ENERGY LEVELS AND WAVE FUNCTIONS OF PIONIC ATOMS

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A B S T R A C T

Present data on strong interaction shifts and widths in pionic atoms are analyzed in terms of a non-local effective potential with five parameters. Realistic mass and charge distributions are used and strong interaction effects on the vacuum polarization are included. The dependence of predictions on parameters is discussed; properties of the wave functions are presented for different assumptions of the interaction. The effective potential predicts a change of the attractive p wave interaction to a repulsion in the region $Z \approx 36$. It also suggests important isospin effects in the gradient interaction for very heavy nuclei. The Born approximation of the real isovector part of the potential is in agreement with soft pion theory; this term, as well as the entire real potential, are also close to multiple scattering theory. The analysis suggests that the isoscalar πN amplitude is $-0.01\mu^{-1} \lesssim a_N^{(+)} \lesssim 0$. The absorptive potential is about a factor of two stronger than predicted.

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1. INTRODUCTION

Recently, precision experiments on the strong interaction shifts and level widths of the 1s and 2p orbits of π mesic atoms have become available. Similar, although less detailed, information begins also to be available for states of higher angular momentum, in particular for the 3d and 4f states ¹⁾⁻⁹⁾. It has previously been shown theoretically by M. Ericson ^{10),11)} and in more detail by Ericson-Ericson ^{12),13)} that the strong interaction of pions with nuclei at low energies is closely related to the πN interaction. In fact, it can be made quite plausible that nucleons, bound in the nucleus, scatter low energy pions very similarly to free nucleons, so that the nuclear pion interaction can be described by a multiple scattering theory. Although Refs. 13) and 14) develop such a theory in considerable generality its practical application has so far been based on a pseudopotential approach to s and p wave πN scattering motivated by the short interaction range. The corresponding effective π nuclear interaction is then expressible in terms of a velocity dependent effective potential which depends on πN s and p wave scattering lengths and on the nuclear mass density. Absorptive effects can also be included in a similar way assuming the absorption occurs on two closely correlated nucleons. In this case the relevant density is two-nucleon density. This theory has successfully described the major features of the early experiments on π mesic atoms, notably the change of the repulsive interaction in the 1s state into attraction for higher angular momentum states, the large isospin effects as well as the magnitude of absorption. Although the previous analysis based on incomplete and rather crude data seemed to produce a satisfactory over-all picture it was for technical reasons based on a very oversimplified picture of the nuclear matter distribution (square well), as well as on perturbative expansions. While these approximations can be partly justified, it is certainly necessary to analyze present precision experiments much more accurately since in many cases the pion wave function is strongly distorted by the nucleus. A major obstacle in the analysis is the high accuracy needed in the evaluation of the eigenvalues of the levels of the mesic atom.

For example, the atomic wave functions are changed so much by the strong interaction that even the long range vacuum polarization deviates from the value for a point nucleus. Thus an improper treatment of such a basically minor effect might invalidate a comparison between theory and experiment. We have therefore developed effective numerical methods suitable for dealing with bound state problems with absorptive interactions as has been described elsewhere ¹⁵⁾, and these are applied in the following.

The aim of the present paper is to analyze present data on π mesic atoms in detail so as to obtain good and up to date empirical values for the interaction parameters of the effective potential from experiments. It is, further, to discuss critically the effects in eigenvalues and wave functions produced by changes of these parameters, as well as to compare the empirical parameters to those predicted. Finally, we explore the predictions of our parameters in situations for which less accurate or no experimental information is available, so as to search for indications of new phenomena. In order to be specific and to keep parameters to a minimum we have deliberately aimed at a simple description of the interaction. The form of the effective potential is described in Section 2 as well as our approach to finite size charge effects and vacuum polarization. Section 3 contains a comparison of predicted and experimental data. Section 4 discusses these data in detail for the various mesonic orbits, studies the variation of wave functions and compares parameters to the multiple scattering theory.

2. THE EFFECTIVE π NUCLEAR INTERACTION POTENTIAL

The π nuclear interaction responsible for strong effects in π mesic atoms is assumed to be an effective non-local potential V as suggested by multiple scattering theory [for example, 13), 14)]. The energy levels of the atom are then given by the bound state eigenvalues of the Klein-Gordon equation ($\hbar = c = 1$)

$$\left\{ \vec{\nabla}^2 + [(\epsilon - V_c)^2 - \bar{\mu}^2] \right\} \psi = 2\bar{\mu} V \psi ; \quad (1)$$

where V_c is the Coulomb potential for the extended nuclear charge, E is the pion binding energy including the pion rest mass μ , $\bar{\mu} = \mu(1 + \frac{\mu}{M})^{-1}$ is the reduced pion mass with M the nuclear mass.

Guided by multiple scattering theory for elastic π nuclear interactions ^{13), 16)} we make the following simple ansatz for the potential:

$$2\bar{\mu} V = q(\vec{r}) + (\vec{\nabla} \alpha(\vec{r}) \vec{\nabla}) ; \quad (2)$$

where $q(\vec{r})$ and $\alpha(\vec{r})$ are so far arbitrary functions of $\vec{r}$. It should be noted that the observed repulsive interaction in the 1s states of π mesic atoms changing into attraction in higher angular momentum states makes use of a smooth purely local interaction potential $q(\vec{r})$ extremely awkward ¹⁴⁾, quite apart from the theoretical reasons ^{13), 16)} for a potential of the above type. The potential also has to provide for absorption ^{11), 17)} and therefore will be taken to be complex. By multiple scattering theory $q(\vec{r})$ and $\alpha(\vec{r})$ can be related to πN scattering lengths and π production amplitudes with explicit expressions for these quantities given ¹³⁾. Guided by this description we will take $q(\vec{r})$ to depend linearly on the neutron and proton mass distribution; since absorption is associated with two-nucleon processes we take it proportional to the squared mass density. A similar procedure is followed for $\alpha(\vec{r})$; we want however, to provide for the possibility of a non-linear mass dependence produced by the effective field characteristic of dipole scattering (the Lorentz-Lorenz effect in dielectrics) for which an additional parameter f is introduced.

The following effective potential results:

$$q(\vec{r}) = -4\pi [p_1 b_0 \rho(\vec{r}) + p_1 b_1 (\rho_n(\vec{r}) - \rho_p(\vec{r})) + i p_2 \text{Im} B_0 \rho^2(\vec{r})]; \quad (3a)$$

$$\alpha(\vec{r}) = \alpha_0(\vec{r}) \left[1 + \xi \frac{\alpha_0(\vec{r})}{3} \right]^{-1}; \quad (3b)$$

$$\alpha_0(\vec{r}) = 4\pi [p_1^{-1} c_0 \rho(\vec{r}) + i p_2^{-1} \text{Im} C_0 \rho^2(\vec{r})]. \quad (3c) \quad *)$$

Here b_0 , b_1 , $\text{Im} B_0$, c_0 and $\text{Im} C_0$ are real constants associated with the elementary interaction (see below) and $p_1 = (1 + \mu/m)$ and $p_2 = (1 + \mu/2m)$ where m is the nucleon mass, are kinematical constants; $\rho(\vec{r}) = \rho_n(\vec{r}) + \rho_p(\vec{r})$, $\rho_n(\vec{r})$, and $\rho_p(\vec{r})$ are the average nuclear mass density, the neutron and the proton mass densities respectively. The correlation parameter ξ in Eq. (3b) parametrizes the polarization of the nuclear medium: for a nuclear medium of completely smoothly distributed protons and neutrons we have $\xi = 0$; the granular structure of nuclear matter leads to $\xi \neq 0$ and in particular very short range anti-correlations between nucleons give $\xi = 1$. This is due to the "effective dipole field" at a scatterer which differs from the average field in the medium in analogy to dielectrics.

The form (3) above seems to have sufficient flexibility to parametrize present experiments on π mesic atoms in terms of a small number of constants. Its form is however oversimplified as compared to multiple scattering theory. A number of effects which are quantitatively predicted to be present have been left out or been simplified. Among these are the following:

- a) the effects of nuclear correlations in the local part of the potential are not made explicit although they give the dominant contribution ¹³⁾;

*) Note that this effective potential differs from that in Ref. 13) mainly by a factor $(1 + \mu/m)^{-2}$ which is a small binding correction of kinematical origin not included previously.

- b) all explicit dependence on neutron and proton density ^{*)} is concentrated in the term b_1 . The dependence of other terms on these are expected to be minor ¹³⁾. Thus, there is no dependence on these densities nor in the gradient terms nor in the absorptive terms. The potential is taken to be diagonal in isospin so that nor real nor virtual charge exchange effects by the potential can occur (this is a good approximation for the elastic scattering);
- c) the dispersive effects produced by absorptive processes (i.e., $\text{Re } B_0$ and $\text{Re } C_0$) are not explicit;
- d) the strong interaction hyperfine effects are neglected;
- e) the two-nucleon density in absorptive terms is approximated by the square of the single nucleon distribution.

