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SCATTERING AMPLITUDES THAT SATISFY A MANDELSTAM REPRESENTATION

WITH ONE SUBTRACTION AND UNITARITY

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A B S T R A C T

It is proved that there exist functions which satisfy a Mandelstam representation with one subtraction elastic unitarity and the inelastic unitarity bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ for all energies. We give solutions with positive and with oscillating double spectral functions.

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The study of the Mandelstam representation ¹⁾ to give a crossing symmetric and unitary amplitude has considerably advanced by the work of Atkinson ²⁾⁻⁴⁾. Atkinson proved, for the case without subtractions, that there exist functions which satisfy these properties. The inelastic unitarity is thereby reduced to the inequality which can be written with the two-particle scattering amplitude alone. An extension of this proof to a Mandelstam representation with a finite number of subtractions has been suggested also by Atkinson ⁴⁾, but not even the unitarity bounds $0 \leq \text{Im } f_\ell(s) \leq 1$ have been derived for all partial waves.

That these inequalities $0 \leq \text{Im } f_\ell(s) \leq 1$ for all ℓ are indeed rather restrictive has been shown by Goebel ⁵⁾ and Martin ⁶⁾. If the double spectral function is positive then the total cross-section has to vanish for $s \rightarrow \infty$ ^{5),6)} and the amplitude can always be written in a special form of a Mandelstam representation with two subtractions ⁶⁾.

In a preceding paper ⁷⁾ we have studied the reduced problem of a Mandelstam representation with only one subtraction. The double spectral function $\rho(s,t)$ was not restricted to be positive but we used (among other constraints) bounds for $|\rho(s,t)|$ suggested by Ref. ⁶⁾. The elastic unitarity of such amplitudes was shown, the inelastic inequalities $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ followed only for a finite range of the energy. For higher energies we just obtained $0 \leq \text{Im } f_\ell(s) \leq 1$.

The aim of this paper is to derive the bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ for all energies. To be complete and to give the notations, we indicate the proof of elastic unitarity of Ref. ⁷⁾ in Section 2. The main restrictions on the spectral functions are stated in Section 3 and the inelastic unitarity $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ is obtained in Section 4. The double spectral function may oscillate but in Section 4 the modulus $|\rho(s,t)|$ still satisfies the restrictions for positive double spectral functions in Ref. ⁶⁾. This is no longer

the case for the examples we discuss in Section 5. Here $\rho(s, t)$ increases such that $|\rho(s, t)|$ would require n subtractions ($n=1, 2, 3, \dots$). But the oscillations are so strong that the unitarity bounds are maintained, moreover the amplitude may still be written with one subtraction and hence the total cross-section vanishes for $s \rightarrow \infty$.

2. - THE PROOF OF ELASTIC UNITARITY

We consider the scattering of two equal neutral (pseudo) scalar particles. The scattering amplitude is given by

$$F(s, t) = g(s) + g(t) + g(u) + \frac{s+t}{\pi^2} \int \frac{\rho(s', t') ds' dt'}{(s'-s)(t'-t)(s'+t')} + \\ + \frac{s+u}{\pi^2} \int \frac{\rho(s', u') ds' du'}{(s'-s)(u'-u)(s'+u')} + \frac{t+u}{\pi^2} \int \frac{\rho(t', u') dt' du'}{(t'-t)(u'-u)(t'+u')} \quad (2.1)$$

with
$$g(s) = \frac{1}{\pi} \int \frac{\varphi(s') ds'}{s'-s}$$

The restrictions on $\rho(s, t)$ and $\varphi(s)$ will be given below. To denote the functional dependence on ρ and φ we also write $F[\rho, \varphi]$. The unitarity integral defines a function

$$\phi(s, t) \equiv \phi[\rho, \varphi] = \\ \lambda(s) \frac{1}{8\pi} \left(\frac{s-4}{s} \right)^{\frac{1}{2}} \int_{-1}^{+1} dz' \int_0^{2\pi} d\theta \hat{F}^*(s, z') \hat{F}(s, z z' - (1-z^2)^{\frac{1}{2}} (1-z'^2)^{\frac{1}{2}} \cos \theta), \\ \left(\hat{F}(s, z) = F(s, t) \right) \quad (2.2)$$

Here we have introduced a cut-off function $\lambda(s)$ with

$$\begin{aligned} \lambda(s) &\in C^\infty, \quad \lambda(s) = 1 \quad \text{if } 4 \leq s \leq s_\lambda \quad (> 16) \\ 0 \leq \lambda(s) &\leq 1 \quad \text{if } s_\lambda < s < s_\infty < \infty \\ \lambda(s) &= 0 \quad \text{if } s > s_\infty \end{aligned}$$

There exist uniquely a double spectral function $\rho^{\text{el}}(s, t)$ and a function $\varphi^{\text{el}}(s)$ such that

$$\Im_m F[\rho^{\text{el}}, \varphi^{\text{el}}] = \Phi[s, \varphi] \quad (2.3)$$

The explicit equation for $\rho^{\text{el}}(s, t)$ is ¹⁾

$$\begin{aligned} \rho^{\text{el}}(s, t) &= \lambda(s) \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \times \\ &\int dt_1 dt_2 [t^2 + t_1^2 + t_2^2 - 2(tt_1 + tt_2 + t_1 t_2) - 4 \frac{t t_1 t_2}{s-4}]^{-\frac{1}{2}} \times \\ &\times D^*(s, t_1) D(s, t_2) \end{aligned} \quad (2.4)$$

with the range of integration $t_1 \geq 4, t_2 \geq 4$ and

$$t_1^{\frac{1}{2}} \left(1 + \frac{t_2}{s-4}\right)^{\frac{1}{2}} + t_2^{\frac{1}{2}} \left(1 + \frac{t_1}{s-4}\right)^{\frac{1}{2}} \leq t^{\frac{1}{2}}$$

