Archives

SOME CONSIDERATIONS ON THE EFIMOV - FRADKIN METHOD
IN NON-LINEAR FIELD THEORIES

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A B S T R A C T

We study the method, due to Efimov and Fradkin, of formal summation of the perturbation series in non-linear field theories, in the context of a simple model which bears some similarity to the chiral dynamics Lagrangians.

We address ourselves to the question of the precise mathematical meaning of the "formal sum" of Efimov and Fradkin (in particular, the question of uniqueness of the sum and that of "contours in the auxiliary variable" posed by Delbourgo, Salam and Strathdee), and the analytic properties of the resulting amplitude. It is shown that the summation method of Efimov and Fradkin, with its precise mathematical meaning given in the text, produces a finite function having the correct analyticity properties and imaginary part implied by unitarity requirements. Ambiguities present in summing the perturbation series are not completely eliminated by the considerations of unitarity and analyticity. It is noted in passing that the resulting two-point function is non-local.

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1. - INTRODUCTION

The possibility of formal summation of the perturbation series in non-linear field theories was investigated some time ago by Efimov ¹⁾ and Fradkin ²⁾ who showed that in some cases one may obtain finite results. More recently ^{3),4),5)} the suggestion was made that the Efimov-Fradkin method be applied to give a meaning to theories described by chiral Lagrangians as well as to certain vector theories.

Briefly, the method of Efimov and Fradkin is as follows. In a non-linear field theory given by an interaction Lagrangian of the form

$$\mathcal{L}_I = -g: \phi^n(x) \sum_{m=0}^{\infty} a_m [\int \phi(x)]^m ; \quad (1)$$

the amplitudes of lowest order in the coupling constant g are infinite series in f^2 . The coefficients of f^2 are in general divergent ; however, when a suitable regularization is introduced, they become finite and the infinite series in f^2 may be formally summed. This formal sum, it is argued, may be finite, or may have a finite degree of divergence, even in the limit when the regulator mass tends to infinity.

We shall investigate this proposal in detail in the context of a simple example which, however, bears sufficient similarity to the chiral dynamics Lagrangians to appear interesting. In this simple example the complications due to the derivative and vector couplings of more realistic theories are absent and we can concentrate on answering the question of the precise mathematical meaning of the "formal sums" of Efimov and Fradkin and that of the analyticity properties of the resulting amplitude (essentially a two-point function). The formal sum of Efimov and Fradkin is given in terms of an integral over a parameter : we find that the path of integration can be specified in such a way as to

produce an amplitude in momentum space having the physically correct analyticity properties in the variable $s = p^2$. As long as the regulator mass is kept finite, this amplitude can be expanded in powers of f^2 generating in this way the regularized perturbation expansion which is shown to be the asymptotic expansion in f^2 of the amplitude. In this way a precise meaning to the formal summation is given.

The choice of the integration path over the parameter, alluded to above, can also be described as an appropriate average of the two possible analytic continuations for the sum from negative values of f^2 to the physical positive value. Each of the two analytic continuations, taken alone, would give for positive f^2 an amplitude with a non-vanishing imaginary part in the region for s below the threshold, where the amplitude must be real. The average gives a real function there, in agreement with unitarity, and can also be shown to have the right analytic continuation to values of s above the threshold. Let us compare, for s below the threshold, one of the two complex analytic continuations from negative to positive f^2 , with their real average. For finite regulator mass, these functions both have the same regularized perturbation expansion as their asymptotic expansion in f^2 . This apparent paradox is resolved by the verification that the imaginary part has a vanishing asymptotic expansion in f^2 . As the regulator mass tends to infinity, however, this is no longer true and the imaginary part has a non-vanishing asymptotic expansion.

The summation method of Efimov and Fradkin, with its precise interpretation given here, produces a finite function having the correct analyticity properties and the correct imaginary part, as dictated by unitarity requirements. One may wish to compare the result of this method with the function represented by a dispersion integral over the imaginary part. For the non-linear model considered, the imaginary part increases exponentially for large s , corresponding to real multiple particle production. A dispersion integral, therefore, requires an infinite number of subtractions. The summation method of Efimov and Fradkin does not eliminate completely the resulting ambiguity, because

the averaging procedure mentioned above is not unique. Different ways of averaging give functions which differ by an entire function of s which is real on the real axis and vanishes for $s \rightarrow -\infty$. It is interesting to observe that, for finite regulator mass, this entire function has a vanishing asymptotic expansion in f^2 , so that in any case agreement with the perturbation expansion is obtained. One could eliminate this ambiguity by requiring that the real part of the amplitude should vanish as $s \rightarrow +\infty$, in which case the averaging procedure becomes unique; however, there does not appear to be sufficient motivation for this restriction.