As a consequence of these approximations the parameters used in (3) are to be regarded as effective parameters, which should be kept in mind in the later comparison with predictions of the multiple scattering theory. This is in particular true of the local parameter b_0 .

The effective interaction potential $V(\vec{r})$ depends on the mass densities $\rho(\vec{r})$, $\rho_n(\vec{r})$ and $\rho_p(\vec{r})$. For present purposes it is sufficient to represent these by the two-parameter Fermi-distribution extensively used in the analysis of muonic atoms:

$$\rho^F(r) = N \left(1 + \exp[4 \ln 3 (r - c)/t] \right)^{-1}.$$

Here the normalization constant N is chosen so that the volume integral over $\rho^F(r)$ is the total number of particles (neutrons,

*) This term is given in the form $b_1(\rho_n(\vec{r}) - \rho_p(\vec{r}))$ so as to emphasize that π mesic atoms are sensitive to differences in proton and neutron densities.

protons) ^{*)}. The radial parameter c and the surface thickness parameter t have conventional meanings. The proton distribution is related to the nuclear charge distribution by the conventional correction for the proton size:

$$\langle r^2 \rangle_{p-distr.} = \langle r^2 \rangle_{charge} - \langle r^2 \rangle_p .$$

In the following analysis we will usually keep the shape of the neutron and proton distribution identical, although significant differences are expected in elements of large neutron excess. The reason for this, as well as for most other approximations, is to keep the number of parameters to a minimum. Since our aim is to determine the principal parameters for nuclei in general, we leave this detailed analysis of individual elements to future studies.

The effect of the extended nuclear charge is included in the static Coulomb interaction $V_c(r)$ in the usual way. The effect of vacuum polarization is not included in Eq. (1). It is taken into account as follows in analogy with the procedure for muonic atoms, Refs. 18), 19). To first order, vacuum polarization is described by a long range potential $V_p(\vec{r})$ which for $r < \lambda_e$, the electron Compton wavelength, is given by ^{20), 21)}:

$$V_p(\vec{r}) = - \frac{2\alpha e^2}{3\pi} \int d\vec{r}'^3 \rho_{ch}(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|} \left(\ln \frac{\lambda_e}{c|\vec{r}-\vec{r}'|} - \frac{5}{6} \right) ; \quad (4)$$

with $c = 1.781$. Its contribution to the energy E_n of a bound state n is approximately

$$E_n = E_{vac} = \int d\vec{r}^3 |\psi_n|^2 V_p ;$$

where $\psi_n(\vec{r})$ is the normalized pionic wave function in the presence of the strong interaction. The condition for the use of Eq. (4) is

^{*)} We use the exact normalization:

$$N = (Z/4\pi) \cdot \left\{ \frac{c}{3} \left[\left(\frac{\pi t}{4 \ln 3} \right)^2 + c^2 \right] - 2 \left(\frac{t}{4 \ln 3} \right)^3 \cdot \sum_{n=1}^{\infty} n^{-3} \left[-\exp(4 \ln 3 \frac{c}{t}) \right]^{-n} \right\}^{-1} .$$

essentially $n^2/Z < \frac{1}{2}$, where n is the principal quantum number, so that Eq. (4) is invalid for very light elements or high n .

Since the evaluation of vacuum polarization is very time-consuming in numerical studies we explicitly list the change of the vacuum polarization shift due to the strong interaction in Table V. Since the contributions come nearly entirely from outside the interaction region, the strong interaction effects in the vacuum polarization are nearly model independent. While the vacuum polarization contribution is fairly large, the change in vacuum polarization, due to strong effects, is typically 1-2% of the strong interaction level shifts in all cases considered here. Presently, this effect is thus still outside experimental precision and the point nuclear values can be used.

The effect of higher order vacuum polarization is approximately a shift $\Delta E_n \approx 0.04 \alpha^3 Z^2 E_n$, where E_n is the full binding energy of the state n ²²⁾. This is negligible in all cases considered here.

3. EIGENVALUES AND WAVE FUNCTIONS

A. Transformation of effective potential for spherical field

We have, in general, limited our discussion to nuclei of spherical mass distributions or to those nuclei for which deviations from sphericity produce effects inside of present experimental uncertainties. Consequently we will use $q(\vec{r}) = q(r)$ and $\alpha(\vec{r}) = \alpha(r)$, so that the orbital angular momentum ℓ is a good quantum number. The non-local wave Eq. (1) can then be transformed exactly into an ordinary local potential problem by the substitution

$$\tilde{u}_\ell(r) = (1 - \alpha(r))^{1/2} \cdot r \phi_\ell(r) ;$$

for the radial wave function $\phi_\ell(r)$. The equation becomes

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + [1 - \alpha(r)]^{-1} \left\{ \frac{\alpha'(r)}{r} + \frac{\alpha''(r)}{2} - \left(\frac{\alpha'(r)}{2} \right)^2 [1 - \alpha(r)]^{-1} + \beta(r) \right\} \right] \tilde{u}_\ell = 0 ; \quad (5)$$

with

$$\beta(r) = [(\epsilon - V_c(r))^2 - \bar{\mu}^2] - q(r) .$$

From this explicit local representation it is easily seen that the variation of $\alpha(r)$ at the nuclear surface produces a spread out dipole layer. An explicit display of the dipole character of the gradient interactions with parameters typical of the π nuclear case is given in Fig. 1 (see discussion, Section 3.F). Because of this transformation the numerical solution of Eq. (5) presents no technical difficulties beyond those normally met for potential problems.

B. Numerical method of integration

We have described in detail the method of integration for the bound state eigenvalues of the complex interaction in Ref. 15).

An approximate eigenvalue is taken. The equation is integrated outwards from the origin with correct boundary conditions there out to regions very much larger than the interaction region. For the exact eigenvalue the asymptotic wave function vanishes exponentially for large r ; for an approximate eigenvalue the small admixture of the irregular exponentially increasing wave function soon dominates. These irregular contributions are made to vanish rapidly by an iteration procedure in the eigenvalue until a highly accurate solution is found. The precision is typically about $1:10^6$ in the eigenvalue for three to four iterations.

C. Experimental information

Strong interaction effects on the 1s, 2p, 3d and 4f levels of π mesic atoms have been experimentally investigated ^{1),9)}. The main contribution to the X-ray transition energies comes from the atomic transition energies for a point charge with a small additional shift $\mathcal{E}$ due to strong interactions and spread out charge distribution. It has been an unfortunate but common practice to determine a "strong interaction shift" by comparison of observed energies to the transition energies for a hypothetical nucleus with no strong interaction but with a spread out charge distribution, i.e., $\mathcal{E} = \mathcal{E}_{\text{ch}} + \mathcal{E}_{\text{strong}}$. This model dependent procedure is unfortunate since wave function distortion makes meaningless a separation of charge distribution and strong interaction effects. We will therefore always discuss the full shift $\mathcal{E}$. Although a transition shift in principle depends on the shift in both of the states involved, in practice this shift $\mathcal{E}$ is accurately described by the shift in the more deeply bound of the states between which the transition occurs. The transition broadening Γ is also dominated by the more deeply bound state to high accuracy. In the absence of unresolved fine or hyperfine structure it gives directly the absorption width due to strong interaction (the radiation width is negligible). Finally, the X-ray attenuation of a line determines the absorption width Γ for the higher state in the

transition from a knowledge of the initial population and the radiative transition probability ^{8),9),11),23)}.

There are several precision experiments on shifts and widths of the 1s and 2p orbits ¹⁾⁻⁶⁾. The results are in good agreement for over-all features. There are some discrepancies in isotopic shifts as will be discussed below. For over-all parametrization it is not of major importance which set of data is used. We have presently chosen to compare primarily with the CERN data ^{3),4),6)-9)} which cover a somewhat greater variety of nuclei than the other ones ^{1),2),5)} and which include attenuation widths. We have confined our discussion to pure isotopes with experimentally determined charge radius ²⁴⁾.

D. Parameters

For a comparison with experiments we have used eight elements with measured 2p-1s transitions from ¹⁰B to ²³Na and ten elements with measured 3d-2p transitions from ²⁷Al to ⁵⁹Co, i.e., 36 data points fulfilling our criteria. Assuming full anti-correlations between scatterers at short distance ($\xi = 1$) the following five parameters describe well these data:

$$\begin{aligned} b_0 &= -0.030 \mu^{-1} ; b_1 = -0.080 \mu^{-1} ; \text{Im } B_0 = 0.040 \mu^{-4} ; \\ c_0 &= 0.24 \mu^{-3} ; \text{Im } G_0 = 0.14 \mu^{-6} . \end{aligned} \quad (6)$$

The confrontation of the results for this set with experiments is presented in Tables I-V and is displayed in Figs. 2-5. The uniqueness of the set will be discussed below.

