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A PROOF OF THE VALIDITY OF GENERALIZED
INTERFERENCE MODELS

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ABSTRACT

It is shown that, under fairly general hypotheses, the scattering amplitude $A(s,u,t)$ can always be written as $A(s,u,t) = A_1(s,u,t) + A_2(s,u,t) + A_3(s,u,t)$, where for instance $A_1(s,u,t)$ has resonances only in the t channel, and it is such that for $s \rightarrow \infty$ at fixed t it has a Regge behaviour, whereas it goes to zero faster than any power if $s, t \rightarrow \infty$ at fixed u or if $u, t \rightarrow \infty$ at fixed s .

Lately there has been a growing interest for the problem of the information contained in the finite energy sum rules (FESR) ¹⁾. The concept of duality ¹⁾ has been developed according to which it is possible to extrapolate at intermediate energy, and also at low energy under the most drastic assumption, the behaviour given at high energy by the Regge theory. According to duality, the Regge terms at intermediate energy summarize, on the average, the features of the scattering amplitude, in particular they contain the averaged structure of the resonances. This point of view can be contrasted with the so-called interference model, in which the contribution of the resonances and of the Regge terms are added separately. Models have been proposed which satisfy FESR and crossing symmetry according to two points of view : duality and the interference model. The Veneziano model ²⁾, for instance, is said to be dual, whereas the models proposed by Kupsch ³⁾ and the present author ⁴⁾ seem to be realizations of the interference model. It is clear that, in order to avoid double counting, the interference model must be constructed in such a way that the contribution of the resonances in the direct channel gives zero in the FESR, i.e., the resonances must be contained in a part of the scattering amplitude which is strictly superconvergent in the sum rules of any moment. We shall call this model ⁵⁾ the "generalized interference model".

The aim of this note is to clarify the rather vague concept of duality by showing that, under very general hypotheses, it is always possible to write a decomposition of the scattering amplitude in the form of the generalized interference model. Let us consider first the case of a scattering amplitude having only s and t channels, $A(s,t)$. Later on, we shall extend the results to the case in which the three channels s , t and u are present.

The hypotheses are as follows :

- i) $A(s,t)$ is an analytic function with cuts from s_0 to $+\infty$ in the s variable, and from t_0 to $+\infty$ in the t variable. The behaviour of $A(s,t)$ for $s \rightarrow \infty$ at t fixed or for $t \rightarrow \infty$ at s fixed or for $t \rightarrow \infty$ at s fixed is of the Regge type, e.g., in the first case the behaviour is

$$\sum_i \beta_i \frac{(-s)^{\alpha_i(t)}}{\sin \pi \alpha_i(t)}$$

This hypothesis can be weakened by saying that, for instance, at fixed t the behaviour in s is bounded by a power. In particular we need the assumption that the discontinuity in s , $\Delta_s A(s, t)$ of $A(s, t)$ for s real positive, greater than the elastic threshold s_0 , is such that

$$|\Delta_s A(s, t)| < \psi(s) (|t^{\alpha(s)+1}| + |t^{\alpha^*(s)+1}|) \quad (1)$$

for $|t| > \bar{t}$, where $\psi(s)$ is a function of s which may be growing in an arbitrary way for $s \rightarrow \infty$. $\Delta_s A(s, t)$ will have in general singularities for t on the real positive axis. Nevertheless we make the assumption that $\psi(s)$ is a holomorphic function of s [it is sufficient that $\psi(s)$ in Eq. (1) is everywhere defined in the positive real axis, we can then find a holomorphic $\psi'(s)$ such that on the positive real axis $\psi'(s) > \psi(s)$]. If this is not true for $\Delta_s A(s, t)$, it will be true for a local average $\overline{\Delta_s A(s, t)}$ of $\Delta_s A(s, t)$. In that case we will prove the asymptotic behaviour of our functions for their local average only. Alternatively, our proof of the asymptotic behaviour will be valid for s and t everywhere in the first sheet, but not on the positive real axis.

