w_Y) = [2\pi^2 i]^{-1} \delta^4(\hat{p}_Y - \hat{p}'_Y) \times \\ \times \left[\left(\frac{1}{2} w_Y + p_Y^0 \right)^2 - p_Y^2 - 1 + i\epsilon \right]^{-1} \left[\left(\frac{1}{2} w_Y - p_Y^0 \right)^2 - p_Y^2 - 1 + i\epsilon \right]^{-1} \quad (14)$$

We assume the validity of the separable approximation so that we can write a partial-wave expansion for I_Y in the following way :

$$\langle p_Y | I_Y | p'_Y \rangle = \sum_{\ell} (4\pi)^{-1} (2\ell+1) P_{\ell}(p_Y \cdot p'_Y / p_Y^0 p_Y'^0) \times \\ \times \lambda_{\ell} g_{\ell}(p_Y^0, p_Y) g_{\ell}(p_Y'^0, p_Y') \quad (15)$$

It then follows that the two-body scattering amplitude can be similarly expressed :

$$\langle p_Y | T_Y | p'_Y \rangle = \sum_{\ell} (4\pi)^{-1} (2\ell+1) P_{\ell}(p_Y \cdot p'_Y / p_Y^0 p_Y'^0) \times \\ \times t_{\ell}(w_Y^2) g_{\ell}(p_Y^0, p_Y) g_{\ell}(p_Y'^0, p_Y') \quad (16)$$

The quantities $g_{\ell}(p_Y^0, p_Y)$ are the off-shell form factors for the ℓ wave partial wave amplitude and $t_{\ell}(w_Y^2)$ is the propagator for the ℓ wave interaction, related to λ_{ℓ} and g_{ℓ} by

$$[t_{\ell}(w_Y^2)]^{-1} = \lambda_{\ell}^{-1} - (2\pi^2 i)^{-1} \int_0^{\infty} d p_Y^0 \int_{-\infty}^{\infty} d p_Y^0 p_Y^2 |g_{\ell}(p_Y^0, p_Y)|^2 \times \\ \times \left[\left(\frac{1}{2} w_Y + p_Y^0 \right)^2 - p_Y^2 - 1 + i\epsilon \right]^{-1} \left[\left(\frac{1}{2} w_Y - p_Y^0 \right)^2 - p_Y^2 - 1 + i\epsilon \right]^{-1} \quad (17)$$

An important step in the derivation of the equations is the elimination of the subenergies p_Y^0 from the variables of integration. We shall not discuss that problem here, simply accept the results of Ref. 27) in which it is shown that the relative energies should be evaluated only at their on-shell values $\bar{p}_Y^0 = 0$.

Writing $g_\ell(p_Y)$ for $g_\ell(0, p_Y)$, equation (17) reduces to

$$\left[t_\ell(w_Y^2) \right]^{-1} = \lambda_\ell^{-1} - \frac{1}{2\pi} \int_0^\infty p_Y^2 dp_Y |g_\ell(p_Y)|^2 \left\{ \frac{1}{4} \bar{w}_Y (\bar{w}_Y^2 - w_Y^2 - i\epsilon) \right\}^{-1} \quad (18)$$

At this stage we should ask what form do we take for the form factor $g_\ell(p)$? We have to admit that to a large extent we have to carry out some inspired guesswork. Firstly we know the connection between $g_\ell(p)$ and the phase shift for the ℓ th partial wave

$$\begin{aligned} \frac{p}{8\sqrt{1+p^2}} |g_\ell(p)|^2 \cot \delta_\ell &= \text{Re} \left[t_\ell^{-1}(\bar{w}_Y^2) \right] \\ &= \lambda_\ell^{-1} - (2\pi)^{-1} P \int_0^\infty p'^2 dp' |g_\ell(p')|^2 \left\{ \frac{1}{4} \bar{w}_Y (\bar{w}_Y'^2 - \bar{w}_Y^2 - i\epsilon) \right\}^{-1} \end{aligned} \quad (19)$$