E. Discussion

1s state: shifts

The experimental and calculated 1s level shifts of Table I are shown in Fig. 2a on an absolute scale and in Fig. 2b on a reduced scale (essentially division by Z^3) which removes the major part of the variation with Z . To explore the sensitivity of these shifts to the parameters we first switched off the gradient interaction as well as the absorption altogether (i.e., $\text{Im} B_0 = c_0 = \text{Im} C_0 = 0$). As is seen in Fig. 2a the calculated shifts are nearly unaffected by this operation; the parameters b_0 and b_1 are to high accuracy the same as previously. This confirms the conclusion of Ericson and Ericson¹³⁾ that the 1s shift is dominated by the local interaction. On the reduced scale of Fig. 2b it is clearly seen that the shifts for isospin $T \neq 0$ nuclei deviate systematically from those with $T = 0$. The isospin shifts become particularly evident if we compare the experimental shifts of $T \neq 0$ with calculations with $b_1 = 0$, i.e., assuming the strong interaction is due to a fictitious system of equal number of neutrons and protons. The evidence for the isospin interaction has sometimes been doubted²⁵⁾; its necessity is most vividly displayed by the $T = 1$ nucleus $^{18}_0\text{O}$ which has been accurately measured.

It should be noted that some experimental conflict is still associated with isospin shifts: the relative shift in the $^{10}_\text{B}$ $^{11}_\text{B}$ system as observed by the Berkeley group²⁾ is of the same sign but only half of the CERN result³⁾ so that the results are just compatible within errors. However, the absolute transition energies differ. The corresponding μ mesic transitions measured in Berkeley give imaginary nuclear charge radii. Assuming the same calibration error as in Ref. 2) for μ and π transitions makes the data consistent. The Williamsburg group⁵⁾ found recently accurate shifts which fall close to those of the CERN group (and even closer to theory). This supports this explanation. Heavier elements like $^{19}_\text{F}$ and $^{23}_\text{Na}$ are more

insensitive to isospin effects, but they are consistent with the same value for b_1 . Beyond ^{23}Na there are no measurements of 1s shifts. Calculation of these shifts for the $T=0$ nuclei ^{32}S and ^{40}Ca shows that the predicted shifts for both elements lie on a nearly horizontal line in Fig. 2b. Therefore we predict that the shifts can be extrapolated by $\epsilon_{\text{th}}^{T=0}(1s) \approx -13 \cdot 10^{-3}$. $\cdot Z \cdot E_{\text{point}}(1s)$ for $Z > 16$, where $E_{\text{point}}(1s)$ is the point e.m. binding energy of the 1s state. In the case of neutron excess the shifts will be somewhat more repulsive. We note that the 1s energy shifts seem to be very well described by the potential model; its parameters Eq. (6) coincide closely with those previously given in Ref. 13).

Widths

The 1s widths of Table I are presented in Fig. 3a on an absolute scale and in Fig. 3b on the same reduced scale as used in Fig. 2b. Varying the potential parameters we find that the 1s widths have little dependence on gradient interaction (i.e., we can take $c_0 = \text{Im } C_0 = 0$, with only a minor change in the calculated widths).

In addition, we may try to describe the 1s widths without any real part of the potential (i.e., $b_0 = b_1 = 0$). It is then still possible to reproduce the experimental widths nearly as well as before provided the absorption strength $\text{Im } B_0$ is reduced to about one half of its previous value (see Fig. 3a). This would lead to a very considerable increase in predicted absorption widths for heavier elements like ^{32}S and ^{40}Ca , which are presently unmeasured. The assumption that the absorption strength can be chosen independent of the real potential is, however, physically quite unreasonable: it is seen in Fig. 6 that the repulsive 1s shift reduces the probability of pions inside the nucleus by a factor > 2 for ^{16}O . Since the wave function outside the nucleus is completely determined by the exact eigenvalue, we clearly see that the reduction is produced by the decreased probability at the nuclear surface, which

determines the inside probability in first approximation. Thus the result does not depend much on the detailed structure of the nuclear interaction. In other words, any nuclear interaction leading to correct $1s$ shifts reduces the probability of pions inside the nucleus as indicated in Fig. 6. It is therefore not meaningful to achieve a description of the absorption neglecting the real energy shifts, respectively, the changed probability of pions inside the nucleus.

The $1s$ widths are less precisely determined than the $1s$ shifts and they show a larger scattering with respect to calculated values. A minor anomaly in the widths of ^{10}B - ^{11}B which appears on the reduced scale in Fig. 3b for the CERN data ³⁾ is probably not significant since it is not confirmed by other recent measurements ⁵⁾.

A significant anomaly appears in the ^{16}O - ^{18}O system. It is predicted that ^{18}O has a 0.5 keV smaller width than ^{16}O while it is observed that ^{18}O has a 1.1 keV larger width. While the deviation is formally less than two standard deviations the line shapes of ^{16}O and ^{18}O can be directly compared to higher precision. The predicted smaller width for ^{18}O is mainly due to the increased repulsion in a $T=1$ nucleus and is not very model dependent. We are therefore forced to the conclusion that the addition of two neutrons to ^{16}O increases the intrinsic absorption probability for π^- by about 25%. The origin of this effect is obscure. The microscopic theory of absorption suggests the intrinsic probability to be roughly proportional to mass number times density; it therefore varies inversely with the third power of the mass radius for fixed A [Ref. 13), Eq. (B2)]. Precision muonic X-ray data indicate the ^{18}O radius to be $(3.5 \pm 0.8)\%$ larger than for ^{16}O ²⁶⁾, and electron scattering data are consistent with this result. The difficulty could be made to disappear with the very unpalatable assumption that the ^{18}O radius would be slightly smaller than that of ^{16}O . We feel that it is urgent to obtain an independent

measurement for the difference in ^{16}O - ^{18}O absorption widths, as well as a direct measurement of the mass distribution for these nuclei.

It has been suggested from experimental work that the $1s$ widths may saturate for heavier nuclei. Within the potential analysis no saturation of $1s$ widths is expected below ^{40}Ca . It is not experimentally clear whether a deviation between theory and experiments occurs for $Z > 9$, since the uncertain value for ^{23}Na differs by less than two standard deviations. It would be of great interest to obtain experimental $1s$ widths in this region, or, alternatively, the imaginary part of π -nuclear scattering lengths. A saturation effect in the widths would indicate a major missing feature in our phenomenological description of low energy π nuclear interaction. This would be surprising since, so far, not even a qualitatively satisfactory theoretical argument for such a saturation has been given.

2p state: shifts

The shifts in the $2p$ states are compared to calculated values in Fig. 4a and 4b on an absolute and a reduced scale. The purely local potential which is repulsive cannot describe these attractive shifts. The $2p$ shifts depend only weakly on the absorption interaction although stronger than the $1s$ shifts, due to the repulsive force of the absorption. It is not possible to describe the $2p$ shifts well by a real gradient interaction only ($b_0 = b_1 = \text{Im } B_0 = \text{Im } C_0 = 0$) since the calculated shifts in the heavier elements become much too attractive when normalized to light elements (see Fig. 4a): the local potential introduces a repulsive effect which lowers the calculated values to the observed ones. For more extended nuclei than those which have presently been studied, the local potential is expected to become dominant²⁷⁾. The $2p$ interaction shift would then become repulsive as for $1s$ shifts. Tentative calculations as well as extrapolations on the reduced scale in Fig. 4b

indicate that the cross-over point is around $Z = 36$, with an uncertainty of a few units. This is a charge number only somewhat higher than has been studied experimentally. This important new possibility motivates a special effort to measure $2p$ shifts for higher Z . Isospin effects are not very prominent in the $2p$ states in any very direct way, although heavier elements show influence: if the repulsive isospin strength b_1 is switched off, the $2p$ shift and width of ^{27}Al increases by only 5%. The absence of this term would however produce too attractive shifts in elements like ^{52}Cr , ^{55}Mn etc., which have larger neutron excess. This is therefore an additional argument that b_1 must have the sign and the strength we have used (the isospin term in the gradient interaction is expected to be small; see $4f$ shifts however). We note in particular that the value of Z for which the $2p$ interaction shift changes sign depends importantly on b_1 .