Irrespective of the particular averaging prescription chosen, the amplitude vanishes like s^{-1} as $s \rightarrow -\infty$ and its imaginary part increases exponentially in s as $s \rightarrow +\infty$. This is true in particular of the two-point function. In view of the exponential increase for positive s , the theory under consideration is therefore not "strictly local" ⁶⁾ and is not even "local" ^{7),8)}. On the other hand, the very welcome decrease of the two point function like s^{-1} for negative s suggests that, whenever it is inserted in more complicated graphs, one should minimize the degree of divergence by performing the integrations in the Euclidian way. What becomes then imperative is a very careful verification of unitarity. Perhaps this will show the need for a requirement that the real part of the amplitude should decrease as $s \rightarrow +\infty$, providing thereby a way to restrict the ambiguities mentioned above.

In Section 4, we consider in detail the case where the mass of the particles is zero. In this case one can obtain very simple explicit formulas which provide illustrations for all above-mentioned results.

2. - FORMAL SUMMATION OF SERIES

To investigate the method of Efimov et al. of formally summing the series, we consider a two-point function of the form

$$-i F(s) = \int d^4x e^{ipx} \langle T(V(x)V(0)) \rangle_0 \quad (2)$$

where $s = p^2 = p_0^2 - \underline{p}^2$, and

$$V(x) =: \phi(x) \sum_{l=0}^{\infty} [\alpha \phi^2(x)]^l : \equiv : \frac{\phi(x)}{1 - \alpha \phi^2(x)} : \quad (3)$$

ϕ being a free scalar field of mass μ . The Fourier transform $i\tilde{F}(x)$ of (2) may be expressed as an infinite series in α^2 :

$$\begin{aligned} -i \tilde{F}(x) &= \sum_{l=0}^{\infty} (2l+1)! \alpha^{2l} [i \Delta_F(x, \mu^2)]^{2l+1} \\ &= - \frac{d}{d\alpha} [\alpha i \tilde{G}(x)] \end{aligned}$$

where $\Delta_F(x, \mu^2)$ is the Feynman propagator for a particle of mass μ , and

$$-i \tilde{G}(x) = \sum_{l=0}^{\infty} (2l)! \alpha^{2l} [i \Delta_F(x, \mu^2)]^{2l+1} \quad (4)$$

In the following we shall consider only $i\tilde{G}(x)$ and $iG(s)$, since F and $\tilde{F}$ may be derived from them. The Fourier transform $iG(s)$ is not defined, since the propagator $\Delta_F(x)$ is too singular on the light cone. We define the regularized series

$$-i\tilde{G}_N^R(x, \alpha^2) = \sum_{l=0}^N (2l)! \alpha^{2l} [i\Delta_F(x, \mu^2) - i\Delta_F(x, M^2)]^{2l+1} \quad (5)$$

where $M^2 > \mu^2$. The Fourier transform of (5) now exists, and we write it as

$$G_N^R(s, \alpha^2) = \sum_{l=0}^N (2l)! \alpha^{2l} \frac{4\pi^2}{\sqrt{-s}} \int_0^\infty dr r^2 J_1(\sqrt{-s}r) [\Delta^R(r)]^{2l+1} \quad (6)$$

for $s < 0$. The quantity $\Delta^R(r)$ is

$$\Delta^R(r) = \frac{1}{4\pi^2 r} [\mu K_1(\mu r) - M K_1(Mr)] \quad (7)$$

$K_1(x)$ being a modified Bessel function. It has the integral representation ⁹⁾

$$\Delta^R(r) = \frac{1}{4\pi^2} \int_0^\infty \frac{dt}{t} e^{-\left(\frac{r^2}{t} + \frac{\mu^2}{4}t\right)} \left[1 - e^{-\frac{1}{4}(M^2 - \mu^2)t}\right] \quad (8)$$

For other ranges of s , $G_N^R(s, \alpha^2)$ is defined by the analytic continuation of (6). Thus, in the range $0 \leq s < \mu^2$, it is

$$G_N^R(s, \alpha^2) = \sum_{l=0}^N (2l)! \alpha^{2l} \frac{4\pi^2}{\sqrt{s}} \int_0^\infty dr r^2 I_1(\sqrt{s}r) [\Delta^R(r)]^{2l+1} \quad (9)$$

Before proceeding further, it is useful to note some of the properties of $\Delta^R(r)$ that we shall need :

- (1) it is positive for $0 < r < \infty$:

$$\Delta^R(r) > 0, \text{ for } 0 < r < \infty. \quad (10a)$$

and is monotonically decreasing in the same interval ;

- (2)

$$\lim_{r \rightarrow 0} 4\pi^2 \Delta^R(r) = -\frac{1}{2}(M^2 - \mu^2) \ln r \quad (10b)$$

- (3)

$$\lim_{r \rightarrow \infty} 4\pi^2 \Delta^R(r) = \left(\frac{\pi\mu}{2r^3}\right)^{1/2} e^{-\mu r} \quad (10c)$$

Does the series (6) or (9) has a limit as $N \rightarrow \infty$?

