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A CLASS OF FUNCTIONS THAT SATISFY CROSSING SYMMETRY AND UNITARITY

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A B S T R A C T

Atkinson's proof of the existence of functions that satisfy crossing symmetry and unitarity is extended to amplitudes given by a Mandelstam representation with one subtraction. The inelastic unitarity bounds $\text{Im} f_{\ell}(s) \geq |f_{\ell}(s)|^2$ are obtained only for a finite interval of the energy, for higher values of s we only get $1 \geq \text{Im} f_{\ell}(s) \geq 0$. The proof extends to positive and to oscillating double spectral functions.

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I. INTRODUCTION

An iteration process to construct a crossing symmetric amplitude for two-particle scattering which also satisfies elastic unitarity has been proposed ten years ago by Mandelstam¹⁾. The proof that such an amplitude actually exists and does not violate the inelastic unitarity bounds has been derived just now by Atkinson^{2),3)}. So far the proof applies to a $\pi\pi$ scattering amplitude which is given by an unsubtracted Mandelstam representation with positive double spectral functions. It is possible to extend these arguments to the case of an arbitrary finite number of subtractions if one only cares about crossing symmetry and elastic unitarity. Such a program has also been given by Atkinson⁴⁾. In this paper we realize a different method for one subtraction (but it also applies to any finite number of subtractions). The real difficulty arises with the discussion of the inelastic unitarity. To preserve the upper limit $\text{Im} f_\ell(s) \leq 1$ we restrict the double spectral functions rather drastically by $|\rho(s,t)| \leq C \cdot (s/t + t/s)(\log s \cdot \log t)^{-1-\epsilon}$. Martin has shown that this bound is almost the best possibility for a positive double spectral function⁵⁾. We do not discuss an improvement of this condition in the case of oscillating functions, but we show that within this restriction also a class of oscillating functions satisfies the lower unitarity limit. These techniques are the first step to study double spectral functions which have both positive and negative values. Unfortunately the inelastic unitarity bounds $\text{Im} f_\ell(s) \geq |f_\ell(s)|^2$ can be derived only up to some finite energy $s_\lambda > 16$. For higher energies we obtain just $1 \geq \text{Im} f_\ell(s) \geq 0$.

If the same method is applied to the case without subtractions Atkinson's result $\text{Im} f_\ell(s) \geq |f_\ell(s)|^2$ follows for all energies but now including a class of amplitudes with oscillating double spectral functions.

The proof in this paper concerns only the scattering of equal neutral particles. The extension to charged particles with isospin one is possible but then one needs a lot of additional algebra [as it may be seen in Ref. 3)].

II. THE NON-LINEAR MAPPING FOR ELASTIC UNITARITY

For the elastic scattering of two equal neutral (pseudo) scalar particles we assume a Mandelstam representation

$$F(s, t) = g(s) + g(t) + g(u) + \frac{s+t}{\pi^2} \int \frac{S(s', t') ds' dt'}{(s'-s)(t'-t)(s'+t')} + \\ + \frac{t+u}{\pi^2} \int \frac{S(t', u') dt' du'}{(t'-t)(u'-u)(t'+u')} + \frac{s+u}{\pi^2} \int \frac{S(s', u') ds' du'}{(s'-s)(u'-u)(s'+u')} \quad (1)$$

with

$$g(s) = d + \frac{s}{\pi} \int \frac{\varphi(s') ds'}{s'(s'-s)}$$

which corresponds to the bound $|F(s, t)| \leq C \cdot (1+|s|+|t|)^{1-\epsilon}$. This is more restrictive than the usual assumption $|F(s, t)| \leq C \cdot (1+|s|)^{1-\epsilon} \cdot (1+|t|)^{1-\epsilon}$. The reason for this restriction will be seen by the discussion of the inelastic unitarity bounds. Actually to derive only elastic unitarity we may write any Mandelstam representation with a finite number of subtractions.

As in Atkinson's proof ^{2), 3), 4)} the mapping for the double spectral functions is defined by

$$S(s, t) \rightarrow S'(s, t) \equiv S^{el}(s, t) + S^{el}(t, s) + \omega(s, t) \quad (2)$$