The only other information is that $g_\ell(p)$ should have the correct threshold and asymptotic behaviour 31), 32). Lovelace 27) has suggested an extension of the Hulthén form factor in non-relativistic theory of the form

$$g_\ell(p) = \frac{N p^\ell}{(p^2 + \mu_\ell^2)^{\ell+1}} \quad (20)$$

The major disadvantage of this parametrization, which was pointed out by Barbour and Schult 29), is that the S wave cannot resonate. In our calculations we wish to investigate the effects of widely varying choices for the S wave pion-pion interaction and therefore need

a parametrization which is sufficiently flexible to allow the phase shift to pass through 90° . In this matter we have been influenced by the choice of Barbour and Schult in their calculations. Our parametrization is given by

$$g_\ell(p) = \frac{(1+p^2)^{\frac{1}{4}} p^\ell}{[(\beta_\ell^2 + p^2)(\beta_\ell^{*2} + p^2)]^{\ell+1}} \quad (21)$$

where $\text{Re}(\beta_\ell) \geq 0$.

Putting $\beta_0 = a + ib$, we can write the S wave phase shift in the form

$$\cot \delta_0 = \frac{8(1+p^2)^{\frac{1}{2}}}{p|g_0(p)|^2} \left[\lambda_0^{-1} + \left\{ 64 a^3(a^2+b^2)(a^2+b^2-p^2) \right\} \text{Re} \left\{ \frac{p^2-5a^2-b^2+4ia p}{(a^2+b^2-p^2)-4a^2p^2-4ia p} \right\} \right] \quad (22)$$

and the S wave scattering length is given by

$$a_0^{-1} = 8(a^2+b^2)^4 \lambda_0^{-1} + (16a^3)^{-1} (a^2+b^2)(5a^2+b^2) \quad (23)$$

For the P wave we find that $\beta_1 = 20 + i23.5$ and $\lambda_1 = -2/\pi \times -2.171 \times 10^{21}$ corresponds to a ρ resonance with $m_\rho = 760$ MeV and $\Gamma_\rho = 120$ MeV and the D wave parameters $\beta_2 = 15 + i15$, $\lambda_2 = -2/\pi \times 8.779 \times 10^{27}$ gives an f^0 resonance at $m_{f^0} = 1260$ MeV, $\Gamma_{f^0} = 140$ MeV.

B) - Three-particle states

We begin by defining our potential in exactly the manner of Freedman et al. This potential corresponds essentially to the Peierl's mechanism and whose imaginary part corresponds to overlapping two-particle channel interactions. If we abbreviate the three-particle state representation as $|q_\gamma N_\gamma\rangle$ where N_γ stands for $j_\gamma, T_\gamma, L_\gamma$ then the explicit form of the potential $\bar{Z}_{\alpha\beta}$ is

$$\begin{aligned}
\langle q_\alpha N_\alpha | \bar{Z}_{\alpha\beta}^{JI}(W) | q_\beta N_\beta \rangle = \\
= \frac{1}{2} (1 - \delta_{\alpha\beta}) R^I(T_\alpha, T_\beta) (2J+1)^{-1} \left[(2j_\alpha+1)(2j_\beta+1) \times \right. \\
\times (2L_\alpha+1)(2L_\beta+1) \left. \right]^{\frac{1}{2}} \sum_{M_i} C_{0 \quad M \quad M}^{L_\alpha \quad j_\alpha \quad J} C_{0 \quad M \quad M}^{L_\beta \quad j_\beta \quad J} \times \\
\times \int_{-1}^1 d(\cos \chi_\gamma) d_{MM}^J(\chi_\gamma) d_{M0}^{j_\alpha}(\theta_\alpha) d_{M0}^{j_\beta}(\theta_\beta) \times \\
\times g_{j_\alpha}(p_\alpha) g_{j_\beta}(p_\beta) \frac{\bar{W}}{\bar{q}_\gamma^0} \cdot \frac{1}{\bar{W}^2 - W^2 - i\epsilon}
\end{aligned} \quad (24)$$

where

$$\bar{W} = (1 + q_\alpha^2)^{\frac{1}{2}} + (1 + q_\beta^2)^{\frac{1}{2}} + \bar{q}_\gamma^0 \quad (25)$$

and

$$(\bar{q}_\gamma^0)^2 = 1 + q_\alpha^2 + q_\beta^2 + 2q_\alpha q_\beta \cos \chi_\gamma$$