Widths

It is first to be noted that if there is no absorption by gradient interaction ($\text{Im } C_0 = 0$) the previously determined local absorption strength is much too weak to explain the data. We therefore conclude that $\text{Im } C_0 \neq 0$. On the other extreme it is possible to parametrize the $2p$ absorption widths alone rather well assuming all interaction but gradient absorption turned off ($b_0 = b_1 = \text{Im } B_0 = c_0 = 0$). This parametrization is unphysical since it ignores the observed $2p$ energy shifts. As compared to the best effective potential the absorption strength has, in this case, to be increased from $\text{Im } C_0 = 0.14$ to 0.19 (see Fig. 5a). The reason for this increase is - a) compensation for absorption by the local potential, - b) more importantly, compensation for the increased probability of presence of the pion in the nucleus caused by the attractive $2p$ energy shift (see Fig. 7). This effect is nearly model independent. When the attraction is turned off the widths are roughly described by the increased absorption strength. The agreement between

calculated and observed line broadenings is good, although not as good as for the 2p shifts with the general behaviour well described.

It should be remarked that the lightest element for which the width has been measured, ^{27}Al , has a surprisingly larger width, although the uncertainty is very large. This deformed nucleus has spin $5/2$; the strong interaction may in this case give rise to a substantial strong hyperfine splitting¹³⁾ which would appear as an additional line broadening. An estimate of this effect indicates that $(1/3-1/2)$ of the width may be due to this²⁸⁾.

An interesting independent source of absorption widths are the X-ray intensities which recently have become available⁷⁾⁻⁹⁾. From a knowledge of radiative transition rate and the attenuation factor due to absorption it is in principle straightforward to deduce the absorption width^{11),23)}. It is, however, necessary to know the initial population of the state which requires a knowledge of the mesonic cascade; further, the radiative transition rate has to be correctly deduced in the presence of strong interactions. Because of this, the evidence on absorption widths by this method is only indirect and may have rather important systematic errors. Presently, no direct calibration exists. This is compensated by the possibility it gives to study extremely small widths which are otherwise not measurable so that the range of available nuclei becomes much wider. With this caution we have plotted these widths in Fig. 5b. Although the trend is reasonable as well as the magnitude, there seems to be a clear discontinuity between these widths and the directly measured ones with the attenuation widths systematically smaller. A decrease of the absorption parameter $\text{Im } C_0$ by 30% from $\text{Im } C_0 = 0.14$ to $\text{Im } C_0 \approx 0.10$ would give a reasonable, although not perfect compromise fit to both sets.

We have made the analysis up to this point assuming a potential of the form (2) and (3) with $\xi = 1$. The characteristic non-linear dependence on density of this potential is due to the local effective field which in turn is due to the assumption of very short range anti-correlations between scatterers (the so-called Lorentz-Lorenz effect; see further Section 4). We will now briefly explore the changes which occur in the analysis if we assume no anti-correlations ($\xi = 0$). We have not found any clear effect which proves or disproves the effective field assumption: it is possible to accommodate $\xi = 0$ as a possibility simply by decreasing c_0 from $0.24\mu^{-3}$ to $\approx 0.20\mu^{-3}$; simultaneously it is necessary to decrease the absorption parameter $\text{Im } C_0$ from $0.14\mu^{-6}$ to $\approx 0.07\mu^{-6}$. Although we have not explored the fit with these parameters in detail, it does not appear to be noticeably different from the one for $\xi = 1$ for the 2p states.

Higher states

Once the parameters of the effective potential have been fixed from 1s and 2p experiments, it becomes relevant to ask if the same effective potential is capable of describing the pionic shifts in other angular momentum states or if new effects appear. Tables III and IV compare the potential predictions to observed shifts and widths for 3d and 4f states in selected nuclei of known charge distribution. The purely local potential ($b_0, b_1, \text{Im } B_0$) has now nearly negligible effect so that, in reality, we explore the predictions of the gradient interaction.

The 3d shifts are well described by this potential within uncertainties although there is some evidence for somewhat too low predicted values. Also the 3d widths are satisfactory within uncertainties; we note however a tendency in the three heaviest elements for the experimental widths to be smaller than predictions. For 4f states both shifts and widths are generally larger than predicted. As far as the width is concerned this

is doubtlessly due to the important unresolved hyperfine splitting as known from muonic atoms; indeed the discrepancies occur for the three strongly deformed nuclei ^{232}Th , ^{238}U and ^{239}Pu .

Although no direct evidence exists, the larger than predicted energy shifts, both in the 3d and 4f states, are consistent with the expected isospin dependence of the gradient interaction ¹³⁾; its effect in these heavy elements is not negligible because of the very large neutron excess. The increase is roughly expected to a factor $\sim 2N/A$ which is about 20%. While we believe the effect to be real we have not felt that present data justify the introduction of an additional constant to describe this effect.

The higher states therefore seem to conform qualitatively with the effective potential used. On the quantitative level it seems clear that a more accurate discussion must include isospin effects and hyperfine splittings.

F. Wave functions

We now turn to the explicit representation of the properties of the effective potential (2) and its associated wave functions in the 1s and 2p states. The real potential for ^{40}Ca with standard parameters is shown in Fig. 1 after the transformation (5) into a local potential. It is clearly seen that the gradient interaction curve (III) acts as dipole layer of finite extension at the nuclear surface. For s states which vary little over nuclear dimensions its net effect is nearly nil. The p states are linearly increasing with r and therefore have dominant contributions from the outer, attractive part of the dipole potential. The same phenomenon occurs also for higher orbital angular momentum states. The absorptive potential has a similar behaviour.

Figure 6 shows the probability density $|\phi_{1s}(r)|^2$ for a 1s pion in ^{16}O for various assumptions of potential parameters. There are two groups of curves of which one contains the real local potential and the other one does not. Since the wave function outside the nucleus is completely determined by the observed energy shift which to good accuracy is given by the real local potential only the first group is realistic. We note that the wave function is profoundly changed far outside the nucleus, which is the reason why the long range vacuum polarization shift is influenced by the strong interaction. Since the probability outside the interaction region depends only on a correct eigenvalue independent of interaction, we further note that its model independent value at the nuclear surface determines the probability distribution over the entire nucleus in a nearly model independent way. In the present case the repulsive s wave interaction reduces the probability at the origin to less than one half of its original value. It is obvious that normal perturbation theory (Born approximation) becomes very poor, but it can be very much improved using the effective wave function associated with the energy shift ¹⁰⁾. In the more detailed comparison it is amusing to note that the normally attractive gradient interaction (curve I v. VI) repels the wave from the nuclear interior when used together with a repulsive local interaction (curve IV. II). Thus, the addition of the complex gradient interaction does not necessarily increase the 1s widths.

Figure 7 represents the analogous situation for the density of the 2p state in ^{40}Ca . In contrast to the 1s case the density is now controlled by both the real local and the real non-local interaction strength (b_0 and c_0). We note by comparing the interaction with no absorption (III) to the one with absorption (II) that only minor changes are produced. The density with the full interaction strength differs strongly both from the one with only gradient interaction (V) which is much too high, and from the one with only local interaction (IV) which is much too low. This simply reflects that the interaction shift for (V) is too attractive, while the small density of (IV) reflects the repulsive interaction shift. A relatively good density is obtained with a gradient interaction chosen so as to give approximately the correct eigenvalue (III).

4. COMPARISON WITH THEORETICAL PREDICTIONS

Our main purpose in this paper is to parametrize the experiments on π mesic atoms efficiently and realistically using a few parameters only. For this we have used an approximate version of an effective non-local potential suggested by multiple scattering theory. We will now briefly confront the empirical parameters with the quantitative predictions of this theory and (for the 1s interaction) with current algebra predictions.

We first discuss the real part of the effective potential, that is the parameters b_0 , b_1 and c_0 . The empirical parameters are in excellent agreement with those previously derived^{13),29)} from less precise and much more limited data and using over-simplified nuclear matter distributions. The experimental data are well parametrized with little scattering about predicted values.

The theoretical strength parameters b_0 and b_1 for the isoscalar and isovector local potential can be explicitly expressed in terms of the neutron and proton πN scattering lengths a_n and a_p . The principal contributions are:

$$b_0 = \frac{a_n + a_p}{2} - \langle \frac{1}{r} \rangle_{\text{corr}} \cdot \frac{1}{4} [2(a_n^2 + a_p^2) + (a_n - a_p)^2] + \text{corrections} \quad (7a) \quad *)$$

$$b_1 = \frac{a_p - a_n}{2} + \text{corrections} \quad (7b) \quad *)$$

Here $\langle 1/r \rangle_{\text{corr}}$ is the expectation value of the inverse correlation distance of two identical particles in the nucleus as discussed below.