The function $\omega(s, t)$ has to be symmetric with a support in $s \geq 16$, $t \geq 16$. To obtain $S^{el}(s, t)$ we need the absorptive part in the t channel

$$D(s, t) = \varphi(t) + \frac{s+t}{\pi} \int \frac{S(s', t) ds'}{(s'-s)(s'+t)} + \frac{t+u}{\pi} \int \frac{S(t, u') du'}{(u'-u)(t+u')} \quad (3)$$

and write

$$S^{el}(s, t) = \lambda(s) \frac{1}{2\pi} \left(\frac{s-4}{s} \right)^{\frac{1}{2}} \theta(z - 2z_0^2 + 1) \cdot \\ \int_{z_0}^{z(z, z_0)} dz_1 \int_{z_0}^{z(z, z_1)} dz_2 (z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1)^{-\frac{1}{2}} D^*(s, t_1) D(s, t_2) \quad (4)$$

with the quantities

$$z = 1 + \frac{2t}{s-4}, \quad z_0 = 1 + \frac{8}{s-4},$$

$$t_i = \frac{1}{2}(s-4)(z_i-1), \quad i = 1, 2$$

and

$$\chi(z, z_0) = z z_0 - (z^2 - 1)^{\frac{1}{2}} (z_0^2 - 1)^{\frac{1}{2}}$$

Here we have introduced a cut off $\lambda(s)$ to compensate the increasing behaviour of the D's [compare 4], the assumptions on $\lambda(s)$ are

$$\lambda(s) \in C^\infty, \quad \lambda(s) = 1 \quad \text{if } 4 \leq s \leq s_\lambda (> 16)$$

$$0 \leq \lambda(s) \leq 1 \quad \text{if } s_\lambda \leq s \leq s_\Lambda < \infty$$

$$\lambda(s) = 0 \quad \text{if } s \geq s_\Lambda.$$

In the case of subtractions one has also to determine the subtraction functions that can be done using the original form of the unitarity equation. We write

$$\varphi(s) \rightarrow \varphi'(s) \equiv \varphi^{el}(s) + \chi(s) \quad (5)$$

with a given function $\chi(s)$, $\chi(s) = 0$ if $s < 16$, and the terms $\varphi^{el}(s) = \phi(s, t) - \Psi(s, t)$,

$$\phi(s, t) = \lambda(s) \frac{1}{8\pi} \left(\frac{s-4}{s} \right)^{\frac{1}{2}} \int_{-1}^{+1} dz' \int_0^{2\pi} d\theta \hat{F}^*(s, z') \cdot \hat{F}(s, z z' - (1-z^2)^{\frac{1}{2}} (1-z'^2)^{\frac{1}{2}} \cos \theta),$$

$$(\hat{F}(s, z) = F(s, t))$$

(6)

and

$$\Psi(s,t) = \frac{s+t}{\pi} \int \frac{s^{el}(s,t')}{(t'-t)(s+t')} dt' + \frac{s+u}{\pi} \int \frac{s^{el}(s,u')}{(u'-u)(s+u')} du' \quad (7)$$

By the definition of $s^{el}(s,t)$ the function $\Psi(s,t)$ contains also the cut-off $\lambda(s)$. The t dependence of the unitarity integral (6) is just cancelled by $\Psi(s,t)$. This follows from the construction of $s^{el}(s,t)$, the difference $\phi(s,t) - \Psi(s,t)$ has to be an entire function of t with the same bounds as the subtraction polynomial.

We now have to discuss the mapping $(s, \varphi) \rightarrow (s', \varphi')$ which depends on the subtraction constant d and the functions $\chi(s)$ and $\omega(s,t)$. These functions vanish if $s \leq 16$, and by the definition (4) $s^{el}(s,t) = 0$ if $t \leq 16$. Therefore $s'(s,t) = s^{el}(s,t)$ and $\varphi'(s) = \varphi^{el}(s)$ if $s \leq 16$, i.e., any fixed point of the mapping, $\varphi' = \varphi$, $s' = s$, fulfils elastic unitarity (and crossing symmetry). In the next Section the existence of such fixed points is guaranteed by a study of the above equations in a suitable Banach space.

III. THE EXISTENCE OF A FIXED POINT

In the case without subtractions Atkinson discussed the mapping $\mathcal{F} \rightarrow \mathcal{F}'$ in a Banach space $\mathcal{L}_0$ of Hölder continuous functions $f(s,t)$ vanishing at infinity, $\lim_{s \rightarrow \infty} f(s,t) = \lim_{t \rightarrow \infty} f(s,t) = 0$. The norm of this space is ³⁾

$$\|f(s,t)\|_0 = \sup_{4 < s_1, s_2, t_1, t_2 < \infty}$$

$$\left[\frac{|f(s_1, t_1) - f(s_2, t_2)| \log^2(\Omega s) \log^2(\Omega t)}{\left| \frac{s_1 - s_2}{s_1 s_2} \right|^\mu \bar{s}^{-\mu} + \left| \frac{t_1 - t_2}{t_1 t_2} \right|^\mu \bar{s}^{-\mu}} \right] \quad (8)$$

with $\bar{s} = \min(s_1, s_2)$ and $\bar{t} = \min(t_1, t_2)$. The Hölder index μ , $0 < \mu < \frac{1}{2}$, and $\Omega > 1$ are fixed quantities.