- ii) There exists only a finite number of parent trajectories which describe the resonances. This means that for instance in the s channel the amplitude will have poles whenever one of the analytic functions $\alpha_i(s)$ is equal to a positive integer $\alpha_i(s) = n$, and there is a finite number of functions $\alpha_i(s)$, i.e., $i \leq I$.
- iii) Apart from bound states, the poles are on the second sheet. That is, $\alpha_i(s)$ has a cut from s_0 to $+\infty$ and $\alpha_i(s) = n$ only in its second sheet.
- iv) The residue of $A(s, t)$ at the pole in t is a polynomial in s and vice versa, so that the discontinuity in s of $A(s, t)$, $\Delta_s A(s, t)$, does not have poles in t .

We will also add the hypothesis, which however is not essential, as is discussed in the Appendix, Section b), that there is a linearly rising trajectory, i.e., one of the functions $\alpha_i(s)$ is such that $\lim_{s \rightarrow \infty} \alpha_i(s) \rightarrow \infty$ linearly.

We want to show that, under these hypotheses, it is possible to write

$$A(s,t) = A_1(s,t) + A_2(s,t) \quad (2)$$

where $A_1(s,t)$ does not have resonances in s , it has poles in t with the residue of $A(s,t)$, its asymptotic behaviour for $s \rightarrow \infty$ at fixed t is the behaviour of $A(s,t)$, i.e.,

$$\sum_i \beta_i \frac{(-s)^{\alpha_i(t)}}{\sin \pi \alpha_i(t)}$$

whereas for $t \rightarrow \infty$ at fixed s , $A_1(s,t) \rightarrow 0$ faster than any power ^{*}). The properties of $A_2(s,t)$ are the same with the roles of s and t exchanged.

The part which has resonances in the s channel, for instance, $A_2(s,t)$, goes to zero for $s \rightarrow \infty$ at fixed t faster than any power, so that it is strictly superconvergent for any moment.

We recall that, as is known, the Veneziano model ²⁾, which seems not to be of the type of the interference model, does not satisfy our hypotheses iii) or iv).

We will give the demonstration for the case of one parent $\alpha_s(s)$ [and similarly one parent $\alpha_t(t)$] which we assume to be linearly rising with s . For simplicity we assume here that there are no bound states, i.e., the poles in s are on the second sheet, the bound states being treated in the Appendix, Section c).

^{*}) From here on whenever we say that $f(x) \xrightarrow{x \rightarrow \infty} 0$ we mean faster than any power.

We start by introducing the functions $F(s,t)$, $D_1(s,t)$, $D_2(t,s)$, the existence of which will be demonstrated by means of explicit examples, which have the following properties. All of them have cuts in t and s .

- a) $F(s,t)$ has no poles in s or in t in any sheet, it is such that $F(s,t) \xrightarrow{s \rightarrow \infty} 0$ at t fixed, and $[F(s,t)-1] \xrightarrow{t \rightarrow \infty} 0$ at s fixed, when s or t go to infinity in the first sheet.
- b) $D_1(s,t)$ has poles in s at the position of the resonances and the residue is just the residue of $A(s,t)$. It has no poles in t and $D(s,t) \rightarrow 0$ for $s \rightarrow \infty$ at fixed t and for $t \rightarrow \infty$ at fixed s , when s or t go to infinity in the first sheet. $D_2(t,s)$ has the same properties with s and t exchanged.

We then write the identity

$$A(s,t) = [A(s,t) - D_1(s,t) - \{A(s,t) - D_1(s,t) - D_2(t,s)\}F(s,t)] + [D_2(t,s) + \{A(s,t) - D_1(s,t) - D_2(t,s)\}F(s,t)] \quad (3)$$

One sees immediately that, by calling $A_1(s,t)$ the quantity in the first bracket and $A_2(s,t)$ the quantity in the second bracket, we have the decomposition of Eq. (2); A_1 and A_2 having the required properties. If we want to preserve a crossing symmetry of $A(s,t)$ we can simply start from $A(s,t) = \frac{1}{2}[A(s,t) \pm A(t,s)]$ and write the identity (3) for $A(s,t)$ and $A(t,s)$ exchanging the roles of s and t .