*) Apart from the term in $(a_n - a_p)^2$ in Eq. (7a) (which arises from the previously neglected virtual charge exchange) the expressions above are except from notation identical to those of Eqs. (42) and (47) of Ref. 13).

These multiple scattering expressions can be easily understood. For a short range interaction with a scattering length small compared to internucleon spacing, the normal situation is that the potential strength is dominated by the coherent isoscalar and isovector πN scattering length. The role of the potential is simply to iterate the πN scattering property. In this approximation the nucleus behaves like a uniform medium. For πN interactions the isoscalar term is abnormally small as is well known from soft pion theorems. Higher contributions become then important; in particular it becomes crucial that the nucleus is a granular structure of nucleons. The polarization of the nuclear medium is now important; it is described by the nuclear pair correlations. The uniform medium does not correctly describe rescattering by two nucleons which are so close that their density distribution differs from the uniform one. The leading correction to potential (7) has the formal structure of a double scattering term. The inverse distance between scatterers $\langle 1/r \rangle_{\text{corr}}$, typical of double scattering, is however in this case not associated with nuclear dimensions, but with the correlation distance in the medium. For s wave scattering the main contributions come from the Pauli correlations which have the longest range. We evaluate it for Fermi gas of characteristic momentum p_F . The expectation value $\langle 1/r \rangle_{\text{corr}}$ for the correlation distance using the Pauli correlations for two identical particles of parallel spin in the nucleus is then

$$\langle \frac{1}{r} \rangle_{\text{corr}} = \frac{3}{2\pi} p_F \quad *)$$

It is a characteristic feature of the correlation term that it is always negative (repulsive). Out of the experimental $b_0 = -0.030 \mu^{-1}$ this dominant term contributes $-0.020 \mu^{-1}$ and it is nearly model independent. The remaining part of b_0 , $-0.010 \mu^{-1}$, should be due to the correction term of Eq. (7a) and to the coherent neutron and proton scattering lengths. The main correction comes from the dispersive effect of nuclear pion absorption. No satisfactory evaluation of this term exists. There are however good reasons to expect its contribution to be repulsive (negative) and

*) Although we will here use $g_F = \text{const} \approx 250 \text{ MeV/c}$, in reality $\langle 1/r \rangle_{\text{corr}}$ varies like $g(r)^{1/3}$. This weak dependence on r is neglected.

of order Γ/ε ¹⁷⁾. If we accept this very crude estimate, the dispersive correction is $\approx -0.005\mu^{-1}$ to $-0.010\mu^{-1}$. A small but well-defined correction of $+0.004\mu^{-1}$ is produced by the nucleon Fermi motion in the nucleus, which is attractive (indirect p wave πN interaction). We can from these considerations conclude that the coherent average $a_N^{(+)} = (a_n + a_p)/2$ for πN scattering lengths according to π mesic atoms should be in the range $-0.010\mu^{-1} \lesssim a_N^{(+)} \lesssim 0$. The direct determination of $a_N^{(+)}$ from πN scattering is controversial: the theoretical analyses of πN scattering experiments give values ranging from $a_N^{(+)} = -0.012\mu^{-1}$ ³⁰⁾ to $a_N^{(+)} = +0.023\mu^{-1}$ ³¹⁾. The reason for the differences is presumably that $a_N^{(+)}$ is constructed as a difference between large numbers ^{*)}. Finite mass corrections to soft pion theorems support a negative $a_N^{(+)}$ of the size we deduce in the absence of the so-called δ term ^{34), 35)}. A rather large positive $a_N^{(+)}$ as advocated by Lovelace ³¹⁾ can only, with great difficulty, be reconciled with present data on π mesic atoms.

The situation for the isovector interaction strength is much simpler. The correction terms from correlation and absorption are very small ($\approx -10\%$), since no cancellations occur. The deduced $b_1 = 0.080\mu^{-1}$ is then in excellent agreement with $(a_p - a_n)/2 = 0.087\mu^{-1}$ ³¹⁾ or 0.095 ± 0.005 ³⁶⁾ as deduced from experiments on πN interactions.

This result for the isovector strength can be linked to current algebra predictions in a very interesting way ³⁷⁾. M. Ericson et al. have shown that in this case the transition from soft pions to real pions in an extended system like the nucleus implies that the Born approximation of the interaction (not the amplitude) should be compared to the soft pion predictions. The application of this

*) Even a crude measurement of the πD scattering length would give $a_N^{(+)}$ with far better accuracy directly from experiments, Refs. 32), 33).

to our potential with the empirical value for b_1 leads to excellent agreement with predictions within 10%. The same idea has been generalized to the isoscalar amplitude ³⁸⁾. It can then be shown that the leading soft pion term has main contributions which essentially are those of the Born term for the multiple scattering potential. In the absence of $\bar{O}$ terms, the coherent sum of scattering lengths would be small, so that the double scattering correlation term dominates.

After this discussion of the local parameters we turn to the gradient interaction which can be related to the p wave πN scattering lengths. An elementary derivation of the principal multiple scattering contributions in the absence of absorption can be made as follows [a much more stringent and general discussion which includes higher corrections is given in Ref. 13)]: assume a nucleus of equal number of neutrons and protons with equal probability for spin up and spin down. The pion field induces dipole scattering from the nucleons proportional to the local gradient at a scatterer (the effective field $\vec{E}_1$) and the πN p wave scattering length. The induced dipole strength $\vec{p}_k$ of the k^{th} nucleon is thus

$$\vec{p}_k = 3 a_k^{\ell=1} \vec{E}_{1k}$$

where $a_k^{\ell=1}$ is the p wave scattering length. We now define the average dipole strength density $\vec{p}(\vec{r}) = -\frac{\alpha(\vec{r})}{4\pi} \vec{\nabla} \phi$. When the exciting effective field is independent of the particle $\underline{k}$ and characteristic of the medium only, this dipole strength density is proportional to the coherent average of p wave scattering lengths for neutrons and protons of both spins

$$\begin{aligned} \vec{p}(\vec{r}) &= 3 \frac{a_{n\uparrow}^{\ell=1} + a_{n\downarrow}^{\ell=1} + a_{p\uparrow}^{\ell=1} + a_{p\downarrow}^{\ell=1}}{4} \rho(\vec{r}) \cdot \vec{E}_1(\vec{r}) \\ &= \frac{4\alpha_{33} + 2\alpha_{13} + 2\alpha_{31} + \alpha_{11}}{3} \rho(\vec{r}) \cdot \vec{E}_1(\vec{r}) \\ &= c_0 \rho(\vec{r}) \vec{E}_1(\vec{r}) . \end{aligned}$$

For a dilute medium the local field strength at a scatterer does not differ appreciably from the average field strength in the medium so that $\vec{E}_1(\vec{r}) = \vec{\nabla}\phi$. In this limit we have

$$\vec{p}_{dilute}(\vec{r}) \approx c_0 \rho(\vec{r}) \vec{\nabla}\phi$$

with

$$c_0 = \frac{4\alpha_{33} + 2\alpha_{13} + 2\alpha_{31} + \alpha_{11}}{3}.$$

Hence $\alpha_0(\vec{r}) = -4\pi c_0 \rho(\vec{r})$. For non-dilute systems the difference between the local and average field strength is important as is well known from the theory of the refractive index. In the extreme case of a uniform medium of discrete scatterers, the effective dipole field strength can be calculated simply by considering the scatterer to be placed in the centre of a very small spherical hole cut in the medium. An elementary calculation which takes the induced dipole charge on the walls of the cavity into account gives in the limit of vanishing radius:

$$\vec{E}_1(\vec{r}) = \left(1 + \frac{4\pi}{3} c_0 \rho(\vec{r})\right)^{-1} \vec{\nabla}\phi.$$

By substitution of this expression we see that

$$\alpha(\vec{r}) = \alpha_0(\vec{r}) \left[1 - \frac{\alpha_0(\vec{r})}{3}\right]^{-1};$$

with $\alpha_0(\vec{r}) = -4\pi c_0 \rho(\vec{r})$ ^{*)}. It is possible to extend this

*) It is very interesting to compare this result to the classical Lorentz-Lorenz formula. For pure dipole interaction in an infinite uniform medium, we determine the refractive index from $K = nk$. From the equation $K^2 = (1 + \alpha)^{-1} k^2$ we find, using the above result

$$3 \frac{n^2 - 1}{n^2 + 2} = -4\pi \rho c_0.$$

The classical Lorentz-Lorenz formula for a dense dielectric is

$$3 \frac{n^2 - 1}{n^2 + 2} = 4\pi N\alpha; \quad [N = \text{density}; \alpha = \text{dipole strength}]$$

The fundamental difference between the two expressions is due to the time-dependence and the vector nature of the e.m. field as opposed to the static (pseudo) scalar pion field.

expression to include absorption on nuclear pairs. A more detailed microscopic derivation relates the effective field to nuclear pair correlations¹³⁾ and the above term results in the limit when these have vanishing range. Actual pair correlations have finite range and the effective field effects occur jointly for s and p wave πN interactions. The general expressions for the solution are then expressed in terms of coupled integral equations which up to now have not been explicitly solved for a general case. The expectations are that the correlation range corrections are unimportant for the real π nuclear interaction. For the absorptive interactions the possibility of significant changes in the effective field is a distinct possibility. The deduced value for $c_0 = 0.24$ is in good (but not perfect) agreement with the predicted value $c_0 = 0.21$ with corrections of -10%. The exploration of the deduced value for c_0 in the absence of the effective field effect in the gradient interaction gave $c_0 \approx 0.20$. While this is in even better agreement with predictions, the small difference of only 20% in the two values for c_0 prevents us from arguing both for and against the existence of the effective field.