For our problem it is most convenient to introduce a Banach space $\mathcal{L}$ of all functions $f(s,t)$ such that $(s+t)^{-1}f(s,t)$ belongs to $\mathcal{L}_0$, the norm is defined as

$$\|f(s,t)\| = \|(s+t)^{-1}f(s,t)\|_0 \quad (9)$$

In this way most of the estimates can be reduced to Atkinson's proof. We may characterize the space $\mathcal{L}$ as follows. Let $f(s,t) \in \mathcal{L}$ with $\|f\| = C$ then for $s, t > 4$

$$|f(s,t)| \log^2(\Omega s) \log^2(\Omega t) \leq C (s+t) s^{-\mu} t^{-\mu},$$

$$|f(s_2, t) - f(s_1, t)| \log^2(\Omega s_2) \log^2(\Omega t) \leq$$

$$2C (s_2 + t) \left| \frac{s_2 - s_1}{s_1 s_2} \right|^\mu t^{-\mu} \quad \text{if } s_1 \leq s_2,$$

$$|f(s, t_2) - f(s, t_1)| \log^2(\Omega t_2) \log^2(\Omega s) \leq$$

$$2C (s + t_2) \left| \frac{t_2 - t_1}{t_1 t_2} \right|^\mu s^{-\mu} \quad \text{if } t_1 \leq t_2$$

(10)

is easily obtained. On the other hand a function $f(s,t)$ which fulfills (10) lies in $\mathcal{L}_0$ with $\|f\| \leq 3C$. An equally easy statement follows for the products of a function $f_0(s,t)$ with s , t or u , if $f_0(s,t) \in \mathcal{L}_0$ then $f_1(s,t) = sf_0(s,t)$, $f_2(s,t) = tf_0(s,t)$ and $f_3(s,t) = uf_0(s,t)$ are elements of $\mathcal{L}_0$ with $\|f_i(s,t)\| \leq 3\|f_0(s,t)\|_0$ ($i=1,2,3$).

The proof in the space $\mathcal{L}_0$ gives therefore an estimate for the Hilbert transforms which appear in Eq. (3) and we get

$$\|D(s,t)\| \leq \|\varphi(t)\| + c_1 \|S(s,t)\| \quad (11)$$

If we write the integral in Eq. (4) as

$$\int dz_1 \int dz_2 \frac{(s+t_1)(s+t_2)}{(z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1)^{\frac{1}{2}}} \cdot \frac{D^*(s,t_1)}{s+t_1} \frac{D(s,t_2)}{s+t_2}$$

we see that this is the transform of functions $(D(s,t))/(s+t) \in \mathcal{L}_0$ with a kernel containing the factor $(s+t_1)(s+t_2)$. The integration in (4) is restricted to $t_1, t_2 > 4$,

$$t_1^{\frac{1}{2}} \left(1 + \frac{t_2}{s-4}\right)^{\frac{1}{2}} + t_2^{\frac{1}{2}} \left(1 + \frac{t_1}{s-4}\right)^{\frac{1}{2}} \leq t^{\frac{1}{2}} \quad (12)$$

and from the cut-off $\lambda(s)$ we know $s \leq s_\lambda < \infty$. Hence $4t_1t_2 < (s_\lambda - 4) \cdot t$ and the additional factor $(s+t_1)(s+t_2)$ is bounded by $\text{const} \cdot t$. In the estimates for $|S^{el}(s,t)|$, $|S^{el}(s_1,t) - S^{el}(s_2,t)|$ and $|S^{el}(s,t_1) - S^{el}(s,t_2)|$ we therefore get, in comparison with the case without subtractions, only a factor t . This factor is compensated in the definition of the norm of $S^{el}(s,t)$ in the space $\mathcal{L}_0$ and we obtain again Atkinson's result ³⁾

$$\|S^{el}(s,t)\| \leq c_2 \|D(s,t)\|^2 \quad (13)$$

and

$$\|S_1^{el}(s,t) - S_2^{el}(s,t)\| \leq c_2 (\|D_1(s,t)\| \|D_2(s,t)\|) \|D_1(s,t) - D_2(s,t)\| \quad (14)$$

where S_1^{el} is the transform of D_1 and S_2^{el} the transform of D_2 . If we let the cut-off energy s_2 go to infinity then also $C_2 \rightarrow \infty$.