The function $D(s, t)$ is the absorptive part in the t channel

$$D(s, t) = \varphi(t) + \frac{s+t}{\pi} \int \frac{\rho(s', t) ds'}{(s'-s)(s'+t)} + \frac{t+u}{\pi} \int \frac{\rho(t, u') du'}{(u'-u)(t+u')} \quad (2.5)$$

The single discontinuity $\varphi^{\text{el}}(s)$ has to be determined by the $\ell = 0$ partial wave projection of Eq. (2.3)

$$\begin{aligned} \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \varphi^{\text{el}}(s) + 2 [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt \frac{s+t}{\pi} \int \frac{\rho^{\text{el}}(s, t') dt'}{(t'-t)(s+t')} = \\ \frac{1}{2} \lambda(s) \left| [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt F(s, t) \right|^2 \end{aligned} \quad (2.6)$$

If we take a symmetric function $\omega(s, t)$ with support $s \geq 16, t \geq 16$ and a function $\chi(s)$ with support $s \geq 16$, the mapping

$$\begin{cases} \mathcal{S}(s, t) \rightarrow \mathcal{S}'(s, t) \equiv \mathcal{S}^{el}(s, t) + \mathcal{S}^{el}(t, s) + \omega(s, t) \\ \varphi(s) \rightarrow \varphi'(s) \equiv \varphi^{el}(s) + \chi(s) \end{cases}$$

is well defined. Under certain conditions on $\omega(s, t)$ and $\chi(s)$ this mapping has a fixed point solution $\hat{\mathcal{S}}(s, t), \hat{\varphi}(s)$ (2)-4), 7). The amplitude $F[\hat{\mathcal{S}}, \hat{\varphi}]$ satisfies elastic unitarity and, of course, crossing symmetry.

Following Atkinson (2)-4), we have defined in Ref. 7) a Banach space $\mathcal{L}_0$ of Hölder continuous functions by $\lim_{s \rightarrow \infty} f(s, t) = \lim_{t \rightarrow \infty} f(s, t) = 0$ and the norm $\|f(s, t)\|_0 = \sup_{4 \leq s_1, s_2, t_1, t_2 < \infty}$

$$\left[\frac{|f(s_2, t_2) - f(s_1, t_1)| \log^2(\Omega \bar{s}) \log^2(\Omega \bar{t})}{\left| \frac{s_2 - s_1}{s_2 s_1} \right|^\mu \bar{t}^{-\mu} + \left| \frac{t_2 - t_1}{t_2 t_1} \right|^\mu \bar{s}^{-\mu}} \right] \quad (2.7)$$

with $\bar{s} = \min(s_1, s_2)$, $\bar{t} = \min(t_1, t_2)$, $0 < \mu < \frac{1}{2}$, $\Omega > 1$. To describe increasing functions we have introduced a space $\mathcal{L} : f(s, t) \in \mathcal{L}$ if and only if $(s+t)^{-1} f(s, t) \in \mathcal{L}_0$; the norm is given by $\|f(s, t)\| = \|(s+t)^{-1} f(s, t)\|_0$.

The existence of a fixed point of the mapping (2.7) was shown in Ref. 7) for spectral functions $\varphi(s)$ and $\mathcal{S}(s, t)$ in $\mathcal{L}$ [the subtraction function $g(s)$ in (2.1) was given by $g(s) = d + \frac{s}{\pi} \int \frac{\varphi(s')}{s'(s'-s)} ds'$]. To derive the unitarity bounds $0 \leq \text{Im } f_e(s) \leq 1$ we needed additional restrictions on the spectral functions. To obtain the stronger inequalities $\text{Im } f_e(s) \geq |f_e(s)|^2$ for all energies the functions $\chi(s) \in \mathcal{L}$ and $\omega(s, t) \in \mathcal{L}$ have therefore to satisfy several constraints which are discussed in the following Sections. [Especially $\chi(s)$ has to decrease for $s \rightarrow \infty$ and we can take $g(s) = \frac{1}{\pi} \int \frac{\varphi(s')}{s'(s'-s)} ds'$ in (2.1)].

3. - RESTRICTIONS ON THE SPECTRAL FUNCTIONS

We assume that $\omega(s, t)$ can be written as

$$\omega(s, t) = \psi(s, t) + \psi(t, s) \quad (3.1)$$

where $\psi(s, t)$ is chosen such that $s \cdot \psi(s, t) \in \mathcal{L}$ or, equivalently, $s(s+t)^{-1} \psi(s, t) \in \mathcal{L}_0$. The function $\psi(s, t)$ is then also an element of $\mathcal{L}$ with $\|\psi(s, t)\| \leq \frac{1}{2} \|s \cdot \psi(s, t)\|$. If the function $s \cdot \psi(s, t)$ lies within the sphere $\|s \cdot \psi(s, t)\| \leq B_1$ then $\|\omega(s, t)\| \leq B_1$ and we may substitute the quantity B in Ref. 7) by B_1 . The support of $\psi(s, t)$ has to be restricted to $s \geq 16, t \geq 16$. There are some reasons to take a support of $\omega(s, t)$ in a smaller region than $s \geq 16, t \geq 16$ because $\omega(s, t)$ should contain (for $16 \leq s \leq s_2$ or $16 \leq t \leq s_2$) only the contribution of those Feynmann graphs which cannot be obtained by iteration 3). So one expects a boundary of the support $t = 16 \frac{s-4}{s-16}$. This choice was possible in Ref. 7) and does therefore still apply for s and t below the cut-off energy s_2 [see the arguments following 7), Eq. (38)]. But in our present proof we only reach $t = 16 + 64(s-4)^{-1}$ if $s > s_2$ and $s = 16 + 64(t-4)^{-1}$ if $t > s_2$ (see the end of Section 4).