The multiple scattering theory of Ref. 13) describes pion absorption in the nucleus as taking place on nucleonic pairs at short internucleon distance. In practice the dominant contributions seem to come from pairs of deuteronic quantum numbers. For a short-ranged absorption interaction the s wave absorption rate in the deuteron is proportional to $\beta_{11} |\phi_D(0)|^2$ where β_{11} is a phenomenological constant and $|\phi_D(0)|^2$ the density of nucleons at the origin of the deuteron. On the assumption that the short range behaviour of the absorption is the same for deuteronic pairs in the nucleus as for the deuteron, the local absorption rate per unit volume in the nucleus can be obtained as follows: suppose that the nuclear neutrons and protons have independent density distributions (but for the very short range absorption region already included by the phenomenological constant) and that all spin orientations are equally probable. The local density of deuteron pairs per unit volume is the neutron density $\rho_n(r)$ multiplied by the proton

density $\rho_p(r)$. However, only $3/4$ of the total number of pairs have the deuteron spin 1. Hence the absorption rate per unit volume is proportional to $\beta_{11} \cdot (3/4) \rho_n(r) \rho_p(r) \approx \beta_{11} \cdot (3/16) \rho^2(r)$. But, for minor kinematical corrections this is indeed the expression for pure deuteron absorption given by Eq. (42) of Ref. 13) for the absorption by the local part of the π nuclear potential. A very similar argument can be given for p wave, i.e., gradient absorption. To these absorption rates should be added the smaller absorption rate produced by a pair of nucleons in a relative singlet state.

The numerical predictions for the absorption rates have changed from those given in Ref. 13), mainly due to new experimental information. In particular, the accurate measurement by Rose of the cross-section for $\pi^+ + D \rightarrow p + p$ in the few MeV region ³⁹⁾ has changed the long accepted constant α in the low energy cross-section for the inverse process $p + p \rightarrow \pi^+ + D$:

$$\sigma(p + p \rightarrow \pi^+ + D) = \alpha \eta$$

where η is the pion momentum in units $\mu = 1$. The old value of $\alpha = 0.138$ mb is now $\alpha_{\text{Rose}} = 0.240$ mb, i.e., nearly twice as large. In addition, the deuteron wave function at the origin used in Ref. 13) has been ameliorated leading to a 10% enhancement of the intrinsic absorption strength ⁴⁰⁾. The deuteron strength parameter of Ref. 13) is therefore now $\beta_{11} = 0.077 \mu^{-4}$ instead of $\beta_{11} = 0.044 \mu^{-4}$. The predicted value for the absorption strength of the local interaction is therefore

$$(\text{Im } B_0)_{\text{predicted}} = 0.0174 \mu^{-4},$$

instead of $0.0116 \mu^{-4}$. The singlet contribution is about 20% only. This predicted value is more than a factor two smaller than the value $(\text{Im } B_0)_{\text{exp}} = 0.040 \mu^{-4}$ which we have deduced from the analysis of experiments on nuclei.

A similar situation occurs for the gradient absorption parameter $\text{Im } C_0$. New data on the reaction $n + p \rightarrow \pi^- + p + p$ ⁴¹⁾ show that

the production cross-section for the singlet nucleon state is nearly a factor four smaller than the previous uncertain value: this changes the p wave singlet absorption parameter from $\text{Im } \gamma_{10} = 1.4 \mu^{-6}$ to the new value $\text{Im } \gamma_{10} = 0.4 \mu^{-6}$. Because of the revised deuteron wave function the p wave deuteron strength increases slightly from $\text{Im}(\gamma_{01} + 5 \gamma_{21}) = 2.80 \mu^{-6}$ to $\text{Im}(\gamma_{01} + 5 \gamma_{21}) = 3.10 \mu^{-6}$. The new predicted value for the gradient absorption strength $\text{Im } C_0$ is thus

$$(\text{Im } C_0)_{\text{predicted}} = 0.073 \mu^{-6},$$

instead of 0.088. The singlet term represents about 12% of this term. Once more, we find that the predicted value is a factor two lower than the value we deduce from the present analysis:

$$(\text{Im } C_0)_{\text{exp}} = 0.14 \mu^{-6}.$$

We are therefore faced with the problem that both the absorption parameters are a factor two smaller than expected. We do not know the origin of this effect. It should be observed however that the gradient absorption needed to describe the data can be made consistent with predictions if we assume that there is no effective field effect (i.e., no nuclear polarization). This leaves however the problem intact for the local potential. Since the local potential is dominated by the corresponding nuclear polarization effect for s wave scattering, a consistent neglect of nuclear polarization would lead to serious discrepancies in the real part of the local potential. It would perhaps be more realistic to argue that polarization effects should mainly influence the real potential. This possibility cannot be completely ruled out, since the correlations between the nucleonic pairs and other scatterers in the medium may not be describable as non-penetrating systems.

It seems unlikely that the absorption discrepancy is due to the nuclear density distribution. A more plausible possibility is that the short range absorption operator differs for a nuclear pair as compared to the free deuteron due to different nucleon pair correlations. In our opinion the present discrepancy is quite significant. It indicates that an essential and important feature of the absorption phenomenon still is missing from the theoretical description.

5. CONCLUSION

In the preceding Sections we have analyzed a considerable body of data on energy shifts and widths in π mesic atoms. These were described in terms of a simple effective potential of non-local type. The structure of the potential is independently given by multiple scattering theory for the pions. This theory also links the shape of the potential to nuclear mass distributions and charge distributions, so that little freedom exists in the potential form. The potential strength is predicted from πN scattering and π production in NN collisions, but we have described the strength by free adjustable parameters in the previous comparison. After having made an adjustment of the five relevant strength parameters, using nuclei of known charge distribution only, we have deduced a universal set of potential parameters which describe strong interaction effects in an impressive part of the periodic system. It is indeed very striking that the effective potential parameters seem constant from nucleus to nucleus.

After the potential has been determined in this way we have explored the expected properties of the interaction both in higher angular momentum states (3d and 4f states) as well as for nuclei heavier than those which have been experimentally investigated until now. We find for the 1s state that absorption widths are expected to increase with Z at least up to ^{40}Ca . Although it has been suggested that already ^{23}Na may show signs of width saturation, the present interaction does not lead to any such expectation. An interesting effect may occur in the 2p state for nuclei only slightly heavier than those already explored: for nuclei of $Z \approx 36$ the effective interaction changes from an attractive to a repulsive one. The local repulsive interaction becomes thus very dominant. On the contrary, the 3d and 4f states are expected to remain attractive throughout the periodic system (with a sign change for 3d states just at its end), since their momentum components are larger. The observed shifts of these states are of expected magnitudes and sign. There is some circumstantial evidence from

systematic deviations for the heaviest elements of an isotopic spin dependence in the gradient interaction of the magnitude expected theoretically, but no firm conclusion is yet possible.

With the same potential we have also explored the pion wave functions and density distributions. It is striking that the pion density even inside the nucleus depends mainly on a correct energy eigenvalue. Both the absorption widths and the detailed structure of the interaction region are rather unimportant. This is the more remarkable since it is the interaction region that generates the energy shifts. The separation of the various types of potential contributions is usually clear, particularly between angular momentum states. The wave function distortion is nearly independent of absorption effects. It depends on the real interaction in a rather model independent way. This observation is of importance, since it permits quantitative calculation of distortion effects in a large number of low energy pion phenomena without invoking detailed dynamical models.