To discuss the unitarity integral (6) we first remark that we are interested only in the partial wave contribution with $\ell = 0$. This gives

$$\phi^0(s) = \lambda(s) 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} |f_0(s)|^2 \quad (15)$$

with

$$f_0(s) = [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 F(s,t) dt \quad (16)$$

Since $F(s,t) \in \mathcal{L}$, $\|F(s,t)\| < C(d + \|\varphi\| + \|S\|)$, compare Eq. (10), also $(\lambda(s))^{\frac{1}{2}} f_0(s)$ is a function in $\mathcal{L}$ (the t dependence is the trivial one) and $\|(\lambda(s))^{\frac{1}{2}} f_0(s)\| \leq C'(d + \|\varphi\| + \|S\|)$. But this gives the wanted result $\phi^0(s) \in \mathcal{L}$ and

$$\|\phi^0(s)\| \leq c_3 (d + \|\varphi\| + \|S\|)^2 \quad (17)$$

In these estimates there is no problem with high energy limits because of the compact support of the functions. We only have to look at local Hölder continuity and at bounds within a finite interval.

For the difference of two functions $\phi_1^0(s)$ and $\phi_2^0(s)$ which are constructed by φ_1 and S_1 and φ_2 and S_2 respectively we get

$$\phi_1^0(s) - \phi_2^0(s) = \lambda(s) 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} \operatorname{Re} (f_{01}(s) + f_{02}(s))^* (f_{01}(s) - f_{02}(s))$$

and therefore

$$\|\phi_1^0(s) - \phi_2^0(s)\| \leq c_3 (2d + \|\varphi_1\| + \|\varphi_2\| + \|S_1\| + \|S_2\|) \cdot (\|\varphi_1 - \varphi_2\| + \|S_1 - S_2\|) \quad (18)$$

With the help of (11) and (12) the integrals (7) are easily estimated

$$\|\Psi(s,t)\| \leq c_4 (\|\varphi\| + \|S\|)^2 \quad (19)$$

$$\begin{aligned} \|\Psi_1(s,t) - \Psi_2(s,t)\| \leq c_4 (\|\varphi_1\| + \|\varphi_2\| + \|S_1\| + \|S_2\|) \cdot \\ \cdot (\|\varphi_1 - \varphi_2\| + \|S_1 - S_2\|) \end{aligned} \quad (20)$$

These bounds are also valid for the partial wave projections.

The mapping $(\mathcal{P}, \mathcal{P}) \rightarrow (S', \mathcal{P}')$ depends on the functions $\chi(s)$ and $\omega(s,t)$ [see (2), (5)] the supports of which are restricted to $s \geq 16$ and $s \geq 16, t \geq 16$ respectively. In addition we assume $\chi(s), \omega(s,t) \in \mathcal{L}_2$, $\|\chi(s)\| \leq A$ and $\|\omega(s,t)\| \leq B$.

Gathering our results we know

$$\|S'\| \leq \alpha (\|\varphi\| + \|S\|)^2 + B \quad (21)$$

$$\|\varphi'\| \leq \beta (d + \|\varphi\| + \|S\|)^2 + \gamma (\|\varphi\| + \|S\|)^2 + A \quad (22)$$

$$\begin{aligned} \|S'_1 - S'_2\| \leq \alpha (\|\varphi_1\| + \|\varphi_2\| + \|S_1\| + \|S_2\|) \cdot \\ \cdot (\|\varphi_1 - \varphi_2\| + \|S_1 - S_2\|) \end{aligned} \quad (23)$$

$$\begin{aligned} \|\varphi'_1 - \varphi'_2\| \leq [\beta (2d + \|\varphi_1\| + \|\varphi_2\| + \|S_1\| + \|S_2\|) + \\ + \gamma (\|\varphi_1\| + \|\varphi_2\| + \|S_1\| + \|S_2\|)] (\|\varphi_1 - \varphi_2\| + \|S_1 - S_2\|) \end{aligned} \quad (24)$$

The quantities α, β, γ can be assumed > 1 and are fixed by the type of transforms we discuss. They are unbounded functions of the cut-off energy. Only d, A and B are at our disposal.

Let $S_x \subset \mathcal{S}$ be the sphere $\{s(s,t), \|s\| \leq x\}$ and $T_y \subset \mathcal{T}$ the set $\{y(s), \|y\| \leq y\}$ then $S_x \times T_y$ is mapped into itself if

$$\begin{aligned} \alpha(x+y)^2 + B &\leq x \\ \beta(d+x+y)^2 + \gamma(x+y)^2 + A &\leq y \end{aligned} \quad (25)$$