Since *) $\mathcal{S}^{el}(s, t)$ vanishes for $s > s_2$ the function $s \cdot \mathcal{S}^{el}(s, t)$ belongs to $\mathcal{L}$ and therefore $\mathcal{S}(s, t) = \mathcal{S}^{el}(s, t) + \mathcal{S}^{el}(t, s) + \omega(s, t)$ has also the structure (3.1).

A simple method to obtain (3.1) is by factorization. To this end we introduce the Banach space $\mathcal{B}_0$ of Hölder continuous functions of one variable with the norm

*) In the following $\mathcal{S}(s, t)$, $\mathcal{S}^{el}(s, t)$, $\psi(s)$, $\psi^{el}(s)$ denote the fixed point solution.

$$|f(s)| = \sup_{4 < s_1, s_2 < \infty} \frac{|f(s_2) - f(s_1)| \log^2(\Omega s) s_1^\mu s_2^\mu}{|s_2 - s_1|^\mu} \quad (3.2)$$

and $\lim_{s \rightarrow \infty} f(s) = 0$ [the constants μ and Ω are taken as in (2.7)].

The inclusion $\mathcal{L}_0 \times \mathcal{L}_0 \subset \mathcal{L}_0$ for products of functions is obvious. Now let $\mathcal{L}_{-1}$ be the set of all functions $f(s)$ such that $sf(s) \in \mathcal{L}_0$ and let $\mathcal{L}_1$ contain all functions which satisfy $s^{-1}f(s) \in \mathcal{L}_0$. Then $\mathcal{L}_0 \times \mathcal{L}_1$ and $\mathcal{L}_1 \times \mathcal{L}_0$ are subsets of $\mathcal{L}$, and if $g(s) \in \mathcal{L}_{-1}$ and $h(t) \in \mathcal{L}_1$ the ansatz $\psi(s, t) = g(s) \cdot h(t)$ even fulfils $s \cdot \psi(s, t) \in \mathcal{L}$ and may be used in (3.1).

From (3.1) we get an estimate of $\omega(s, t)$, $|\omega(s, t)| \leq \|s \cdot \psi(s, t)\| (2 + \frac{s}{t} + \frac{t}{s}) s^{-\mu} \log^{-2}(\Omega s) t^{-\mu} \log^{-2}(\Omega t)$ which is somewhat more restrictive than condition (32) of Ref. 7).

The form (3.1) of a double spectral function also implies some bounds for dispersion integrals. The integral

$$D^1(s, t) = \frac{s+t}{\pi} \int \frac{\omega(s', t)}{(s'-s)(s'+t)} ds'$$

is estimated by (see Appendix A)

$$|D^1(s, t)| \leq C \cdot \left[1 + \frac{|s|}{t} (1 + |s|)^{-\mu} \log^{-2}(\Omega (1 + |s|)) + \frac{t}{1 + |s|} \right] \times t^{-\mu} \log^{-2}(\Omega t) \quad (3.4)$$

for $t > 16$ and all s with $C \leq \text{const} \|s \cdot \psi(s, t)\|$. The double integral

$$F^1(s, t) = \frac{s+t}{\pi^2} \int \frac{ds'}{s'-s} \frac{dt'}{t'-t} \frac{\omega(s', t')}{s'+t'}$$

can be majorized by

$$|F'(s,t)| \leq C \cdot \left[1 + \frac{|t|^{1-\mu} \log^{-2}(\Omega(1+|t|))}{1+|s|} + \frac{|s|^{1-\mu} \log^{-2}(\Omega(1+|s|))}{1+|t|} \right] \quad (3.5)$$

for all s and t with $C \leq \text{const } B_1$.

The discontinuity $\chi(s) \in \mathcal{L}$ is assumed to vanish at infinity, or more explicitly $s^{-\varepsilon} \chi(s)$ has to belong to $\mathcal{L}_0$ for some ε , $0 < 2\varepsilon < \mu$. Now the mapping $\varphi(s) \rightarrow g(s) = \frac{1}{\pi} \int \frac{\varphi(s')}{s'-s} ds'$ is no longer continuous in $\mathcal{L}$. But for the proof of elastic unitarity ⁷⁾ we only need a mapping of the type $\varphi(s) \rightarrow \lambda(s) g(s)$ and for functions $\chi(s)$ with $\|s^{-\varepsilon} \chi(s)\| \leq A_1$, $\varphi(s) = \varphi^{\text{el}}(s) + \chi(s)$, this mapping is continuous in the norm of $\mathcal{L}$. The estimates in Ref. ⁷⁾ follow therefore also in the present case with the substitution $d=0$ and a somewhat different β , so we have, e.g., $\|\varphi^{\text{el}}(s,t)\| \leq \leq 16 \alpha (A+B)^2$ from ⁷⁾, Eq. (30). Moreover A and B may be substituted by A_1 and B_1 .