The answer is no, and it can be seen as follows. Consider, for example, the series in (9) which is valid for $0 \leq s < \mu^2$. The quantity

$$\mathcal{J}_l = \int_0^\infty dr r^2 \frac{I_l(\sqrt{s}r)}{\sqrt{s}} [\Delta^R(r)]^{2l+1}$$

satisfies an inequality

$$\mathcal{J}_l > \int_0^{r_0} dr r^2 \frac{I_l(\sqrt{s}r)}{\sqrt{s}} [\Delta^R(r)]^{2l+1}$$

for any r_0 , $0 < r_0 < \infty$, since the integrand is positive definite. Let us choose r_0 such that

$$\Delta^R(r_0) = \mu^2.$$

Then

$$J_l > \frac{\mu^{2(2l+1)}}{2} \int_0^{r_0} dr r^3 = \mu^{2(2l+1)} r_0^4 / 8$$

because $I_1(x) > x/2$ for $x \geq 0$. We see therefore that $G_\infty^R(s, \alpha^2)$ is a divergent series in α^2 .

We may apply Borel's method of "summation" ¹⁰⁾ to (9), and define a function G_B^R by

$$G_B^R(s, \alpha^2) = 4\pi^2 \int_0^\infty d\xi e^{-\xi} \sum_{l=0}^\infty (\alpha\xi)^{2l} J_l \quad (11)$$

However, the sum in (11)

$$\sum_{l=0}^\infty (\alpha\xi)^{2l} J_l$$

cannot be convergent for all values of ξ in the range of integration. The method of Efimov and Fradkin ^{2), 3)} consists of defining a new function $G_E^R(s, \alpha^2)$ by

$$G_E^R(s, \alpha^2) = 4\pi^2 \int_0^\infty d\xi e^{-\xi} \int_0^\infty dr r^2 \frac{I_1(\sqrt{s}r)}{\sqrt{s}} \frac{\Delta^R(r)}{1 - \alpha^2 \xi^2 [\Delta^R(r)]^2}, \quad (12)$$

$$0 \leq s < \mu^2$$

whose formal series expansion in α^2 corresponds to Eq. (11). Note, however, that the integral representation (12) is not well-defined for real α . For $s \leq 0$, $I_1(\sqrt{s}r)/\sqrt{s}$ is to be replaced by $J_1(\sqrt{-s}r)/\sqrt{-s}$ in Eq. (12).

The function $G_E^R(s, \alpha^2)$ is analytic in α^2 , with a cut running from $\alpha^2 = 0$ to $+\infty$. Equation (12) can be analytically continued in α^2 , however, beyond the first sheet by deforming the contour suitably.

What is the precise mathematical relation between $G_E^R(s, \alpha^2)$ and the series expansion (9)? Let

$$\varepsilon < \arg \alpha^2 < 2\pi - \varepsilon$$

and consider

$$\begin{aligned} & |G_E^R(s, \alpha^2) - G_N^R(s, \alpha^2)| \\ &= 4\pi^2 |\alpha^2|^{N+1} \left| \int_0^\infty d\xi e^{-\xi} \xi^{2(N+1)} \int_0^\infty dn n^2 \frac{I_1(\sqrt{s}n)}{\sqrt{s}} \frac{[\Delta^R(n)]^{2N+3}}{1 - \alpha^2 \xi^2 [\Delta^R(n)]^2} \right| \end{aligned}$$

for fixed N . Since

$$\frac{1}{|1 - \alpha^2 \xi^2 [\Delta^R(n)]^2|} \leq |\operatorname{cosec} \phi|,$$

$$\phi = \arg \alpha^2$$

we see immediately that

$$|G_E^R(s, \alpha^2) - G_N^R(s, \alpha^2)| < (2N+1)! |\alpha^2|^{N+1} |\operatorname{cosec} \phi| A_N \quad (13)$$

where

$$A_N = 4\pi^2 \int_0^\infty dn n^2 \frac{I_1(\sqrt{s}n)}{\sqrt{s}} [\Delta^R(n)]^{2N+3}$$

is independent of α^2 . Thus we have established that the series expansion G_N^R is the asymptotic expansion of G_E^R for small $|\alpha^2|$, provided that $\varepsilon < \arg \alpha^2 < 2\pi - \varepsilon$.