$R^I(T_\alpha, T_\beta)$ is the isospin coefficient given by

$$R^I(T_\alpha, T_\beta) = (-1)^{2I - T_\gamma} \left[(2T_\alpha+1)(2T_\beta+1) \right]^{\frac{1}{2}} \left\{ \begin{matrix} 1 & 1 & T_\beta \\ 1 & I & T_\alpha \end{matrix} \right\} \quad (26)$$

where the last bracket is the well-known 6-j symbol according to the convention of Edmonds³³⁾. Graphically, the potential of Eq. (24) corresponds to Fig. 1.

In addition we need a "potential" to correspond to Fig. 2. To do this we need a reduced form factor of the three-particle state A, $g_0(p_\gamma, q_\gamma)$ whose explicit form we leave undefined for the moment. We then define this potential as

$$\begin{aligned}
\langle 0 | \bar{Z}_{\alpha\beta}^{JI}(W) | q_\beta N_\beta \rangle &= \\
&= \frac{1}{2} \int_0^\infty \frac{p_\beta^2 dp_\beta g_0(p_\beta, q_\beta) g_{JA}(p_\beta)}{\frac{1}{4} w_\beta (\bar{w}_\beta^2 - \tilde{w}_\beta^2 - i\epsilon)}
\end{aligned} \quad (27)$$

where

$$\tilde{w}_\alpha^2 = [W - \bar{q}_\alpha^0]^2 - q_\alpha^2$$

The fundamental relationship between the decay amplitudes for $A \rightarrow 3\pi$ which we denote as $\langle 0 | \bar{X}_{\alpha\gamma}^{JI} | q_\gamma N_\gamma \rangle$ and the potentials above constitute the modified form of the Faddeev equations :

$$\begin{aligned}
\langle 0 | \bar{X}_{\alpha\beta}^{JI} | q_\beta N_\beta \rangle &= \langle 0 | \bar{Z}_{\alpha\beta}^{JI} | q_\beta N_\beta \rangle + \\
&+ \sum_{\delta \neq \beta} \sum_{N'_\delta} \frac{1}{2\pi} \int_0^\infty \frac{q_\delta'^2 dq'_\delta}{\sqrt{1+q_\delta'^2}} \langle 0 | \bar{X}_{\alpha\delta}^{JI} | q'_\delta N'_\delta \rangle t_{N'_\delta}(\tilde{w}_\delta'^2) \langle q'_\delta N'_\delta | \bar{Z}_{\delta\beta}^{JI} | q_\beta N_\beta \rangle
\end{aligned} \quad (28)$$

The amplitude $\bar{X}_{\alpha\gamma}$ describes the process in which A decays into pion γ and an $\alpha\beta$ bound state but since all three final particles are identical (apart from isospin labels) we can immediately simplify the equations (28). We can drop all indices α, β etc., and simply rewrite the equations in the form

$$\begin{aligned}
\langle 0 | \bar{X}^{JI}(W) | q N \rangle &= \langle 0 | \bar{Z}_{\alpha}^{JI}(W) | q N \rangle + \\
&+ 2 \sum_{N'} \frac{1}{2\pi} \int_0^\infty \frac{q'^2 dq'}{\sqrt{1+q'^2}} \langle 0 | \bar{X}^{JI}(W) | q' N' \rangle t_{N'}(\tilde{w}'^2) \langle q' N' | \bar{Z}^{JI}(W) | q N \rangle
\end{aligned} \quad (29)$$