4. - THE INELASTIC UNITARITY BOUNDS

Within the limitations of the previous Section the functions $\omega(s,t)$ and $\chi(s)$ can be arranged so that the amplitude (2.1) also satisfies the inelastic unitarity bounds

$$\text{Im } f_\ell(s) \geq |f_\ell(s)|^2, \quad \ell = 0, 2, 4, \dots \quad (4.1)$$

for all energies $s > 16$. This proof is divided in two parts, the case $\ell = 0$ and the case $\ell > 0$.

A) - Angular momentum $\ell = 0$

To obtain $\text{Im } f_0(s) \geq |f_0(s)|^2$ we have to show [see (2.6)]

$$\begin{aligned} & \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \chi(s) + 2 [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt \frac{s+t}{\pi} \int_4^{\infty} \frac{\mathcal{P}^{\text{el}}(t',s) + \omega(s,t')}{(t'-t)(s+t')} dt' \geq \\ & \geq \frac{1}{2} (1 - \lambda(s)) [s(s-4)]^{-1} \left| \int_{4-s}^0 F(s,t) dt \right|^2 \end{aligned} \quad (4.2)$$

The integral over the amplitude (2.1) may be written as

$$\begin{aligned} \int_{4-s}^0 F(s,t) dt &= (s-4) g(s) + 2 \int_{4-s}^0 dt \left[g(t) + \frac{s+t}{\pi^2} \int \frac{\mathcal{P}(s',t') ds' dt'}{(s'-s)(t'-t)(s'+t')} \right] + \\ &+ 2 \int_{2-\frac{s}{2}}^0 dt \frac{t+u}{\pi^2} \int \frac{\mathcal{P}(t',u') dt' du'}{(t'-t)(u'-u)(t'+u')} \end{aligned} \quad (4.3)$$

The second term in (4.2) decreases faster than $s^{-\mu}$ and the right-hand side faster than $s^{2(\varepsilon-\mu)}$ (see below). The function $\chi(s)$ with $|s^{-\varepsilon} \chi(s)| \leq A_1$, $2\varepsilon < \mu$, can be chosen to have the asymptotic behaviour $A_1 \cdot s^{\varepsilon-\mu} \log^{-2}(\Omega s)$. On account of the quadratic structure of (4.2) a positive $\chi(s)$ can then dominate for all energies if the values of A_1 and B_1 $A_1 > B_1$ are appropriately small.

To get the requested bounds we first consider those terms in (4.2) and (4.3) containing the double spectral function $\mathcal{P}(s,t)$ which can be estimated with the help of (3.4) and (3.5). These inequalities are valid also for $\mathcal{P}(s,t) = \mathcal{P}^{\text{el}}(s,t) + \mathcal{P}^{\text{el}}(t,s) + \omega(s,t)$, only the constant C has to be replaced by $\text{const } (B_1 + (A_1 + B_1)^2)$ to account for the contribution of $\mathcal{P}^{\text{el}}(s,t)$ and $\mathcal{P}^{\text{el}}(t,s)$ [compare the arguments following (32) in ⁷⁾]. The integrals $\int_{4-s}^0 dt$ in (4.2) and (4.3) can be estimated with the help of (3.4) and (3.5), they yield the bound

$s^{1-\mu}$, so all terms in (4.2) originated by the double spectral function are majorized by $\text{const } (B_1 + (A_1 + B_1)^2) \cdot s^{-\mu}$.

On the other hand we know from $\|s^{-\varepsilon} \chi(s)\| \leq A_1$ that $|g(s)| \leq \text{const} \cdot A_1 (1+|s|)^{\varepsilon-\mu}$ for high values of s . Since $g(s)$ or $g(t)$ show up only in the quadratic terms, (4.2) follows with appropriate values of A_1 and B_1 .

B) - Angular momentum $\ell \geq 2$

The partial wave amplitude is now given by the Froissart-Gribov formula

$$f_\ell(s) = \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_4^\infty Q_\ell(1 + \frac{2t}{s-4}) \mathcal{D}(s,t) dt \quad (4.4)$$

To obtain (4.1) we need some positivity properties of $\omega(s,t)$. As in Ref. ⁷⁾ we assume that $\omega(s,t)$ is positive along the boundary of its support and that this part dominates in the sense of Eqs. (39)-(41) of Ref. ⁷⁾. An ansatz which satisfies these restrictions is

$$\omega(s,t) = \omega_1(s,t) + \omega_2(s,t) \quad (4.5)$$

with $\omega_1(s,t), \omega_2(s,t)$ of the form (3.1),

$$\Psi_1(s,t) = h(s) (t-16)^{1-\mu} \log^{-2}(\Omega t)$$

$$h(s) \in \mathcal{C}_{-1}, \quad h(s) = \begin{cases} = b_1(s-16)^\mu, & b_1 > 0, & \text{if } 16 < s < 17 \\ > 0 & & \text{if } 17 < s < 18 \\ = 0 & & \text{if } s > 18 \end{cases}$$

and $\|s \cdot \Psi_2(s,t)\| < \|s \cdot \Psi_1(s,t)\|$.

In the following we assume for simplicity this ansatz with $\|s \cdot \Psi_2(s,t)\| \leq \frac{1}{2} \|s \cdot \Psi_1(s,t)\|$. The norm of $\omega_1(s,t)$ can be changed by the parameter b_1 .