Actually this fact holds for a wider range of $\arg \alpha^2$. Consider, for example, the case in which α^2 is positive. There are two possible analytic continuations of $G_E^R(s, \alpha^2)$ from, say, a negative value of α^2 to the positive α^2 . These two analytic continuations are achieved by rotating the ξ integration contour by an angle θ , $0 < \theta < \pi/2$ (or $0 > \theta > -\pi/2$) depending on whether $G_E^R(s, \alpha^2)$ is being continued to the positive real α^2 axis from below (or above). Take for example the latter case. We write

$$G_E^R(s, \alpha^2 + i0) = 4\pi^2 \int_0^{\infty} d\xi e^{-\xi} e^{-i\theta} \int_0^{\infty} dn n^2 \frac{I_1(\sqrt{s}n)}{\sqrt{s}} \frac{\Delta^R(n)}{1 - \alpha^2 \xi^2 [\Delta^R(n)]^2} \quad (14)$$

$$0 < \theta < \pi/2$$

where the contour of ξ integration is given by $\xi = \zeta e^{-i\theta}$ with $0 \leq \zeta \leq \infty$. In this case we have

$$\begin{aligned} & |G_E^R(s, \alpha^2 + i0) - G_N^R(s, \alpha^2)| \\ &= 4\pi^2 \alpha^{2(N+1)} \int_0^{\infty} d\zeta e^{-\zeta \cos \theta} \zeta^{2N+2} \\ &\quad \times \int_0^{\infty} dn n^2 \frac{I_1(\sqrt{s}n)}{\sqrt{s}} \frac{[\Delta^R(n)]^{2N+3}}{|1 - \alpha^2 \zeta^2 e^{-2i\theta} [\Delta^R(n)]^2|} \end{aligned}$$

Since

$$\frac{1}{|1 - \alpha^2 \zeta^2 e^{-2i\theta} [\Delta^R(n)]^2|} \leq |\operatorname{cosec} 2\theta|$$

we obtain the inequality

$$| G_E^R(s, \alpha^2 + i0) - G_N^R(s, \alpha^2) | = (2N+3)! |\sec \theta|^{2N+3} |\operatorname{cosec} 2\theta| \alpha^{2N+2} A_N$$

and we have proved that G_N^R is asymptotic to G_E^R even for positive α^2 . In fact, a slight extension of the above argument shows that G_N^R is asymptotic to G_E^R for α^2 such that

$$-\frac{\pi}{2} + \epsilon \leq \operatorname{Arg} \alpha^2 \leq 2\pi + \frac{\pi}{2} - \epsilon$$

[The ranges $-\frac{\pi}{2} < \arg \alpha^2 < 0$ and $2\pi < \arg \alpha^2 < 2\pi + \frac{\pi}{2}$ are obtained by analytic continuation of Eq. (12) as in Eq. (14), for example.]

We have seemingly arrived at a paradox at this point : the functions $G_E^R(s, \alpha^2 + i0)$ and $G_E^R(s, \alpha^2 - i0)$, $\alpha^2 > 0$, which are complex conjugates of each other with non-vanishing imaginary parts are represented by the same asymptotic series (9). The paradox is resolved when we realize that the imaginary part of $G_E^R(s, \alpha^2 + i0)$:

$$\begin{aligned} 2i D^R(s, \alpha^2) &= 2i \operatorname{Im} G_E^R(s, \alpha^2 + i0) \\ &= G_E^R(s, \alpha^2 + i0) - G_E^R(s, \alpha^2 - i0) \end{aligned} \tag{15}$$

is a function of α^2 whose asymptotic expansion for small $\alpha^2 > 0$ is identically zero. An example of such functions is e^{-C/α^2} with $C > 0$. In the next Section, we shall show that $D^R(s, \alpha^2)$ has this property as long as the cut-off is kept finite.

3. -

DEFINITION OF THE PHYSICAL AMPLITUDES

Physically, for positive α^2 , the contour of the integration must be chosen so that the function $G_E^R(s, \alpha^2)$ has all the desired analytic properties in s . In particular, in the region $s \leq \mu^2$, the function must be real. As has been shown in the last Section, $G_E^R(s, \alpha^2)$ is a real analytic function in α^2 with a cut from $\alpha^2 = 0$ to ∞ for $s \leq \mu^2$. Therefore the combination

$$\lambda G_E^R(s, \alpha^2 + i0) + \lambda^* G_E^R(s, \alpha^2 - i0)$$

will be real, and its formal expansion in α^2 will agree with the defining series (9) provided that $\text{Re } \lambda = \frac{1}{2}$. Writing $\lambda = \frac{1}{2} + ib$, we define the real amplitude $G^R(s, \alpha^2; b)$ by

$$\begin{aligned} G^R(s, \alpha^2; b) &= \frac{1}{2} [G_E^R(s, \alpha^2 + i0) + G_E^R(s, \alpha^2 - i0)] \\ &\quad + ib [G_E^R(s, \alpha^2 + i0) - G_E^R(s, \alpha^2 - i0)] \\ &= 2\pi^2 \left[\int_0^{\infty} e^{i\theta} + \int_0^{\infty} e^{-i\theta} \right] d\xi e^{-\xi} \int_0^{\infty} d\alpha \alpha^2 \frac{I_1(\sqrt{s}\alpha)}{\sqrt{s}} \frac{\Delta^R(\alpha)}{1 - \alpha^2 \xi^2 [\Delta^R(\alpha)]^2} \\ &\quad - 2b D^R(s, \alpha^2). \end{aligned} \tag{16}$$