The full S matrix element which describes the decay into three free pions is then a sum of three terms :

$$\begin{aligned} \langle A | S_M^{II}(\vec{w}_1, \vec{w}_2, \vec{w}_3) | 3\pi \rangle = \sum_{\alpha N_\alpha} K_{T_\alpha}^I \left[\frac{(2j_\alpha+1)(2L_\alpha+1)}{4\pi} \right]^{\frac{1}{2}} \times \\ \times C_{0 \ M \ M}^{L_\alpha \ j_\alpha \ J} \alpha_{M0}^{J_\alpha}(\theta_\alpha) \langle 0 | \bar{X}^{II}(w) | q_\alpha N_\alpha \rangle \epsilon_{N_\alpha}(\vec{w}_\alpha^2) g_{N_\alpha}(p_\alpha) \end{aligned} \quad (30)$$

where the isospin factor $K_{T_\alpha}^I$ is given by

$$K_{T_\alpha}^I = C \begin{pmatrix} 1 & 1 & T_\alpha \\ i_\alpha & i_\beta & i_\alpha+i_\beta \end{pmatrix} C \begin{pmatrix} T_\alpha & 1 & I \\ i_\alpha+i_\beta & i_\gamma & i_\alpha+i_\beta+i_\gamma \end{pmatrix} \quad (31)$$

The full amplitude of Eq. (30) is then the amplitude we require for the Dalitz plot density in Eq. (12). The actual solving of Eq. (29) can be performed only numerically which is a non-trivial task. In the next Section we explain our procedure for obtaining the numerical solutions.

4. - SOLVING THE EQUATIONS

The numerical solution of the Faddeev equations, even in the simplified form of (29) presents a formidable task. Our method is an extension of the one originated by Hetherington and Schick²⁸⁾ later adapted by Barbour and Schult²⁹⁾ for the case of τ decay. The only difficulties of extending the method arise from our relativistic treatment of the problem.

Equations (29) may be written in the form

$$\bar{X}_N(w, q) = \bar{Z}_0(w, q) + \frac{1}{\pi} \sum_{N'} \int_0^\infty \frac{q'^2 dq'}{\sqrt{1+q'^2}} \bar{X}_{N'}(w, q') K_{NN'}(w, q, q') \quad (32)$$

where the kernel is given by

$$K_{NN'}(W, q, q') = \epsilon_{N'}(\tilde{W}^{1,2}) \bar{Z}(W, q, q') \quad (33)$$

From (24) and (25) we see that this kernel has several logarithmic branch points. These occur, in the q' plane, at

$$1) \quad q' = \pm \frac{1}{2} q \pm \left[\frac{1}{4} q^2 + F \right]^{\frac{1}{2}}$$

where $F = \frac{E^2 - q^2}{4} - \frac{E^2}{E^2 - q^2}$

and $E = W - \sqrt{1 + q^2} + i\epsilon$ (34)

$$2) \quad \begin{aligned} q' &= \pm \frac{1}{2} q \pm i\beta_\ell & q' &= \pm \frac{1}{2} q \pm i\beta_\ell^* \\ q' &= \pm 2q \pm i\beta_\ell & q' &= \pm 2q \pm i\beta_\ell^* \end{aligned} \quad (35)$$

One of these singularities - at $\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + F}$ - will always lie arbitrarily close to the positive real axis in the q' plane. Following the new standard procedure, we avoid this singularity by rotating the contour of integration into the fourth quadrant of the q' plane (since the singularity lies above the real axis). In order to solve Eq. (19) numerically it is necessary to consider the equation at a set of points (q) which coincides with the set of points (q'); that is we must solve the equations for values of q lying on the same rotated contour as q' . We can then substitute this solution back into the same equation to obtain a solution for real physical values of q .