There is a solution with positive x and y if

$$\begin{aligned} 4\alpha B &< 1 \\ 2\beta d &< 1 \\ A + \beta d^2 + B &< \frac{1}{4} \frac{(1 - 2\beta d)^2}{\alpha + \beta + \gamma} \end{aligned} \quad (26)$$

which is possible for small values of d , A and B . The solutions with extremal values of x and y are

$$x_{1,2} = \alpha z_{1,2}^2 + B, \quad y_{1,2} = z_{1,2} - \alpha z_{1,2}^2 - B$$

with

$$z_{1,2} = \frac{1}{2} \frac{1 - 2\beta d}{\alpha + \beta + \gamma} \left[1 \pm \sqrt{1 - 4 \frac{\alpha + \beta + \gamma}{(1 - 2\beta d)^2} (A + \beta d^2 + B)} \right]^{\frac{1}{2}}$$

For the minimal solution x_2, y_2 we get the estimates

$$\begin{aligned} B &\leq x_2 \leq z_2 \leq 2 \frac{A + \beta d^2 + B}{1 - 2\beta d} \\ A &\leq y_2 \leq z_2 \leq 2 \frac{A + \beta d^2 + B}{1 - 2\beta d} \end{aligned} \quad (27)$$

If we further restrict x and y to

$$\begin{aligned} 2\alpha(x+y) &< 1 \\ 2\beta(d+x+y) + 2\gamma(x+y) &< 1 \end{aligned} \quad (28)$$

then by (23) and (24) the mapping $(\mathcal{S}, \mathcal{P}) \rightarrow (\mathcal{S}', \mathcal{P}')$ is a contraction mapping with a unique fixed point. If d , A and B satisfy (26) the conditions (28) are fulfilled at least for the minimal solution x_2, y_2 . So the existence of a unique fixed point is guaranteed for the set $S_{x_2} \times T_{y_2}$. Actually, the above equations give a solution for a larger set, but we will not discuss this extension. [Also, we do not discuss the Schauder principle, see Refs. 2) and 3), which proves the existence of a fixed point under weaker conditions.] In any case the fixed point $(\mathcal{S}, \mathcal{P})$ has to belong to the minimal set $T_{x_2} \times S_{y_2}$ and we know therefore ^{*)}

$$\begin{aligned} \|\mathcal{S}\| &\leq 2 \frac{A + \beta d^2 + B}{1 - 2\beta d} \\ \|\mathcal{P}\| &\leq 2 \frac{A + \beta d^2 + B}{1 - 2\beta d} \end{aligned} \quad (29)$$

To summarize, for any given d , $\chi(s)$ and $\omega(s, t)$ with $\|\chi(s)\| \leq A$ and $\|\omega(s, t)\| \leq B$ such that (25) is fulfilled, we have found a fixed point solution $(\mathcal{S}, \mathcal{P}) = (\mathcal{S}', \mathcal{P}')$. These functions $\mathcal{P}(s) = \mathcal{P}^{el}(s) + \chi(s)$ and $\mathcal{S}(s, t) = \mathcal{S}^{el}(s, t) + \mathcal{S}^{el}(t, s) + \omega(s, t)$ are estimated by (29) and by

$$\begin{aligned} \|\mathcal{S}^{el}(s, t)\| &\leq 16 \alpha \left(\frac{A + \beta d^2 + B}{1 - 2\beta d} \right)^2 \\ \|\mathcal{P}^{el}(s)\| &\leq 16(\beta + \gamma) \left(\frac{A + B + d}{1 - 2\beta d} \right)^2 \end{aligned} \quad (30)$$

The generalization of this part of the proof to the case of an arbitrary finite number of subtractions is obvious. The Banach space has to include the higher powers of s and t and we have to evaluate more partial waves of Eqs. (6) and (7). This is also

^{*)} In the following $\mathcal{S}$, $\mathcal{P}$ etc., denote the fixed point solution.

no problem to include isospin invariance for charged particles. The difficulties appear with the discussion of the inelastic unitarity bounds.

IV. THE INELASTIC UNITARITY BOUNDS

The amplitude constructed so far fulfils crossing symmetry and elastic unitarity. We now show that it also satisfies the inelastic unitarity bounds

$$\lambda(s) |f_\ell(s)|^2 \leq \text{Im } f_\ell(s) \leq 1, \quad \ell = 0, 2, 4, \dots \quad (31)$$

if the constant d and the functions $\chi(s)$ and $\omega(s, t)$ are chosen suitably. The inequalities (31) imply $0 \leq \text{Im } f_\ell(s) \leq 1$ for all energies but $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ only for $s \leq s_\lambda < \infty$. This is an unsatisfactory point in our proof and further work should try to remove it.

The restrictions on $\omega(s, t)$ we shall use to derive (31) [i.e., the following conditions (32) and (39)-(41)] are sufficient but not necessary. They have a comparatively simple form, the more complicated generalizations are not discussed.