Before proceeding any further, let us remark that, in the expression

$$G_E^R(s, \alpha^2 \pm i0) = 4\pi^2 \int_0^{\infty} e^{\pm i\theta} d\xi e^{-\xi} \int_0^{\infty} d\alpha \alpha^2 \frac{I_1(\sqrt{s}\alpha)}{\sqrt{s}} \frac{\Delta^R(\alpha)}{1 - \alpha^2 \xi^2 [\Delta^R(\alpha)]^2} \tag{14'}$$

the order of the ξ and r integrations may be interchanged. This may be seen by making the change of variables

$$\xi = e^{-\xi} \quad ; \quad x = (1+r)^{-1},$$

which maps the range of integrations to a finite one and noting that the resulting integrand in ξ and x is finite throughout the domain of integrations.

Let us first study $D^R(s, \alpha^2)$. It can be written as

$$D^R(s, \alpha^2) = 2\pi^2 \int_0^\infty r^2 dr \frac{I_1(\sqrt{s}r)}{\sqrt{s}} \Delta^R(r) \oint d\xi e^{-\xi} \frac{1}{1 - \alpha^2 \xi^2 [\Delta^R(r)]^2}$$

where the contour integral in ξ is around the point

$$\xi = [\alpha \Delta^R(r)]^{-1} \quad ; \quad \alpha = \sqrt{\alpha^2} > 0$$

Consequently, we have

$$D^R(s, \alpha^2) = -2\pi^2 \int_0^\infty dr r^2 \frac{I_1(\sqrt{s}r)}{\sqrt{s}} \frac{\pi}{\alpha} e^{-1/\alpha \Delta^R(r)} \quad (17)$$

The integral in Eq. (17) is absolutely convergent, since for small r

$$e^{-1/\alpha \Delta^R(r)} \sim \exp \left[\frac{8\pi^2}{\alpha} \frac{1}{M^2 - \mu^2} \frac{1}{\ln r} \right] \quad (18a)$$

as long as M^2 is finite, and as $r \rightarrow \infty$

$$e^{-1/\alpha \Delta^R(r)} \sim \exp \left[- \left(\frac{4\pi^2}{\alpha} \right) \left(\frac{2r^3}{\pi\mu} \right)^{1/2} e^{\mu r} \right] \quad (18b)$$

Now write Eq. (17) in two parts :

$$\begin{aligned} D^R(s, \alpha^2) = & -2\pi^2 \int_0^{r_0} r^2 dr \frac{I_1(\sqrt{s}r)}{\sqrt{s}} \frac{\pi}{\alpha} e^{-1/\alpha \Delta^R(r)} \\ & - 2\pi^2 \int_{r_0}^{\infty} r^2 dr \frac{I_1(\sqrt{s}r)}{\sqrt{s}} \frac{\pi}{\alpha} e^{-1/\alpha \Delta^R(r)} \end{aligned} \quad (19)$$

where $r_0 \ll \frac{1}{M}$ is so chosen that, for $r < r_0$, $\exp[-1/\alpha \Delta^R(r)]$ is bounded by $\exp[(C/\alpha)(1/\ln r)]$ for some $C > 0$. The second term on the right of Eq. (19) vanishes faster than any power of α as $\alpha \rightarrow +0$ as can be seen by repeated differentiation with respect to α . The first term may be bounded, in magnitude, by

$$\frac{2\pi^3}{\alpha} \int_0^{r_0} dr r^3 \exp\left[-\frac{C}{\alpha} \frac{1}{\ln(1/r)}\right]$$

By substitution : $\zeta = -\ln r$, the integral is transformed into

$$\begin{aligned} & \frac{2\pi^3}{\alpha} \int_{\ln(1/r_0)}^{\infty} d\zeta \exp\left[-4\zeta - \frac{C}{\alpha} \zeta^{-1}\right] \\ & < \frac{2\pi^3}{\alpha} \int_0^{\infty} d\zeta \exp\left(-4\zeta - \frac{C}{\alpha} \zeta^{-1}\right) \\ & = \frac{2\pi^3}{\alpha} \cdot 2 K_0\left(4\sqrt{\frac{C}{\alpha}}\right) \rightarrow \alpha^{-3/4} e^{-4\sqrt{\frac{C}{\alpha}}} \end{aligned}$$

as $\alpha \rightarrow +0$. We see that the first term on the right of Eq. (19) is also bound by a function which has a vanishing asymptotic expansion (i.e., the term vanishes faster than any power of α as $\alpha \rightarrow +0$).