Under somewhat more general conditions than those given here we have derived in Ref. ⁷⁾ the unitarity bounds (4.1) up to the energy s_λ , and $0 \leq \text{Im } f_\ell(s) \leq 1$ for all energies. The energy s_λ is determined by the cut-off $\lambda(s)$ in (2.2) and is taken larger than 18. From the details of this proof we remember (see ⁷⁾, IV, 2)

$$|\text{Im } f_\ell(s)| \leq \frac{1}{2} \quad (4.6)$$

for all s , $\ell = 2, 4, \dots$ if B_1 is chosen small enough (this bound is also implicitly contained in the following estimates) and

$$S^{\text{el}}(s, t) \geq 0 \quad (4.7)$$

for all t if $4 \leq s \leq 16$. To obtain (4.1), we have now to prove

$$\text{Im } f_\ell(s) \geq 2 |\text{Re } f_\ell(s)|^2, \quad \ell = 2, 4, \dots, s > s_\lambda \quad (4.8)$$

From (4.7) we know

$$\text{Im } f_\ell(s) \geq \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{16}^{\infty} Q_\ell(1 + \frac{2t}{s-4}) S(s, t) dt$$

since $S(s, t) = S^{\text{el}}(t, s)$ for $t < 16$. The function $S^{\text{el}}(s, t)$ depends quadratically on $\omega(s, t)$ and $\chi(s)$ therefore we get with small norms

$$\text{Im } f_\ell(s) \geq \frac{1}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{16}^{\infty} Q_\ell(1 + \frac{2t}{s-4}) \omega_1(s, t) dt$$

where we only take into account the dominant part $\omega_1(s, t)$ of (4.5). For values of $s > 18$, this gives

$$\begin{aligned} \text{Im } f_\ell(s) \geq & \frac{1}{\pi} [s(s-4)]^{-\frac{1}{2}} (s-16)^{1-\mu} \log^{-2}(\sqrt{s}) \times \\ & \times \int_{16}^{18} Q_\ell(1 + \frac{2t}{s-4}) h(t) dt \end{aligned} \quad (4.9)$$

The real part of (4.4) is divided in

$$R_e^{(1)}(s) = \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_4^{17} Q_e(1+\frac{2t}{s-4}) \operatorname{Re} D(s,t) dt$$

and

$$R_e^{(2)}(s) = \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{17}^{\infty} Q_e(1+\frac{2t}{s-4}) \operatorname{Re} D(s,t) dt$$

$$\operatorname{Re} f_e(s) = R_e^{(1)}(s) + R_e^{(2)}(s) .$$

The proof of (4.8) is then reduced to

$$\operatorname{Im} f_e(s) \geq 4 |R_e^{(1,2)}(s)|^2, \quad s > s_2, \quad e=2, 4, \dots$$

Let us first discuss $R^{(2)}(s)$. The contribution of $\varphi(t)$ does not exceed

$$\frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{17}^{\infty} Q_e(1+\frac{2t}{s-4}) |\varphi(t)| dt \leq$$

$$\operatorname{const} (A_1 + (A_1 + B_1)^2) s^{\varepsilon-\mu} \int_v^{\infty} Q_e(1+x) x^{\varepsilon-\mu} dx$$

with $v = \frac{34}{s-4}$, and for the first dispersion integral in (2.5) the estimate (3.4) gives

$$\begin{aligned} & \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_{17}^{\infty} Q_e(1+\frac{2t}{s-4}) |D^1(s,t)| dt \leq \\ & \operatorname{const} (B_1 + (A_1 + B_1)^2) \left\{ s^{-\mu} \int_v^{\infty} Q_e(1+x) x^{-\mu} dx + \right. \\ & \left. + s^{-\mu} \log^{-2}(2s) Q_2(1+\frac{34}{s-4}) \int_{17}^{\infty} t^{-1-\mu} dt + s^{-\mu} \int_v^{\infty} Q_e(1+x) x^{1-\mu} dx \right\} \end{aligned}$$

The second dispersion integral has the same bound. The function $|R^{(2)}(s)|^2$ decreases therefore at least as $s^{2(\varepsilon-\mu)}$ if s goes to infinity. Since $0 < 2\varepsilon < \mu$ this is dominated by (4.9). The behaviour of the above expressions is determined by $Q_e(1+\frac{34}{s-4})$

[we may write for all the integrals $\int_{1+v}^{\infty} Q_{\ell}(z) f(z) dz =$
 $= \int Q_{\ell}(z) z^2 z^{-2} f(z) dz \leq Q_{\ell}(1+v) (1+v)^2 \int_{1+v}^{\infty} z^{-2} f(z) dz$

because $z^2 Q_{\ell}(z)$ is decreasing in z for $\ell \geq 2$. Hence the dependence of $|R_{\ell}^{(2)}(s)|^2$ is estimated by

$$\left[Q_{\ell}\left(1+\frac{34}{s-4}\right) \right]^2 < Q_{\ell}\left(1+\frac{34}{s-4}\right) \cdot Q_0\left(1+\frac{34}{s-4}\right)$$

and there is no problem to obtain

$$\operatorname{Im} f_{\ell}(s) \geq 4 |R_{\ell}^{(2)}(s)|^2$$

for $s > s_{\lambda}$, $\ell = 2, 4, \dots$ if the values of A_1 and B_1 are small enough.