To obtain the unitarity bounds we first demand [cf. Ref. 5]

$$|\omega(s, t)| \leq C \cdot \left(\frac{s}{t} + \frac{t}{s}\right) (\log s \cdot \log t)^{-1-\varepsilon}, \quad 0 < \varepsilon < 1 \quad (32)$$

Then the function $\omega(s, t)$ decreases for $s, t \rightarrow \infty$ in all directions not parallel to the s or t axis. Since the support of $S^{el}(s, t)$ is bounded in s we also get, compare (10),

$$\begin{aligned} |S^{el}(s, t)| &\leq \|S^{el}(s, t)\| (s+t) s^{-\kappa} t^{-\kappa} \log^{-2}(\Omega s) \log^{-2}(\Omega t) \\ &\leq \|S^{el}(s, t)\| s_\lambda \left(1 + \frac{t}{s}\right) (\log s \cdot \log t)^{-1-\varepsilon} \end{aligned}$$

Hence $S(s, t) = S^{el}(s, t) + S^{el}(t, s) + \omega(s, t)$ satisfies the condition (32) with C replaced by $C_s = C + 32s_\lambda \propto \left(\frac{A + \beta d^2 + B}{1 - 2\beta d}\right)^2$.

The discussion of (31) is divided in two parts, the case $\ell = 0$ which concerns $\chi(s)$ and $\omega(s, t)$, and the case $\ell > 0$ which will require further restrictions on $\omega(s, t)$.

1) Angular momentum $\ell = 0$

The imaginary part of the partial wave amplitude is given by

$$\text{Im } f_0(s) = \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \varphi(s) + 2 [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt \frac{s+t}{\pi} \int_4^{\infty} \frac{S(s, t')}{(t'-t)(s+t')} dt' \quad (33)$$

Moreover we know from the derivation of the elastic unitarity in Section III

$$\begin{aligned} \text{Im } f_0(s) - \lambda(s) |f_0(s)|^2 &= \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \chi(s) + \\ &+ 2 [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt \frac{s+t}{\pi} \int_4^{\infty} \frac{S^{el}(t', s) + \omega(s, t')}{(t'-t)(s+t')} dt' \end{aligned} \quad (34)$$

There is no problem to get an estimate on the dispersion integrals for the region $s \geq 4$, $4-s \leq t \leq 0$. Since $S(s, t)$ is restricted by (32) the integral

$$\int \frac{S(s, t')}{(t'-t)(s+t')} dt'$$

is bounded by

$$C_S \left(\frac{1}{4-t} + \frac{1}{s}\right) (\log s)^{-1-\varepsilon} \int_4^{\infty} \frac{dt'}{t'} (\log t')^{-1-\varepsilon}$$

Hence the second term in (33) does not exceed

$$\frac{C_1}{[s(s-4)]^{\frac{1}{2}}} C_S (\log s)^{-1-\varepsilon} \int_0^{s-4} dt (s-t) \left(\frac{1}{t+4} + \frac{1}{s}\right) \leq C_2 \cdot C_S$$

In the first term of (33), $\varphi(s) = \varphi^{el}(s) + \chi(s)$, the function $\varphi^{el}(s)$ has compact support $4 \leq s \leq s_{\Lambda}$, therefore, see (10), (30), $|\varphi^{el}(s)| \leq \text{const} \left(\frac{A+B+d}{1-2\beta d}\right)^2$, and $\chi(s)$ is only restricted by $\|\chi(s)\| \leq A$. If we choose, d , A , B and C small enough and take a suitable bounded function $\chi(s)$, the inequality $\text{Im } f_0(s) \leq 1$ is obviously obtained.

It is quite easy to get the expression (34) positive for $s > 16$ since a positive $\chi(s)$ can dominate the bounded integral (if only d , A , B and C are small enough with a choice $A > B$, C). Under the stronger conditions on $\omega(s, t)$ in the case $\ell > 0$ even the integral term alone becomes positive at least in a region $16 < s < 16 + \delta$.

So the unitarity limits (31) are proved for $\ell = 0$ if d , A , B and C are chosen appropriately small and $\chi(s)$ is a positive bounded function.

2) Angular momentum $\ell \geq 2$

The imaginary part of the partial wave amplitude is given by the Froissart-Gribov formula

$$\text{Im } f_\ell(s) = \frac{4}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{\sigma_0(s)}^{\infty} Q_\ell\left(1 + \frac{2t}{s-4}\right) S(s, t) dt \quad (35)$$

where $\sigma_0(s)$ determines the boundary of the double spectral region $\sigma_0(s) = \min\left(\frac{4s}{s-16}, \frac{16s}{s-4}\right)$. The estimate $\text{Im } f_\ell(s) \leq 1$ is certainly obtained if (32) is satisfied and d , A , B and C are chosen small enough [see Ref. 5].