On the other hand, in the limit $M^2 \rightarrow \infty$, the estimate of (18a) is not valid, but we have instead

$$\lim_{r \rightarrow \infty} e^{-1/\alpha \Delta^R(r)} = e^{-\frac{4\pi^2}{\alpha} r^2} \quad (18c)$$

Again we split up the integral in Eq. (17) into two parts, just as we did in Eq. (19); one from 0 to r_1 and the other from r_1 to ∞ . Just as before, one can show that the integral from r_1 to ∞ vanishes faster than any power of α as $\alpha \rightarrow +0$. The integral from 0 to r_1

$$I = -\frac{2\pi^3}{\alpha} \int_0^{r_1} r^2 dr \frac{I_1(\sqrt{s}r)}{\sqrt{s}} e^{-1/\alpha \Delta(r)} \quad (20)$$

may be estimated if we choose $r_1 \ll \frac{1}{\mu}$ so that $\exp[-1/\alpha \Delta(r)]$ may be well approximated by Eq. (18c). We then write

$$I \sim \frac{2\pi^3}{\alpha} \int_0^{r_1} dr r^2 \frac{I_1(\sqrt{s}r)}{\sqrt{s}} e^{-4\pi^2 r^2/\alpha}$$

$$\xrightarrow{\alpha \rightarrow +0} -\frac{\pi}{8} \left(\frac{\alpha}{4\pi^2} \right) e^{-\frac{1}{4} \left(\frac{\alpha}{4\pi^2} \right) s}, \text{ for } \alpha \ll 4\pi r_1^2,$$

which has a well-defined power series expansion in α .

Let us consider the analytic properties of $D^R(s, \alpha^2)$ in s . The integral in Eq. (19) is convergent for all complex values of s , since, according to Eq. (18b), the factor $\exp[-1/\alpha \Delta^R(r)]$ provides a sufficient convergence at $r \rightarrow \infty$. The same is true even when $M^2 \rightarrow \infty$ [see Eq. (18c)]. Therefore we conclude that $D^R(s, \alpha^2)$ and $D(s, \alpha^2)$ are both entire in s . The same conclusion holds for $\mu^2 = 0$ since, even though Eq. (18b) is not valid in this case, $\exp[-1/\alpha \Delta(r)] = \exp(-4\pi^2 r^2/\alpha)$.

With this much knowledge on $D^R(s, \alpha^2)$, we can conclude that $G^R(s, \alpha^2; b)$ defined by Eq. (16) satisfies the following conditions :

- (1) its asymptotic expansion for small $\alpha^2 \gg 0$ coincides with the series (9) for any real b as long as the regulator mass M is finite ; [the question as to whether this remains true in the limit $M^2 \rightarrow \infty$ is meaningless since the series (9) is undefined in this case] ;
- (2) it is real for $s < \mu^2$; [to see that this is so for $s < 0$, replace $I_1(\sqrt{s} r)/\sqrt{s}$ by its analytic continuation $J_1(\sqrt{-s} r)/(\sqrt{-s})$; this property remains in the limit $M^2 \rightarrow \infty$;
- (3) it is analytic in the cut s plane, except for a pole at $s = \mu^2$ and a cut from $s = 9\mu^2$ to ∞ .

To appreciate the last point, we first recall that $D^R(s, \alpha^2)$ is entire in s . Next we may write

$$\begin{aligned}
 G^R(s, \alpha^2) &= G^R(s, \alpha^2; b=0) \\
 &= 4\pi^2 \int_0^\infty n^2 dn \frac{I_1(\sqrt{s}n)}{\sqrt{s}} \Delta^R(n) P. \int_0^\infty d\xi \frac{e^{-\xi}}{1 - \alpha^2 \xi^2 [\Delta^R(n)]^2}
 \end{aligned} \tag{21}$$

The behaviour of the function

$$K(n) = P. \int_0^\infty d\xi \frac{e^{-\xi}}{1 - \alpha^2 \xi^2 [\Delta^R(n)]^2}$$

can be studied easily if we express it in terms of exponential integrals ¹¹⁾ :

$$K(r) = -\frac{1}{2\alpha\Delta^R(r)} \left[e^{1/\alpha\Delta^R(r)} E_i\left(\frac{1}{\alpha\Delta^R(r)}\right) + e^{-1/\alpha\Delta^R(r)} E^*\left(\frac{1}{\alpha\Delta^R(r)}\right) \right]$$

In particular, for large r , [small $\Delta^R(r)$], $K(r)$ behaves as

$$K(r) \sim \sum_{l=1}^N (2l)! \alpha^{2l} [\Delta^R(r)]^{2l} + O[(\Delta^R(r))^{2N+2}]$$

The singularities in s of (21) arise from the divergences at the upper limit of the r integration. We see therefore that the singularity structure of the expression (15') in s is exactly the same as that of perturbation theory.