The maximum angle of rotation of the contour is determined by the condition that no singularity of the integrand is crossed in performing the solution. Thus we transform the variables

$$q = v e^{-i\varphi} \quad q' = u e^{-i\varphi} \quad (36)$$

with the condition that $\tan \varphi < \text{Re} \beta / \text{Im} \beta$ for all $\beta \in \mathcal{L}$. The second step is not quite so straightforward; i.e., to find $\bar{X}_N(W, q)$ for real q . We may write the second factor of the kernel in the form

$$\bar{Z}(q, q') = \int_{-1}^1 dx F(x) \left[-A + (B + Cx)^{\frac{1}{2}} \right]^{-1} \quad (37)$$

where

$$A = W - \sqrt{1+q^2} - \sqrt{1+q'^2}$$

$$B = 1 + q^2 + q'^2$$

$$C = 2qq'$$

Thus $\bar{Z}(q, q')$ has two cuts in the q' plane, given by

$$q'_{\pm} = \left[\frac{E^2 - q^2}{E^2 - q^2 x^2} \right] \left[\frac{1}{2} q x \pm \frac{1}{2} E \left\{ 1 - \frac{4(E^2 - q^2 x^2)}{(E^2 - q^2)^2} \right\}^{\frac{1}{2}} \right] \quad (38)$$

where $-1 \leq x \leq 1$ and q'_+ lies above the real axis and q'_- lies below. For some values of q , however, the cut q'_- lies partly in the fourth quadrant and the branch point lies on the real axis. It is necessary, therefore, to distort the contour of integration.

One branch point lies in the fourth quadrant if q lies in the range $q_{\text{crit}} \leq q \leq q_{\text{max}}$ where q_{max} is the maximum physical value, $q_{\text{max}} = (1/2W) \left[(W^2 - 1)(W^2 - 9) \right]^{\frac{1}{2}}$, and $q_{\text{crit}} = \frac{1}{2} \left[(W+1)(W-3) \right]^{\frac{1}{2}}$ so that when $q < q_{\text{crit}}$ both branch points of q' lie in the third quadrant. For a typical value of q in the range $q_{\text{crit}} \leq q \leq q_{\text{max}}$, the cuts have the shape shown in Fig. 3.

Such a shape is extremely awkward to handle. In order to obtain a more suitable form we change the contour of integration for the variable x in Eq. (37). Instead of integrating from -1 to $+1$ along the real axis, we go -1 to $-\infty$ along the real axis, $-\infty$ to $+\infty$ by a clockwise semi-circular arc and $+\infty$ to $+1$ along the real axis. The resulting cuts are shown in Fig. 4.

The dotted line in Fig. 4 indicates the new contour for q' . The integral to be evaluated along this contour is

$$\frac{i}{\pi} \int_0^\pi \frac{q'^2 dq'}{\sqrt{1+q'^2}} \bar{X}_{N'}(W, q) \epsilon_{N'}(\tilde{W}'^2) \bar{Z}_{NN'}(W, q, q') \quad (39)$$

which can be split into two parts : (i) for which the contour goes from near the origin into the fourth quadrant and (ii) for which the contour goes around the cut. The latter part can be expressed as

$$\frac{i}{\pi} \int_0^P \frac{q'^2 dq'}{\sqrt{1+q'^2}} \bar{X}_{N'}(W, q') \epsilon_{N'}(\tilde{W}'^2) \Delta [\bar{Z}_{NN'}(q, q')] \quad (40)$$

where $\Delta [\bar{Z}]$ is the discontinuity of $\bar{Z}$ across the cut in the q' plane and P is the relevant branch point (which is always to the left of q_{crit}).