Examples of the functions $F(s,t)$, $D_1(s,t)$ are as follows :

$$F(s,t) = e^{-(s_0-s)^{1/4}} \exp \left\{ - \frac{(t_0-t)^{1/4}}{(s_0-s)^{1/8}} \right\} \quad (4)$$

$$D_2(s, t) = \frac{1}{2\pi i} \int_{s_0}^{+\infty} \frac{dx}{x-s} \left[A(x_+, t) - A(x_-, t) \right] e^{-\frac{2(\psi(x) + (s_0-s)^{1/4} + (t_0-t)^{1/4})}{\Gamma(-\alpha_s(x_+))\Gamma(-\alpha_s(x_-))}} \quad (5)$$

where the cut of $(z_0 - z)^{1/4}$ is chosen from z_0 to $+\infty$ in such a way that $(z_0 - z)^{1/4}$ is real for z real, less than z_0 , $\psi(x)$ is the function defined in Eq. (1), x_+ and x_- mean "just above the cut" and "just below the cut", respectively. We recall that on the cut $\alpha(x_+) = (\alpha(x_-))^*$. $D_2(t, s)$ is defined analogously to $D_1(s, t)$. These functions are analyzed in detail in the Appendix, Sections a) and b).

In order to consider the general case of a scattering amplitude depending on the three Mandelstam variables s , t and u , $A(s, u, t)$, we note that $F(s, t)$ goes to zero (faster than any power) when $s \rightarrow \infty$ at fixed u . We have to modify the definition of $D_1(s, t)$ by taking the discontinuity of $A(s, u, t)$ across the right-hand side cut in s from s_0 to $+\infty$. This cut is picked out at some value of t , when the s and u cuts are disconnected, and then it is assumed that an analytic continuation in t is possible to define $D_1(s, t)$ everywhere. Hence $D_1(s, t)$ is given by

$$D_2(s, t) = \frac{1}{2\pi i} \int_{s_0}^{+\infty} \frac{dx}{x-s} \left[A(x_+, u(x_+, t), t) - A(x_-, u(x_-, t), t) \right] e^{-\frac{2(\psi(x) + (s_0-s)^{1/4} + (t_0-t)^{1/4})}{\Gamma(-\alpha_s(x_+))\Gamma(-\alpha_s(x_-))}} \quad (6)$$

We notice that also $D_1(s, t)$ goes to zero faster than any power for $s \rightarrow \infty$ at fixed u . $D_1(s, t)$ has no poles in t nor in u , because, following the hypothesis iv) and its generalization to the case of the three channels s, t, u , the poles in u or t have residues which are polynomials in s . Hence $[A(x_+, u(x_+, t), t) - A(x_-, u(x_-, t), t)]$ has no poles in t nor in u .

We define similarly $D_2(t,s)$, $D_3(u,t)$. The decomposition is now as follows, in analogy with Eqs. (2) and (3) :

$$\begin{aligned}
 A(s, u, t) &= A_1(s, u, t) + A_2(s, u, t) + A_3(s, u, t) = \\
 &= \left[A(s, u, t) - D_1(s, t) - D_3(u, t) - \{A(s, u, t) - D_1(s, t) - D_2(t, s) - D_3(u, t)\} \cdot \right. \\
 &\quad \left. \times (F(s, t) + F(u, t)) \right] + \\
 &+ \left[D_1(s, t) + \{A(s, u, t) - D_1(s, t) - D_2(t, s) - D_3(u, t)\} F(s, t) \right] + \quad (7) \\
 &+ \left[D_3(u, t) + \{A(s, u, t) - D_1(s, t) - D_2(t, s) - D_3(u, t)\} F(u, t) \right]
 \end{aligned}$$

where A_1 , A_2 and A_3 are defined to be the first, the second and the third quantity put in square brackets in the last term of the identity. $A_1(s, u, t)$ has poles in t with the residue of $A(s, u, t)$, it has no resonances in s or in u . It has the behaviour of $A(s, u, t)$ for $s \rightarrow \infty$ at fixed t , that is, by hypothesis, the Regge behaviour

$$\beta \frac{(-s)^{\alpha(t)}}{\sin \pi \alpha(t)} + \gamma \frac{(-u)^{\alpha(t)}}{\sin \pi \alpha(t)}$$

because $F(s, t), F(u, t) \rightarrow 0$. It goes to zero faster than any power for $s \rightarrow \infty$, $t \rightarrow \infty$ at fixed u , because $F(s, t) \rightarrow 0$, $F(u, t) \rightarrow 1$, and for $u \rightarrow \infty$, $t \rightarrow \infty$ at fixed s , because $F(s, t) \rightarrow 1$, $F(u, t) \rightarrow 0$. A_2 and A_3 have the properties obtained by cyclic permutation $t \rightarrow s \rightarrow u$.