Furthermore the expression (16) remains finite as $M^2 \rightarrow \infty$, as can be seen easily by replacing $\Delta^R(r)$ by

$$\Delta(r) = \frac{\mu}{4\pi^2 r} K_1(\mu r)$$

in Eqs. (16) and (17). We see also from Eq. (16) that $G^R(s, \alpha^2)$ is analytic in α^2 for $\text{Re } \alpha^2 > 0$.

While our consideration has been based on a particularly simple model given by Eq. (2), it is obvious that the same discussion applies also for the example discussed by Delbourgo, Salam and Strathdee, since the amplitudes considered by them can be obtained from $G_E^R(s, \alpha^2)$ by differentiations in α^2 and algebraic operations. More generally, let us consider the sum of self-energy diagrams represented by Fig. 2 of Efimov's paper ¹²⁾ due to the interaction

$$L_I = g \int d^4x : U[\alpha\phi(x)] :$$

$$U(z) = \sum_N c_N z^N$$

where $U(z)$ has a finite radius of convergence in z (i.e., $C_N/C_{N-1} \rightarrow \text{constant as } N \rightarrow \infty$). It is given by a series :

$$\Sigma(s) = g^2 \sum_{N=3}^{\infty} N(N!) C_N^2 \frac{4\pi^2}{\sqrt{s}} \int_0^{\infty} d\lambda \lambda^2 I_1(\sqrt{s}\lambda) [\alpha^2 \Delta^R(\lambda)]^N$$

The hermiticity of the Lagrangian requires C_N to be real so that C_N^2 is positive. We write $\Sigma(s)$ as

$$\Sigma(s) = g^2 \alpha^2 \frac{\partial}{\partial \alpha^2} H(s, \alpha^2)$$

where

$$H(s, \alpha^2) = \sum_{N=3}^{\infty} \int_0^{\infty} d\xi e^{-\xi} C_N^2 4\pi^2 \int_0^{\infty} d\lambda \lambda^2 \frac{I_1(\sqrt{s}\lambda)}{\sqrt{s}} [\alpha^2 \xi \Delta^R(\lambda)]^N$$

The Efimov-Fradkin function corresponding to the series H is

$$H_E(s, \alpha^2) = \int_0^{\infty} d\xi e^{-\xi} \frac{4\pi^2}{\sqrt{s}} \int_0^{\infty} d\lambda \lambda^2 I_1(\sqrt{s}\lambda) h_E(\alpha^2 \xi \Delta^R(\lambda))$$

where $h_E(z)$ is the analytic continuation of the series

$$h_E(z) \sim \sum_{N=3}^{\infty} C_N^2 z^N$$

As Fried⁴⁾ has emphasized, $h_E(z)$ and $U(z)$ have the same radius of convergence, and the function $h_E(z)$ has a singularity on the positive real axis at the radius of convergence, since the expansion coefficients of $h_E(z)$ are all positive. Therefore $H_E(s, \alpha^2)$, for $s < 9\mu^2$, has a cut in α^2 from 0 to ∞ . The physically acceptable definition of the Efimov-Fradkin function is

$$[H_E(s, \alpha^2)]_{phys.} = \left(\frac{1}{2} + i\epsilon\right) H_E(s, \alpha^2 + i0) + \left(\frac{1}{2} - i\epsilon\right) H_E(s, \alpha^2 - i0)$$

The prescription we have given in Eq. (16) of taking a certain weighted average of two analytic continuations of the Efimov-Fradkin function defined for non-positive α^2 is tailored to make the physical amplitude real in the Euclidean region ¹³⁾. As we have shown, the physical amplitude so defined has the correct analyticity properties that follow from unitarity. This prescription leads to a unique amplitude up to an additive entire function in s , expressed by $-bD(s, \alpha^2)$ in Eq. (16).

In the next Section we consider a case in which various results of this Section can be verified by direct computations.