To evaluate this discontinuity we study the integrand of (37) in the x plane. A point on the cut corresponds to a pole in the x plane at

$$x = x_0 = \frac{A^2 - B}{C} \quad (41)$$

and the discontinuity $\bar{Z}(q, q' + i\epsilon) - \bar{Z}(q, q' - i\epsilon)$ is just the contribution from integrating around this pole in an anti-clockwise direction. The residue of the pole is $F(x_0) \cdot 2A/C$. $F(x)$ contains the factor $(W/q_\gamma^0) \cdot (1/W + \bar{W})$ and at $x = x_0$, $W = \bar{W}$, $q_\gamma^0 = A$ giving

$$\begin{aligned} \Delta [\bar{Z}_{\alpha\beta}^{JI}(W, q, q')] &= \frac{1}{2} (1 - \delta_{\alpha\beta}) R^I(T_\alpha, T_\beta) (2J+1)^{-1} \times \\ &\times \sum_M [(2j_\alpha+1)(2j_\beta+1)(2L_\alpha+1)(2L_\beta+1)]^{\frac{1}{2}} \begin{Bmatrix} L_\alpha & j_\alpha & J \\ 0 & M & M \end{Bmatrix} \begin{Bmatrix} L_\beta & j_\beta & J \\ 0 & M & M \end{Bmatrix} \times \\ &\times \frac{i\pi}{q_\gamma^0} g_{N_\alpha}(p_\alpha) g_{N_\beta}(p_\beta) d_{MM}^J(\chi_\gamma^0) d_{M0}^{j_\alpha}(\theta_\alpha^0) d_{M0}^{j_\beta}(\theta_\beta^0) \end{aligned}$$

(42)

The suffix zero on χ_γ etc., denotes that these quantities are evaluated at $\cos \chi_\gamma^0 = x_0$.

As pointed by Barbour and Schult this last integration along the real axis has a physical relevance in connection with the so-called triangular singularities ³⁴⁾. We could easily examine the effect of these terms by allowing the A mass to vary such that the singularity approached the boundary of the physical region. Our conclusions were that no observable effects were produced in the Dalitz plot by this limiting procedure.

5. - APPLICATION TO THE REACTION $\bar{p}p \rightarrow 3\pi$

So far we have discussed in general terms the decay $A \rightarrow 3\pi$ where A may be any three-pion state with definite quantum numbers - e.g., a three pion resonance. Equally well, A can be the state formed by proton-anti-proton annihilation at rest and in this Section we confine our attention to this example.

The advantages of studying $\bar{p}p \rightarrow 3\pi$ in order to obtain information about $\pi\pi$ phase shifts are quite obvious. The only final state interactions, which can occur, are those we wish to study ; there is no contamination by pion-nucleon interaction and so this method is "cleaner" than say the study of $\bar{K}N \rightarrow 2\pi N$.

Since the annihilation is at rest, the $\bar{p}p$ can exist in two spin states, the singlet and triplet corresponding to 0^- and 1^- respectively. Since the G parity of the final state is -1 , then the isospin of the whole system is associated with the spin state. The two possible initial states are $I^G(J^P) = 1^-(0^-)$ and $0^-(1^-)$.

In order to avoid an excessive number of parameters in the problem, we consider only four different pion-pion interactions, namely :

N (label)	J_α	T_α	L_α
1 (ρ)	1	1	1
2 (f^0)	2	0	2
3 (σ)	0	0	0
4 (σ')	0	2	0

From the singlet state of $\bar{p}p$ all four channels may occur, but only the first channel is allowed from the triplet state. In this paper we shall concern ourselves with the reaction $\bar{p}p \rightarrow \pi^+ \pi^- \pi^0$, in particular we analyze the recent CERN data ²⁴⁾ where over 3800 events were obtained. The estimated contamination by multi-neutral events was 14 % and we suppose such events to be distributed evenly over the Dalitz plot. The main assumption of our analysis is that the reaction proceeds via two-particle interactions of the type listed above. By using the methods described in the previous Sections we ensure that all orders of rescattering are taken into account.

According to Eq. (19) and the particular dependence of g_ℓ on the momentum p given by Eq. (20) we are restricted to a study only of phase shifts which never pass through 0 or 180°. We wish, however, to consider a larger class of phase shifts in which CDD poles may occur. This implies that λ_ℓ must be generalized from simply a constant to a polynomial in w^2 ^{35), 36)}. We have, therefore, proceeded in a slightly different manner by simply multiplying $t(w^2)$ by $(w^2 - w_c^2)$ where w_c is the position of the CDD pole. In this way we successfully imitate the P wave solutions studied by Pisut and Roos ³⁷⁾. Similarly we explored the effects of allowing CDD poles in the $I = T = 0$ interaction.