Finally, we compared the empirical parameters of the potential to predictions of multiple scattering theory and of current algebra in Section 4. We found that the dominant contribution to the isoscalar potential comes from the fluctuation scattering with virtual excitation of the nucleus: this appears as a nuclear polarization due to correlations between nucleons. From the remaining small contribution of the experimental strength we can conclude that the isoscalar πN amplitude $a_N^{(+)}$ consistent with the π mesic data is in the range $-0.010 \mu^{-1} \lesssim a_N^{(+)} \lesssim 0$. The analyses of πN scattering data are controversial for the value of this quantity. The isovector potential is in excellent agreement with the value from multiple scattering theory. M. Ericson et al.³⁷⁾ have recently shown that for the isovector amplitude the proper way to discuss soft pion result for real mass pions for an extended system is essentially to compare the Born approximation to the soft pion limit. This interesting result is strikingly confirmed experimentally by very good agreement to predictions using our isovector potential.

The gradient interaction is also well described by the multiple scattering approach. It is very unfortunate however that the very interesting local field effects which are expected in this term seem numerically to be too small to be displayed.

As a whole the multiple scattering description is quite successful as a framework for a description of experiments for the real interaction on the quantitative level; for the absorptive interaction it seems only to be qualitatively valid. It seems however that we now begin to reach a level of precision which goes beyond this theory in the simplified form in which it is presently used in the analysis, and this is particularly true of the absorption effects.

A major problem is posed by the absorption parameters, which, both for the local and gradient interaction, are twice as large as predicted. This clearly indicates that a major feature is still missing in the theoretical description on this point.

The potential which we have used here for the study of π mesic atoms is presently sufficiently well determined that it becomes relevant to use it for studies of nuclear matter distributions. For this the potential can be used in π mesic atoms as here or even for the low energy pion scattering. The approach we have used has a much wider range of applicability than pionic atoms. Clearly, as the pion energy approaches that of the (33) - resonance, our approach will break down or the parameters will be radically changed. Even before this happens new and interesting effects may occur in the scattering experiments. An outstanding feature of the present potential, as viewed from the standpoint of multiple scattering theory, is its very strong dependence on the non-uniform nucleon distribution in the nucleus: in both the local and gradient real potential the virtual excitation of the medium is of great importance as is shown by the appearance of effects of nuclear pair correlations. This strong dependence on the nuclear graininess is modified by pions of shorter wavelength. It is quite reasonable to believe that even

pions below 100 MeV will provoke noticeable changes in the interaction strengths because the pionic wavelength is then already shorter than the longest range correlations. The energy variation of the interaction is therefore likely to throw considerable light on the way these correlations manifest themselves in a many-body system.

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Table I

Shifts and widths of the 1s states.

Notations: Energies in keV. ϵ_{ch} the expected electromagnetic shift in the absence of strong interactions, ϵ_{th} and Γ_{th} the calculated full shift (including finite size effect) and width according to the parameters of eq.(6); ϵ_{exp} and Γ_{exp} corresponding experimental values.

Element	$[E_{point}^{(2p)} - E_{point}^{(1s)}]^*$	$\langle r^2 \rangle^{1/2}$ [fm]	t [fm]	$\epsilon_{ch}^{(1s)}$	$\epsilon_{th}^{(1s)+}$	$\epsilon_{exp}^{(1s)}$	$\Gamma_{th}^{(1s)}$	$\Gamma_{exp}^{(1s)}$
${}^3\text{Li}^6$	24.56	2.43	1.00	- 0.05	- 0.51	- 0.38±.06 ^b	0.14	0.15±.05 ^b
${}^3\text{Li}^7$	24.65	2.37	1.00	- 0.05	- 0.74	- 0.64±.06 ^b	0.17	0.19±.05 ^b
${}^4\text{Be}^9$	44.09	2.40	1.00	- 0.16	- 2.00	- 1.77±.05 ^b	0.56	0.58±.05 ^b
${}^5\text{B}^{10}$	69.12	2.45	1.00	- 0.40	- 3.51	- 3.18±.18 ^a	1.30	1.27±.25 ^a
${}^5\text{B}^{11}$	69.21	2.42	1.00	- 0.39	- 4.30	- 4.23±.18 ^a	1.38	1.87±.25 ^a
${}^6\text{C}^{12}$	99.93	2.42	1.00	- 0.81	- 6.80	- 6.99±.15 ^a	2.89	2.96±.25 ^a
${}^7\text{N}^{14}$	136.48	2.48	1.77	- 1.58	-11.77	-11.74±.15 ^a	5.19	4.48±.30 ^a
${}^8\text{O}^{16}$	178.80	2.71	1.83	- 3.16	-19.43	-18.85±.25 ^a	6.80	7.56±.50 ^a
${}^8\text{O}^{18}$	178.98	2.77	1.87	- 3.30	-23.85	-23.97±.25 ^a	6.27	8.67±.70 ^a
${}^9\text{F}^{19}$	227.02	2.85	2.00	- 5.51	-32.58	-31.12±0.5 ^a	9.21	9.4±1.5 ^a
${}^{11}\text{Na}^{23}$	340.94	2.94	2.00	-12.62	-63.37	-64.74±1.0 ^a	17.9	10.3±4.0 ^a
${}^{16}\text{S}^{32}$	731.66	3.12	2.50	-57.61	-203.5		61.1	
${}^{20}\text{Ca}^{40}$	1158.97	3.41	2.70	-149.1	-419.2		99.1	

*) includes vacuumpolarization

+) includes the modification of the vacuumpolarization by strong interaction (see table V)

a). see ref.[3]

b) see ref.[5]

Table II

Shifts and widths of 2p states.

Notation as in table I.

Element	$[E_{\text{point}}^{(3d)} - E_{\text{point}}^{(2p)}]^*$	$\langle r^2 \rangle^{\frac{1}{2}}$ [fm]	t [fm]	$\epsilon_{\text{ch}}^{(2p)}$	$\epsilon_{\text{th}}^{(2p)+}$	$\epsilon_{\text{exp}}^{(2p)}$	$\Gamma_{\text{th}}^{(2p)}$	$\Gamma_{\text{exp}}^{(2p)}$
${}^7\text{N}^{14}$	25.064	2.48	1.77	0	0.006		4.0eV	$2.1 \pm 0.3 \text{eV}^{\text{d}}$
${}^8\text{O}^{16}$	32.797	2.71	1.83	0	0.012		7.5eV	$4.7 \pm 0.8 \text{eV}^{\text{d}}$
${}^8\text{O}^{18}$	32.831	2.77	1.87	0	0.012		8.2 eV	$3.8 \pm 0.7 \text{eV}^{\text{d}}$
${}^9\text{F}^{19}$	41.594	2.85	2.00	0	0.023		16 eV	$11.2 \pm 1.9 \text{eV}^{\text{d}}$
${}^{11}\text{Na}^{23}$	62.295	2.94	2.00	0	0.069		56 eV	$34.6 \pm 7.6 \text{eV}^{\text{d}}$
${}^{13}\text{Al}^{27}$	87.20	2.91	2.28	- 0.01	0.17	0.20 ± 0.10	0.18	$0.36 \pm 0.15^{\text{c}}$
${}^{15}\text{P}^{31}$	116.33	3.07	2.45	- 0.02	0.36	0.45 ± 0.10	0.41	$0.43 \pm 0.15^{\text{c}}$
${}^{16}\text{S}^{32}$	132.46	3.12	2.50	- 0.02	0.52	0.60 ± 0.10	0.59	$0.79 \pm 0.15^{\text{c}}$
${}^{20}\text{Ca}^{40}$	207.72	3.41	2.70	- 0.11	1.52	1.94 ± 0.18	2.03	$2.00 \pm 0.25^{\text{c}}$
${}^{22}\text{Ti}^{48}$	251.83	3.56	2.20	- 0.20	1.94	2.15 ± 0.20	3.37	$2.89 \pm 0.25^{\text{c}}$
${}^{23}\text{V}^{51}$	275.50	3.58	2.10	- 0.26	2.26	2.35 ± 0.20	4.43	$3.66 \pm 0.25^{\text{c}}$
${}^{24}\text{Cr}^{52}$	300.21	3.66	2.30	- 0.36	2.74	2.55 ± 0.25	5.41	$4.46 \pm 0.35^{\text{c}}$
${}^{25}\text{Mn}^{55}$	326.04	3.72	2.20	- 0.48	2.98	3.08 ± 0.25	6.69	$6.38 \pm 0.40^{\text{c}}$
${}^{26}\text{Fe}^{56}$	352.92	3.75	2.50	- 0.63	3.65	3.51 ± 0.30	8.34	$8.65 \pm 0.60^{\text{c}}$
${}^{27}\text{Co}^{59}$	380.93	3.83	2.50	- 0.84	3.72	3.81 ± 0.35	9.91	$7.37 \pm 0.70^{\text{c}}$
${}^{38}\text{Sr}^{88}$	762.49	4.14	2.30	- 7.12	-3.68		55.2	
${}^{49}\text{In}^{115}$	1283.87	4.49	2.30	-35.2	-70.8		148.5	
${}^{73}\text{Ta}^{181}$	2956.21	5.50	2.80	-404.	-885.		353.	