To discuss $\text{Im } f_\ell(s) \geq \lambda(s) |f_\ell(s)|^2$ we first remark that

$$\frac{4}{\pi} [s(s-4)]^{-\frac{1}{2}} \int Q_\ell\left(1 + \frac{2t}{s-4}\right) S^{e\ell}(s, t) dt = \lambda(s) |f_\ell(s)|^2$$

by construction of $S^{e\ell}(s, t)$. The problem therefore reduces to get

$$\int Q_\ell\left(1 + \frac{2t}{s-4}\right) (S^{e\ell}(t, s) + \omega(s, t)) dt \geq 0$$

or

$$\int Q_\ell\left(1 + \frac{2s}{t-4}\right) (S^{e\ell}(s, t) + \omega(s, t)) ds \geq 0 \quad (36)$$

for $t > 16$ and $\ell = 2, 4, \dots$

These inequalities certainly follow if $S^{el}(s,t) + \omega(s,t)$ is "sufficiently positive".

As a first step we discuss some implications from the elastic unitarity of our amplitude. Following Mahoux and Martin ⁶⁾ it is shown that $S^{el}(s,t)$ is positive in a certain region. Outside the support of the double spectral function, i.e., $s \leq \sigma_0(t)$, the discontinuity in the t channel can be written as converging Legendre series

$$D(s,t) = 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} \sum_{\ell=0}^{\infty} (2\ell+1) \operatorname{Im} f_{\ell}(t) P_{\ell} \left(1 + \frac{2s}{t-4} \right) \quad (37)$$

From the elastic unitarity in the t channel we know that $\operatorname{Im} f_{\ell}(t) \geq 0$ if $4 \leq t \leq 16$. Since the Legendre polynomials $P_{\ell}(z)$ are positive for real $z \geq 1$ the discontinuity is positive if $4 < t \leq 16$ and $0 < s < \sigma_0(t)$. In Eq. (4) the integration is restricted to values of t_1 and t_2 which fulfil the inequality (12). So, only positive values of $D(s,t)$ determine $S^{el}(s,t)$ in the region

$$\begin{aligned} \frac{16s}{s-4} &\leq t \leq \min(\sigma_1(s), \sigma_2(s)) \\ \sigma_1(s) &= 20 + \frac{128}{s-4} + 16 \left(1 + \frac{4}{s-4} \right)^{\frac{1}{2}} \left(1 + \frac{16}{s-4} \right)^{\frac{1}{2}} \\ \sigma_2(s) &= \left[\left(\frac{4s}{s-16} + \frac{16s}{(s-4)(s-16)} \right)^{\frac{1}{2}} + 2 \left(1 + \frac{4s}{(s-4)(s-16)} \right)^{\frac{1}{2}} \right]^2 \end{aligned} \quad (38)$$

where $S^{el}(s,t)$ is therefore positive. The result which we shall need to derive (36) reads $S^{el}(s,t) \geq 0$ if $4 \leq s \leq 16$, $16 \leq t \leq \sigma_1(16) = 58.9$.

The estimate (38) would allow a support of $\omega(s,t)$ in a smaller region than $s \leq 16$, $t \leq 16$ without interfering with the proof. The support can be taken in any symmetric region, the boundary of which lies below the curve $s = \sigma_2(t)$, $s \geq t$, and

$t = \sigma_2(s)$, $t \geq s$. But to be definite we assume the area $s \geq 16$, $t \geq 16$.

To get further results we have to impose some conditions on $\omega(s, t)$. The following set of assumptions is sufficient to derive (36). The symmetric function $\omega(s, t)$, $\|\omega(s, t)\| \leq B$, has to be "maximal" positive in a strip $16 < s < 16 + \delta$, $t > 16$ and $16 < t < 16 + \delta$, $s > 16$

$$\omega(s, t) > b \cdot B (s-16)^\mu (t-16)^\mu (s+t)^{1-2\mu} \log^{-2}(\Omega s) \log^{-2}(\Omega t) \quad (39)$$

which is possible by the definition of the norm (9) for some fixed $b < 1$ and does not violate (32) for a small B . This part of the function $\omega(s, t)$ has to dominate the integral

$$\int_{16}^{\infty} (2s+t-4)^{-2} \omega(s, t) ds : \\ \int_{16}^{16+\delta} (2s+t-4)^{-2} \omega(s, t) ds \geq \kappa \int_{16+\delta}^{\infty} (2s+t-4)^{-2} |\omega(s, t)| ds \quad (40)$$

with $\kappa > 1$ and for all $t > 16 + \delta$. A weaker assumption is:

let $h_0(t) = 16 < h_1(t) < h_2(t) < \dots$ be those values of s where $\omega(s, t)$ changes its sign, $\omega(s, t) \geq 0$ if $h_{2n}(t) \leq s \leq h_{2n+1}(t)$ and $\omega(s, t) \leq 0$ if $h_{2n+1}(t) \leq s \leq h_{2n+2}(t)$, $n = 0, 1, 2, \dots$, then

$$\int_{h_n}^{h_{n+1}} (2s+t-4)^{-2} |\omega(s, t)| ds \geq \kappa \int_{h_{n+1}}^{h_{n+2}} (2s+t-4)^{-2} |\omega(s, t)| ds \quad (41)$$

has to be valid for a $\kappa > 1$ and all $t > 16 + \delta$. From (40) or (41) one easily derives

$$\int_{16}^{\infty} Q_\ell \left(1 + \frac{2s}{t-4}\right) \omega(s, t) dt > 0$$

$\ell = 2, 4, \dots$, since $z^2 Q_1(z)$ and $Q_{\ell+1}(z)/Q_\ell(z)$ are decreasing functions of z , $z > 1$ [see Ref. 5] and so the part with $16 < s < 16 + \delta$ dominates all integrals. But regarding (39) we even get

$$\int_{16}^{\infty} Q_\ell \left(1 + \frac{2s}{t-4}\right) (s^\ell \omega(s, t) + \omega(s, t)) ds > 0$$

if d , A and B are chosen small enough because $|S^{el}(s,t)|$ is bounded by a term proportional to $(A+\beta d^2+B)^2$ which can be taken arbitrarily smaller than B . From the elastic unitarity we know $S^{el}(s,t) \geq 0$ in the area $s \leq 16$, $t \leq \sigma_1(16) = 58.9$, the inequalities (36) therefore follow for $t \leq \sigma_1(16)$.

This proof extends to all values of t if $S^{el}(s,t) \geq 0$ is satisfied in the whole region $s \leq 16$, $t \geq 4$. Our assumptions (39)-(41) are still strong enough to obtain this property of $S^{el}(s,t)$. From Eq. (4) we see that it is sufficient to show $\text{Re}(D^*(s_1, t_1)D(s_1, t_2)) \geq 0$ for $4 \leq s \leq 16$ and any $t_1, t_2 \geq 4$.

We know from the unitarity in the t channel up to $t = \sigma_1(16)$ that the discontinuity is real and positive for $0 < s < \sigma_0(t)$, $4 < t \leq \sigma_1(16)$ [compare (37)]. This region includes $4 \leq s \leq 16$, $4 < t < \sigma_0(16)$.

To discuss higher t values we remark that the part of $D(s,t)$ originated by $\omega(s,t)$

$$D_\omega(s,t) = \frac{1}{\pi} \int_{16}^{\infty} \frac{1}{s'+t} \left[\frac{s+t}{s'-s} - \frac{s-4}{s'+s+t-4} \right] \omega(s',t) ds'$$

is positive for $4 \leq s \leq 16$. This follows from our assumptions (39) and (40) or (41), and we obtain an estimate $D_\omega(s,t) > \text{const.} \cdot B \cdot t^{1-\mu} \log^{-2}(\Omega t)$ if $t \geq \sigma_0(16)$, $4 \leq s \leq 16$. Also the contribution $\mathcal{K}(t)$ is chosen to be positive in the first part of this Section.

On the other hand those terms coming from $\varphi^{el}(t)$ and $S^{el}(s,t) + S^{el}(t,s)$ are bounded by

$$\text{const} \left(\frac{A+\beta d^2+B}{1-2\beta d} \right)^2 t^{1-\mu} \log^{-2}(\Omega t)$$

So we get $\text{Re}(D^*(s, t_1)D(s, t_2)) \geq 0$ for $4 \leq s \leq 16$ and $t_1, t_2 \geq \sigma_0(16)$ if d , A and B are small. This completes the proof.

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REFERENCES

- 1) S. Mandelstam, Phys.Rev. 112, 1344 (1958); 115, 1741, 1756 (1959).
- 2) D. Atkinson, Nuclear Phys. E7, 375 (1968).
- 3) D. Atkinson, Nuclear Phys. B8, 377 (1968).
- 4) D. Atkinson, The application of fixed point theorems to S matrix theory [Preliminary version of lectures to be delivered at the eleventh Boulder Summer Institute for Theoretical Physics (1968)], University of Bonn preprint (1968).
- 5) A. Martin, CERN preprint TH. 968 (1968).
- 6) G. Mahoux and A. Martin, Nuovo Cimento 33, 883 (1964).