If we consider the P and D wave parameters as input to the problem, we are then left with the parameters of both S waves together with relative amounts of singlet and triplet states as adjustable parameters.

In view of the long time required by even a large computer to evaluate the Dalitz plot density over the whole region, it is impossible to carry out any sophisticated minimization procedure. We had to resort to trial and error methods. In the table below we list values of the parameters corresponding to the best fit obtained :

Channel 1	(ρ)	$\beta = 20.0 + i23.5$	$\lambda = -1.38 \times 10^{21}$
Channel 2	(f^0)	$\beta = 15.0 + i15.0$	$\lambda = -5.58 \times 10^{27}$
Channel 3	(σ)	$\beta = 5.0 + i20.0$	$\lambda = -2.29 \times 10^9$
together with a factor of $(22.5 - q^2)/15.75$ in $t(w^2)$			
Channel 4	(σ')	$\beta = 5.0 + i 8.0$	$\lambda = -2.80 \times 10^{11}$
% singlet contribution = 25.6			
% triplet contribution = 74.4			

Figures 3 and 4 show the behaviour of each phase shift as a function of the energy of the $\pi\pi$ system. Of most interest is the solution for δ_0^0 . An S wave resonance occurs at 780 MeV with width 480 MeV and a CDD pole at 1360 MeV. The $I = 0$ and $I = 2$ S wave scattering lengths are estimated to be $0.63 \mu^{-1}$ and $-0.12 \mu^{-1}$ respectively. The behaviour of δ_0^0 near the position of the ρ meson confirms the results of Schlein, Malumud ¹⁾ and Clegg ²⁾ from $\pi N \rightarrow \pi\pi N$ analyses, namely that the phase shift increases very slowly. Our solution, with a huge width for the resonance, is consistent with analysis of Lovelace ^{20), 38)} on backward pion-nucleon dispersion relations. Our values for the scattering lengths are somewhat larger than Weinberg's ³⁹⁾. However, recent current algebra calculations ⁴⁰⁾, which attempt to take into account S wave interactions, yield solutions consistent with ours.

We also attempted to analyze the Syracuse-Rome data ⁴¹⁾ for the reaction $\bar{p}n \rightarrow 2\pi^-\pi^+$, which can proceed via the singlet state only. We were unable to achieve any kind of reasonable fit to the data because ρ production appears to be severely damped. We failed to find any S wave solutions which could effectively reduce the ρ contribution with respect to f^0 production.

ACKNOWLEDGEMENTS

It is a pleasure to thank Dr. C. Lovelace who suggested this analysis and subsequently provided valuable ideas. We are grateful to Dr. P. Villemees for discussions on the experimental data and to Dr. I.M. Barbour for helpful correspondence.

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FIGURE CAPTIONS

- Figure_1 : Graph corresponding to the potential of Eq. (24).
- Figure_2 : Graph corresponding to the potential of Eq. (27).
- Figure_3 : Cuts in the q' plane corresponding to a x integration from -1 to $+1$.
- Figure_4 : Cuts in the q' plane corresponding to an x integration from -1 to $-\infty$ and $+\infty$ to $+1$.
- Figure_5 : P and D wave solutions which correspond to ρ and f^0 .
- Figure_6 : S wave phase solutions obtained from a best fit to the data of Foster et al. ²⁴⁾. The vertical lines correspond to the fits of Walker et al. ¹⁴⁾ to $\pi N \rightarrow \pi \pi N$ data.
- Figure_7 : The projections of the Dalitz plot from Ref. ²⁴⁾ together with the solution obtained with the phase shifts of Figures 5 and 6.

FIG.1

FIG.2

FIG.3

FIG.4

FIG.5

FIG.6

FIG.7