More difficult arguments are needed for $R_{\ell}^{(1)}(s)$. We know

$$\operatorname{Re} D(s, t) \leq C (t-4)^{\mu} s^{1-\mu} \log^{-2}(\Omega s)$$

for $4 \leq t \leq 17$ with

$$C \leq \text{const} (A_1 + B_1 + (A_1 + B_1)^2)$$

This implies

$$\begin{aligned} |R_{\ell}^{(1)}(s)| &\leq \text{const} (A_1 + B_1 + (A_1 + B_1)^2)^2 s^{-2\mu} \log^{-4}(\Omega s) \times \\ &\times \left| \int_4^{17} (t-4)^{\mu} Q_{\ell}\left(1+\frac{2t}{s-4}\right) dt \right|^2 \end{aligned}$$

The last integral can be estimated by (see Appendix B)

$$\left| \int_4^{17} (t-4)^{\mu} Q_{\ell}\left(1+\frac{2t}{s-4}\right) dt \right|^2 \leq \text{const} \cdot \log s \cdot \int_{16}^{17} (t-16)^{\mu} Q_{\ell}\left(1+\frac{2t}{s-4}\right) dt$$

But this gives the wanted result

$$|R_{\ell}^{(1)}(s)|^2 \leq \frac{1}{4} \operatorname{Im} f_{\ell}(s)$$

for small values of A_1 and B_1 .

The proof in Appendix B shows that we may substitute the lower integration limit $t = 16$ by $t = 16 + 64(s-4)^{-1}$. Therefore the support of $\omega(s, t)$ may be restricted to $t \geq 16 + 64(s-4)^{-1}$ if $s > s_2$ and $s \geq 16 + 64(t-4)^{-1}$ if $t > s_2$ and we still obtain (4.1) (for $16 \leq s, t \leq s_2$ see Ref. 7). An even smaller support of $\omega(s, t)$ seems possible if one takes more carefully into account the contributions of $\mathcal{S}^{\text{el}}(t, s)$ to $\text{Im } f_e(s)$. But that would require to improve the lower bound for $\mathcal{S}^{\text{el}}(t, s)$. We do not follow this possibility.

5. - MORE GENERAL CASES

In this Section, we give examples of crossing symmetric and unitary amplitudes the double spectral functions of which do not lie in $\mathcal{L}$. The representation (2.1) is still valid because of the strong oscillations of $\mathcal{S}(s, t)$, but $|\mathcal{S}(s, t)|$ would require n subtractions.

The ansatz for $\omega(s, t)$ is

$$\omega(s, t) = \omega_1(s, t) + \omega_2(s, t) \quad (5.1)$$

with $\omega_1(s, t)$ as in (4.5) and

$$\begin{aligned} \omega_2(s, t) = & b_2 \theta(s-20) \theta(t-20) (s-20)^n (t-20)^n \times \\ & \times \sin \pi [(s-20)(t-20)]^{n+1} \end{aligned} \quad (5.2)$$

In Appendix C, we study

$$\mathcal{D}_2(s, t) = \frac{1}{\pi} \int_{20}^{\infty} \frac{\omega_2(s', t)}{s' - s} ds'$$

and obtain

$$\begin{aligned} \operatorname{Re} D_2(s, t) = & b_2 \Theta(s-20) \Theta(t-20) (s-20)^n (t-20)^n \times \\ & \times \cos \pi [(s-20)(t-20)]^{n+1} + g(s, t) \end{aligned} \quad (5.3)$$

with a differentiable function $g(s, t)$

$$|g(s, t)| \leq \text{const } |b_2| (1+|s|)^{-1} (1+|t|)^{-1}$$

$$\left| \frac{\partial g(s, t)}{\partial s} \right| \leq \text{const } |b_2| (1+|s|)^{-2} (1+|t|)^{-1}$$

$$\left| \frac{\partial g(s, t)}{\partial t} \right| \leq \text{const } |b_2| (1+|s|)^{-1} (1+|t|)^{-2}$$

The contribution of $\omega_2(s, t)$ to (2.1) and (2.5) may therefore be written in an unsubtracted form, i.e.,

$$F(s, t) = F[s_1, \varphi] + \frac{1}{\pi^2} \int \frac{\omega_2(s', t') ds' dt'}{(s'-s)(t'-t)} + \text{crossed terms}$$

where $F[s_1, \varphi]$ is the amplitude (2.1) with

$$s_1(s, t) = s^{\text{el}}(s, t) + s^{\text{el}}(t, s) + \omega_1(s, t)$$

and

$$D(s, t) = D[s_1, \varphi] + D_1(s, t) + D_2(s, t)$$

With these definitions, the proof of elastic unitarity goes through without considerable modifications. [The norm of $\omega(s, t)$ has now to be substituted by $\|\omega_1(s, t)\| + |b_2|$, or $|b_1| + |b_2|$.] In this case it is convenient to choose the cut-off energies $s_2 = 18$ and $s_1 = 20$ such that the oscillating part of $D_2(s, t)$ does not show up in (2.2)-(2.4).

The contribution of $\omega_2(s, t)$ to the partial waves is given by

$$f_e^{(2)}(s) = \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_4^{\infty} Q_e \left(1 + \frac{2t}{s-4}\right) (D_2(s,t) + D_2(4-s-t, t)) dt$$

$$l = 0, 2, 4, \dots$$

These functions can be majorized by partial waves of the type considered in the last Section. The only trouble could arise from the increasing (and oscillating) part of $D_2(s,t)$, but the estimate

$$\left| \int_{20}^{\infty} Q_e \left(1 + \frac{2t}{s-4}\right) \omega_2(s,t) dt \right| \leq Q_e \left(1 + \frac{40}{s-4}\right) \int_{16}^{16+s^{-1}} |\omega_2(s,t)| dt \leq$$

$$\leq \text{const } |b_2| s^{-1} Q_e \left(1 + \frac{40}{s-4}\right)$$

shows that there is no difficulty. So we obtain the inelastic unitarity bounds (4.1) for a small b_2 .