*) includes vacuum polarization

+) includes the modification of the vacuum polarization by strong interaction

c) see ref.[4]

d) see reff.[8,9]

Table III

Shifts and widths of 3d states.

Parameterset from tables I and II (eq.(6)). Notation as in table I.

Element	$[E_{\text{point}}^{(4f)} - E_{\text{point}}^{(3d)}]^*$	$\langle r^2 \rangle^{\frac{1}{2}}$ [fm]	t [fm]	$\varepsilon_{\text{ch}}^{(3d)}$	$\varepsilon_{\text{th}}^{(3d)+}$	$\varepsilon_{\text{exp}}^{(3d)}$	$\Gamma_{\text{th}}^{(3d)}$	$\Gamma_{\text{exp}}^{(3d)}$
$^{49}\text{In}^{115}$	440.31	4.49	2.30	- .08	2.1	2.6 ± 0.5	2.5	$2.6 \pm .6^e$
$^{51}\text{Sb}^{122}$	477.52	4.67	2.40	- .14	2.7		3.3	
$^{53}\text{I}^{127}$	516.28	4.74	2.00	- .19	3.5	4.5 ± 0.8	4.7	4.4 ± 1.5^e
$^{55}\text{Cs}^{133}$	556.62	4.80	2.50	- .28	4.3	5.4 ± 1.5	5.9	3.3 ± 1.5^e
$^{57}\text{La}^{139}$	598.54	4.86	2.50	- .39	5.2	6.4 ± 2.0	7.7	6.2 ± 2.0^e
$^{59}\text{Pr}^{141}$	642.04	4.88	2.50	- .51	6.6	6.1 ± 2.0	10.1	5.4 ± 2.5^e

Table IV

Shifts and widths of 4f states.

Parameterset from tables I and II (eq.(6)). Notation as in table I.

Element	$[E_{\text{point}}^{(5g)} - E_{\text{point}}^{(4f)}]^*$	$\langle r^2 \rangle^{\frac{1}{2}}$ [fm]	t [fm]	$\varepsilon_{\text{ch}}^{(4f)}$	$\varepsilon_{\text{th}}^{(4f)+}$	$\varepsilon_{\text{exp}}^{(4f)}$	$\Gamma_{\text{th}}^{(4f)}$	$\Gamma_{\text{exp}}^{(4f)}$
$^{73}\text{Ta}^{181}$	453.03	5.50^{++}	2.80	- .01	0.48	0.87 ± 0.2	0.34	0.5 ± 0.2^e
$^{79}\text{Au}^{197}$	531.87	5.32	2.32	- .02	0.90	1.29 ± 0.2	0.85	1.0 ± 0.3^e
$^{83}\text{Bi}^{209}$	588.11	5.50	2.10	- .04	1.45	1.95 ± 0.3	1.46	1.7 ± 0.5^e
$^{90}\text{Th}^{232}$	693.68	5.80^{++}	2.80	- .13	2.89	4.32 ± 0.6	3.01	6.0 ± 0.9^f
"	"	5.80^{++}	2.00	- .11	3.01	"	3.36	" f
"	"	5.70^{++}	2.00	- .10	2.93	"	3.39	" f
$^{92}\text{U}^{238}$	725.54	5.87^{++}	2.80	- .17	3.48	5.96 ± 1.1	3.74	6.1 ± 1.0^f
$^{94}\text{Pu}^{239}$	758.14	5.85^{++}	2.80	- .20	4.15	8.06 ± 1.6	4.61	9.1 ± 2.5^f

*) includes vacuum polarization

+) includes the modification of the vacuum polarization by strong interaction

++) strongly deformed nuclei; rms radii and diffuseness t chosen somewhat arbitrary

e) see ref.[6] f) see ref.[2]

Table V

Strong Vacuum polarization Shifts (in keV)

Parameters for strong interaction are those of equation (6).

Element	$\delta\mathcal{E}_{\text{vac}}(1s)$	$\delta\mathcal{E}_{\text{vac}}(2p)$	$\delta\mathcal{E}_{\text{vac}}(3d)$
${}^3\text{Li}^6$	- .004	0	0
${}^4\text{Be}^9$	- .017	0	0
${}^5\text{B}^{10}$	- .032	0	0
${}^5\text{B}^{11}$	- .040	0	0
${}^6\text{C}^{12}$	- .064	0	0
${}^7\text{N}^{14}$	- .114	0	0
${}^8\text{O}^{16}$	- .184	0	0
${}^8\text{O}^{18}$	- .231	0	0
${}^9\text{F}^{19}$	- .309	0	0
${}^{11}\text{Na}^{23}$	- .590	0	0
${}^{13}\text{Al}^{27}$		.003	0
${}^{15}\text{P}^{31}$		.006	0
${}^{16}\text{S}^{32}$	-1.703	.009	0
${}^{20}\text{Ca}^{40}$	-2.993	.027	0
${}^{22}\text{Ti}^{48}$		.036	0
${}^{23}\text{V}^{51}$		.042	0
${}^{24}\text{Cr}^{52}$		.054	0
${}^{25}\text{Mn}^{55}$		.060	0
${}^{26}\text{Fe}^{56}$		.075	0
${}^{27}\text{Co}^{59}$		.068	0
${}^{38}\text{Sr}^{88}$		.050	.008
${}^{49}\text{In}^{115}$		- .627	.049
${}^{73}\text{Ta}^{181}$		-6.304	.395

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FIGURE CAPTIONS

- Figure 1 Various potential terms of Eq. (5) in local representation (the local representation of the gradient term corresponds to modified wave functions !). The dipole character of potential is striking.
- Figure 2a Pionic 1s interaction shift $\mathcal{E}$ vs. atomic number A (see Table I). The solid curve refers to calculated values using the best interaction potential (6). The dashed lines are calculated values for the real term of the local part of the potential only. No significant effect of other terms is observed.
- Figure 2b Same observed 1s shifts as in Fig. 2a, on a reduced scale vs. A . The lines marked $T=0$, $\frac{1}{2}$ and 1 are calculated with best interaction (6) and refer to target nuclei of different isospin. The necessity for the isospin interaction is evident.
- Figure 3a Observed line broadenings in the pionic 1s states vs. A . Comparison of the effect to the full interaction (6) (solid line) to best fit to local absorption only (dashed line) shows how the local repulsive potential decreases widths for large A .
- Figure 3b Same observed 1s line broadenings as in Fig. 2a. The solid curve is calculated with best potential (6).
- Figure 4a Pionic 2p interaction shifts vs. nuclear charge Z . Solid curve is calculated from the best interaction (6). Dashed curve is best fit using a real gradient interaction only. The lack of repulsion for higher Z is apparent for the latter case.

Figure 4b Same shifts as in Fig. 4a, vs. $\log Z$ on a reduced scale. Solid curve corresponds to best interaction (6). Note prediction for heavier elements with change of sign of shift.

Figure 5a Observed 2p pionic widths vs. Z . Solid line is from best interaction (6) ; dashed line is fit using only the absorptive part of the gradient interaction.

Figure 5b Pionic 2p widths from observed line broadening and deduced from attenuation vs. A on a reduced scale. Full line gives prediction from best interaction potential. The predictions for heavy elements do not indicate saturation of the widths. Attenuation widths are systematically small ; a fit would require a smaller absorption parameter $\text{Im } C_0$ (dashed line).

Figure 6 Pion density for a typical 1s state (^{16}O) for various interactions. Only the one group (II, III, V) has correct level shift due to the local repulsive potential.

Figure 7 Pion density for a typical 2p state (^{40}Ca) for various interactions. Note the strong attraction of pure gradient interaction (V v. I) which is partially off set by the repulsive local interaction (II).

Fig.1: POTENTIALTERMS FOR $^{40}_{20}\text{Ca}$

Fig.2a:

SHIFTS OF 2p-1s TRANSITIONS

Fig. 2b

Fig.3a: WIDTHS OF 2p-1s TRANSITIONS

Fig.3b

Fig.4a: SHIFTS OF 3d-2p TRANSITIONS

Fig. 4 b

Fig.5a

WIDTHS OF 3d-2p TRANSITIONS

Fig. 5b

Fig.6: PION - DENSITY IN THE 1s - STATE OF $^{16}_8\text{O}$

Fig.7: PION-DENSITY IN THE 2p-STATE OF $^{40}_{20}\text{Ca}$