In conclusion, we have shown the validity of the generalized interference models under hypotheses which seem to be fairly general. Of course if one wants to make a practical application, one has to construct models for $A(s, u, t)$, which in our reasoning is any given amplitude satisfying some general conditions. We have shown in particular that without loss of generality, models of the interference type can be used as a starting point for successive approximation of the scattering amplitude $A(s, u, t)$.

A P P E N D I X

- a) We examine first the properties of $F(s, t)$. The cut of $(z_0 - z)^{1/r}$ is chosen from z_0 to $+\infty$, $(z_0 - z)^{1/r}$ is real for z real, less than z_0 , so that if $\arg(s_0 - s) = \varphi - \pi$, $\arg(t_0 - t) = \psi - \pi$ with $0 \leq \varphi \leq 2\pi$, $0 \leq \psi \leq 2\pi$

$$|\arg(s_0 - s)^{1/4}| \leq \frac{\pi}{4} \quad \left| \arg \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} \right| < \frac{3}{16} \pi.$$

Then

$$\operatorname{Re}(s_0 - s)^{1/4} > 0, \quad \operatorname{Re} \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} > 0.$$

When $s \rightarrow \infty$ at fixed t

$$\exp \left\{ - \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} \right\} \rightarrow 1,$$

and $F(s, t) \rightarrow e^{-(s_0 - s)^{1/4}} \rightarrow 0$.

When $t \rightarrow \infty$ at fixed s

$$\exp \left\{ - \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} \right\} \rightarrow 0$$

and

$$F(s, t) \rightarrow 1 - (s_0 - s)^{1/4} \exp \left\{ - \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} \right\} \rightarrow 1 + 0.$$

When $s, t \rightarrow \infty$ at fixed s/t , in particular at fixed u ,

$$\exp \left\{ - \frac{(t_0 - t)^{1/6}}{(s_0 - s)^{1/8}} \right\} \rightarrow 1$$

and $F(s, t) \rightarrow e^{-(s_0 - s)^{1/4}} \rightarrow 0$. Of course there are infinitely many examples of functions of this type. $(s_0 - s)^{1/4}$ may well be $(s_0 - s)^{1/r}$, $r > 2$, and so on.

b) We examine the properties of $D_1(s, t)$. On the cut $\alpha(x_+) = (\alpha(x_-))^*$ and

$$e^{-\frac{1}{\Gamma(-\alpha(x_+))\Gamma(-\alpha(x_-))}} = e^{-\frac{1}{|\Gamma(-\alpha(x_+))|^2}}$$

which goes to zero for $x \rightarrow \infty$ as $\exp[-|\exp \alpha \log \alpha|^2]$.

The integral defining $D_1(s, t)$ is always convergent. It has no poles in t because $(A(x_+, t) - A(x_-, t))$ has no poles in t due to the hypothesis iii). If we go down in the second sheet with s to reach the point where $\alpha(s) = n$, we have to deform the integration path in x in order to pick out the circle around the pole at $x = s$ of the integrand and we get

$$D_1(s, t) = \frac{1}{2\pi i} \int_{s_0}^{+\infty} \frac{dx}{x-s} [A(x_+, t) - A(x_-, t)] e^{-\frac{2(\psi(x) + (s_0-s)^{1/4} + (t_0-t)^{1/4})}{|\Gamma(-\alpha(x_+))|^2}} \\ + [A(s_+, t) - A(s_-, t)] e^{-\frac{2(\psi(s) + (s_0-s)^{1/4} + (t_0-t)^{1/4})}{\Gamma(-\alpha(s_+))\Gamma(-\alpha(s_-))}}$$

$A(s_+, t)$ has a pole, it is equal to the continuation in the second sheet of $A(s, t)$, $A(s_-, t)$ does not have poles because it is $A(s, t)$ on the first sheet. On the pole $(1/\Gamma(-\alpha(s_+))) = 0$, and the exponential becomes equal to 1. Then at the pole in the second sheet $D_1(s, t) \approx A(s, t)$.