In this example, the oscillations are so strong that we do not need subtractions for $\omega_2(s,t)$ and the contribution to the partial waves is much smaller than allowed by the proof. This is no longer the case if we take

$$\omega_2(s,t) = b_2 (s+t)^{1+\delta} (s-20)^n (t-20)^n \sin \pi [(s-20)(t-20)]^{n+1}$$

$$(0 < \delta < 1-\mu)$$

Here we have to use the subtracted dispersion integrals of Section 2. A calculation as before provides a unitary amplitude for small b_2 .

6. - CONCLUDING REMARKS

We have obtained a class of functions which satisfy a Mandelstam representation with one subtraction elastic unitarity and the inelastic unitarity bounds $\text{Im } f_{\ell}(s) \geq |f_{\ell}(s)|^2$. The double spectral functions $\rho(s, t)$ are positive for $s \leq 17$ or $t \leq 17$ and may oscillate elsewhere. But this strip where $\rho(s, t)$ is positive still prevents the amplitude to develop Regge asymptotics. The proof includes polynomially increasing double spectral functions with rather pathological oscillations (their phase varies with s^n).

For all solutions the total cross-section is bounded by $\text{const} \cdot s^{-\mu} (\log s)^{-1}$ and can be arranged to behave like $s^{-\mu} (\log s)^{-1}$. Since we may take $\mu > 0$ arbitrarily small this represents one step further to a realistic $\pi\pi$ amplitude. However, this proof does not allow $\mu = 0$, and still more work has to be done to obtain a non-vanishing total cross-section.

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- APPENDIX A -

To derive the estimate (3.4) we consider

$$J_1(s, t) = \int_{16}^{\infty} \frac{\psi(s', t) ds'}{(s'-s)(s'+t)} \quad \text{and} \quad J_2(s, t) = \int_{16}^{\infty} \frac{\psi(t, s') ds'}{(s'-s)(s'+t)}$$

with $s \cdot \psi(s, t) \in \mathcal{L}_0$, $\|s \cdot \psi(s, t)\| \leq B_1$

The first term of the right-hand side of

$$s \cdot J_1(s, t) = \int \frac{ds'}{s'-s} \frac{s' \psi(s', t)}{s'+t} + \int ds' \frac{\psi(s', t)}{s'+t}$$

is a function in $\mathcal{L}_0$ and the second term belongs to $\mathcal{L}_0$ which follows from

$$\left| \frac{\psi(s, t_2)}{s+t_2} - \frac{\psi(s, t_1)}{s+t_1} \right| \leq C \left| \frac{t_2 - t_1}{t_2 t_1} \right|^\mu s^{-1-\mu} \log^{-2}(\Omega s) \log^{-2}(\Omega t)$$

by integration. Therefore $J_1(s, t)$ is majorized by

$$|J_1(s, t)| \leq C \cdot s^{-1} t^{-\mu} \log^{-2}(\Omega t), \quad s > 4, t > 4 \quad (\text{A.1})$$

with $C = \text{const} \cdot B_1$

The integral $J_2(s, t)$ is easily estimated :

$$|J_2(s, t)| \leq C \cdot s^{-\mu} \log^{-2}(\Omega s) t^{-1-\mu} \log^{-2}(\Omega t) \quad s > 4, t > 4. \quad (\text{A.2})$$

These results can be extended to all real values of s (see ²⁾), we have only to replace s by $1 + |s|$ and to take a somewhat larger C . Then (3.4) follows.

The same technique can be applied to the double integral

$$J_3(s, t) = \int \frac{\psi(s', t') ds' dt'}{(s'-s)(t'-t)(s'+t')} ,$$

$$s \cdot J_3(s, t) = \int \frac{s' \psi(s', t') ds' dt'}{(s'-s)(t'-t)(s'+t')} + \int \frac{\psi(s', t') ds' dt'}{(t'-t)(s'+t')}$$

The first term of the right-hand side is a function in $\mathcal{L}_0$, the second term a function in $\mathcal{L}_0$. Hence we get an estimate of the type (A.1). If we add the crossed term and extend the result to all values of s and t the inequality (3.5) follows.

- APPENDIX B -

In this Appendix, we prove

$$\left| \int_4^{17} (t-4)^{\ell} Q_{\ell} \left(1 + \frac{2t}{s-4}\right) dt \right|^2 \leq \text{const} \cdot \log s \times$$

$$\times \int_{\sigma(s)}^{\sigma(s)+1} (t-\sigma(s))^{\ell} Q_{\ell} \left(1 + \frac{2t}{s-4}\right) dt$$

for $\ell = 0, 1, 2, \dots$ and $s > 4$, with any function $\sigma(s)$,
 $16 \leq \sigma(s) \leq 16 + 64(s-4)^{-1}$. This problem is equivalent to

$$\left| \int_4^5 (t-4)^{\ell} Q_{\ell} \left(1 + \frac{2t}{s-4}\right) dt \right|^2 \leq \text{const} \cdot \log s \times$$

$$\times \int_{\sigma(s)}^{\sigma(s)+1} (t-\sigma(s))^{\ell} Q_{\ell} \left(1 + \frac{2t}{s-4}\right) dt$$

(B.1)

since $(t-4)^{\ell}$ is bounded in $5 \leq t \leq 17$ and $Q_{\ell} \left(1 + \frac{2t}{s-4}\right)$ is decreasing.

As a first step we derive

$$[Q_{\ell}(z)]^2 \leq 3 z Q_0(z) Q_{\ell}(2z^2-1)$$

(B.2)

for $\ell = 0, 1, \dots$ and $1 < z < \infty$.