If we let $s \rightarrow \infty$ in the first sheet, $e^{-(s_0-s)^{1/4}/\Gamma\Gamma}$ in the integrand goes to zero faster than any power, because

$$\frac{1}{\Gamma(-\alpha(x_+))\Gamma(-\alpha(x_-))} > 0$$

and it is an even growing function of x . We can take $\psi(x) > 1$, and the rest of the integrand

$$\Delta_s A(x, t) e^{-\frac{(s_0-s)^{1/4} + 2\psi(x) + 2(t_0-t)^{1/4}}{\Gamma\Gamma}}$$

is such that the integral converges. $D_1(s, t)$ goes then to zero for $s \rightarrow \infty$ faster than any power, as can be seen by majorizing the modulus of the integrand, if s is not on the real positive axis. [If s is on the positive real axis the reasoning is more delicate. All we need, however, is to find a deformed integration path along which $\operatorname{Re}(1/\Gamma\Gamma) > \operatorname{Im}(1/\Gamma\Gamma)$ and such that along it

$$\left| \frac{1}{x-s} \Delta_s A e^{-\frac{2\psi(x)}{\Gamma\Gamma}} \right|$$

is again an integrable function.]

When $t \rightarrow \infty$, for x sufficiently large

$$e^{-\frac{\psi(x)}{\Gamma\Gamma}} < \psi(x)$$

and

$$\left| \Delta_s A(x, t) e^{-\frac{\psi(x)}{\Gamma\Gamma}} e^{-2\frac{(t_0-t)^{1/4}}{\Gamma\Gamma}} \right| < \left| e^{(\alpha(x)+1)\frac{1}{2}t - \frac{2(t_0-t)^{1/4}}{\Gamma\Gamma}} \right| < e^{-\frac{(t_0-t)^{1/4}}{\Gamma\Gamma}}$$

for $|t| > \bar{t}$ independently of x . $e^{-(t_0-t)^{1/4}/\Gamma\Gamma}$ goes to zero faster than any power, and the rest of the integrand

$$e^{-\frac{\psi(x)}{\Gamma\Gamma} - 2\frac{(s_0-s)^{1/4}}{\Gamma\Gamma}}$$

makes the integral convergent.

When $t, s \rightarrow \infty$ at fixed u , we can repeat the reasoning we have made for the case of $t \rightarrow \infty$ at fixed s , and we reach the conclusion that $D_1(s, t)$ goes to zero faster than any power.

We notice that the hypothesis that $\alpha(s)$ is rising with s is not essential, because in order to improve the convergence of the integral defining $D_1(s, t)$ we can replace

$$\exp \left[- \frac{1}{\Gamma(-\alpha(x_+))\Gamma(-\alpha(x_-))} \right]$$

by $\exp[-(f(x)/\Gamma\Gamma)]$, where $f(x)$ is a positive rising function, without modifying the residue at the pole.

c) Finally, if there is a bound state, say at $s=s_1 < s_0$, the residue of the pole being the polynomial in t $\Pi(t)$, we can consider the function

$$g(s,t) = e^{-(s_0-s)^{1/4}/(s_0-s_1)^{1/4}} \frac{\Pi(t)}{s-s_1}$$

which goes to zero faster than any power for $s \rightarrow \infty$, it has a pole with residue $\Pi(t)$. We decompose for instance $A(s,t)$ in the following way :

$$\begin{aligned} A(s,t) &= A_1(s,t) + A_2(s,t) = \\ &= \left[A(s,t) - D_1(s,t) - g(s,t) - \left\{ A(s,t) - D_1(s,t) - D_2(t,s) - g(s,t) \right\} F(s,t) \right] + \\ &+ \left[D_1(s,t) + g(s,t) + \left\{ A(s,t) - D_1(s,t) - D_2(s,t) - g(s,t) \right\} F(s,t) \right] \end{aligned}$$

A_1 and A_2 again satisfy the requirements specified below Eq. (2).

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R E F E R E N C E S

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