If we apply Schwarz's inequality to the integral representation

$$Q_{\ell}(z) = \int_0^{\infty} (z + \sqrt{z^2-1} \cosh \theta)^{-\ell-1} d\theta$$

we obtain

$$[Q_e(z)]^2 \leq Q_0(z) \cdot Q_{2e}(z) \quad (\text{B.3})$$

and $[Q_e(z)]^2 \leq Q_{e-1}(z) \cdot Q_{e+1}(z)$

From the last relation we infer

$$\frac{Q_e(z)}{Q_{e+1}(z)} \leq \frac{Q_{e-1}(z)}{Q_e(z)} \leq \dots \leq \frac{Q_0(z)}{Q_1(z)} \leq 3z$$

and hence

$$Q_{2e}(z) \leq 3z \cdot Q_{2e+1}(z) \quad (\text{B.4})$$

The integral for $Q_{2e+1}(z)$ may be estimated as

$$\begin{aligned} Q_{2e+1}(z) &= \int_0^\infty [(z + (z^2-1)^{\frac{1}{2}} \cosh \theta)^2]^{-e-1} d\theta \\ &= \int_0^\infty [u + (u^2-1)^{\frac{1}{2}} \cosh \theta + (z^2-1) \sinh^2 \theta]^{-e-1} d\theta \\ &\leq \int_0^\infty (u + (u^2-1)^{\frac{1}{2}} \cosh \theta)^{-e-1} d\theta = Q_e(u) \end{aligned}$$

where we have introduced $u = 2z^2-1$. Taking account of (B.3) and (B.4) this implies the relation (B.2).

For $z = 1 + \frac{8}{s-4}$ we have $2z^2-1 = 1 + \frac{2}{s-4}(16 + \frac{64}{s-4})$ and therefore

$$\begin{aligned} &[Q_e(1 + \frac{8+2\tau}{s-4})]^2 \leq \\ &\leq \frac{3}{2} \log s \cdot Q_e(1 + \frac{2}{s-4}(\sigma(s) + \tau)) \end{aligned} \quad (\text{3.5})$$

for any $\sigma(s)$ with $16 \leq \sigma(s) \leq 16 + \frac{64}{s-4}$ and $\tau > 0$.

The estimate (B.1) follows now easily from Schwarz's inequality

$$\left| \int_4^5 (t-4)^{\mu} Q_e \left(1 + \frac{2t}{s-4} \right) dt \right|^2 \leq \int_4^5 (t-4)^{2\mu} \left[Q_e \left(1 + \frac{2t}{s-4} \right) \right]^2 dt$$

and (B.5).

- APPENDIX C -

For the function

$$f(x, y) = \theta(x) \theta(y) (xy)^{n-1} \sin \pi (xy)^n$$

$n > 1$ fixed, the Hilbert transform may be defined by

$$h(x, y) = \lim_{\tau \rightarrow \infty} \frac{1}{\pi} P \int_0^{\tau} \frac{f(x', y)}{x' - x} dx' \quad (C.1)$$

This definition gives immediately $h(x, y) = 0$ if $y < 0$ and the functional relation

$$h(x, y) = h(xy, 1) \quad (C.2)$$

if $y > 0$. Let us consider the function

$$F(z) = \lim_{x \rightarrow \infty} \frac{1}{\pi} \int_0^x \frac{\xi^{n-1} \exp(i \xi^n)}{\xi - z} d\xi$$

This integral is related to $h(x, y)$ by

$$h(x, 1) = \frac{1}{2} \text{Im} [F(x + i\epsilon) + F(x - i\epsilon)] \quad (C.3)$$

Within the sector $0 < \arg \xi \leq \frac{\pi}{2n}$ the function $\exp(i \xi^n)$ decreases to zero if $|\xi| \rightarrow \infty$. We may therefore use the Cauchy theorem for

$$\frac{1}{\pi} \int \frac{\xi^{n-1} \exp(i \xi^n)}{\xi - z} d\xi$$

with the integration along $0 \rightarrow +\infty \rightarrow \infty \cdot \exp(i \frac{\pi}{2n}) \rightarrow 0$. If $|\text{Im } z|$ is rather small, we obtain

$$F(z) = \Theta(\operatorname{Re} z) \Theta(\operatorname{Im} z) 2i z^{n-1} \exp(i z^n) + \\ + \int_0^\infty \frac{(x\gamma)^{n-1} \exp(-x^n)}{x\gamma - z} dx, \quad \gamma = \exp(i \frac{\pi}{2n}).$$

For real z the last integral is bounded by $\operatorname{const}(1+|z|)^{-1}$ and its derivative does not exceed $\operatorname{const}(1+|z|)^{-2}$. The relations (C.2) and (C.3) yield then the result

$$h(x, \gamma) = \Theta(x) \Theta(\gamma) (x\gamma)^{n-1} \cos \pi (x\gamma)^n + g(x, \gamma)$$

with

$$|g(x, \gamma)| \leq \operatorname{const} (1+|x|)^{-1} (1+|\gamma|)^{-1}$$

and

$$\left| \frac{\partial g(x, \gamma)}{\partial x} \right| \leq \operatorname{const} (1+|x|)^{-2} (1+|\gamma|)^{-1},$$

$$\left| \frac{\partial g(x, \gamma)}{\partial \gamma} \right| \leq \operatorname{const} (1+|x|)^{-1} (1+|\gamma|)^{-2}$$

which is equivalent to (5.3).

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