4. - AN EXAMPLE

The function $G^R(s, \alpha^2)$ may be explicitly evaluated, if we take the limit $\mu^2 \rightarrow 0$ and $M^2 \rightarrow \infty$, with $\alpha^2 > 0$ and $s < 0$. In this case we have

$$\Delta^R(n) \rightarrow \Delta(n) = \frac{1}{4\pi^2 n^2} \quad (22)$$

so that

$$\begin{aligned} G(s, \alpha^2; b) &= \int_0^\infty d\xi e^{-\xi} \int_0^\infty n^6 dn \frac{J_1(\sqrt{s}n)}{\sqrt{s}} \\ &\quad \times \left[\left(\frac{1}{2} + ib\right) \frac{1}{n^4 - \xi^2 \beta^2 - i\epsilon} + \left(\frac{1}{2} - ib\right) \frac{1}{n^4 - \xi^2 \beta^2 + i\epsilon} \right] \\ &= \frac{1}{-s} - (-s) \frac{\beta^{1/2}}{2} \int_0^\infty d\xi e^{-\xi} \int_0^\infty \frac{dz}{z^{1/2}} J_1(\sqrt{\beta \xi z}) \\ &\quad \times \left[\frac{P.}{z^2 - s^2} - 2\pi b \delta(z^2 - s^2) \right] \end{aligned} \quad (23)$$

where $\beta = (\alpha/4\pi^2)$ and P. denotes the Cauchy principal value integral.

Making use of the identity ¹⁴⁾

$$\begin{aligned} \frac{1}{2} \int_0^\infty d\xi \xi^{1/2} e^{-\xi} J_1(\sqrt{\beta z \xi}) &= \int_0^\infty d\xi e^{-\xi} s^2 J_1(\sqrt{\beta z} \xi) \\ &= \frac{1}{2} \sqrt{\beta z} e^{-\beta z/4} \end{aligned}$$

We write Eq. (23) as

$$G(s, \alpha^2; b) = G(s, \alpha^2; 0) - 2b D(s, \alpha^2)$$

where

$$\begin{aligned} G(s, \alpha^2) &\equiv G(s, \alpha^2; 0) \\ &= -\frac{1}{s} - \frac{\beta}{8} \left[\int_0^\infty \frac{ds'}{s'-s} e^{-\beta s'/4} + \mathcal{P} \int_{-\infty}^0 \frac{ds'}{s'-s} e^{\beta s'/4} \right] \end{aligned} \quad (24)$$

and

$$D(s, \alpha^2) = -\pi \frac{\beta}{8} e^{\beta s/4} \quad (25)$$

As advertized, $D(s, \alpha^2)$ is an entire function in s . To investigate the analytic properties in s of $G(s, \alpha^2; b)$, it suffices to consider $G(s, \alpha^2)$. Equation (24) is defined for $s < 0$. The analytic continuation of $G(s, \alpha^2)$ is a real analytic function in s with a cut from 0 to ∞ . To see this, we note that we may write, for $s < 0$

$$\begin{aligned} \mathcal{P} \int_{-\infty}^0 \frac{ds'}{s'-s} e^{\beta s'/4} &= \int_{-\infty}^0 \frac{ds'}{s'-s-i\epsilon} e^{\beta s'/4} - i\pi e^{\beta s/4} \\ &= \int_{-\infty}^0 \frac{ds'}{s'-s} \left[e^{\beta s'/4} - \frac{s-1}{s'-1} e^{\beta s'/4} \right] \\ &\quad + e^{\beta s/4} (\log s - i\pi) \end{aligned} \quad (26)$$

where the first integral is analytic in the entire s plane, and defines an entire function in s , and the second term can be continued to the cut s plane, cut from $s = 0$ to ∞ , by writing

$$e^{\beta s/4} (\log s - i\pi) = e^{\beta s/4} \ln(-s)$$

$$-\pi < \arg(-s) < \pi$$

In fact, Eq. (18) can be evaluated explicitly. Let $s' - s = t$. Then ¹¹⁾

$$\begin{aligned} \mathcal{P} \int_{-\infty}^0 ds' \frac{e^{\beta s'/4}}{s' - s} &= e^{\beta s/4} \mathcal{P} \int_{-\infty}^{-s} dt \frac{e^{\beta t/4}}{t} \\ &= e^{\beta s/4} E^*(-\beta s/4) \equiv e^{\beta s/4} [-\operatorname{Re} E_1(\beta s/4)] \end{aligned}$$

where the exponential integral $E_1(x)$ has a representation ¹¹⁾

$$E_1(x) = \int_x^{\infty} dt \frac{e^{-t}}{t} = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-x)^n}{n(n!)} \quad (27)$$

We may therefore write

$$\begin{aligned} \mathcal{P} \int_{-\infty}^0 ds' \frac{e^{\beta s'/4}}{s' - s} &= e^{\beta s/4} \left[\gamma + \log(\beta/4) + \sum_{n=1}^{\infty} \frac{1}{n(n!)} \left(-\frac{\beta s}{4}\right)^n \right] \\ &\quad + e^{\beta s/4} \log(-s) \end{aligned} \quad (28)$$

Combining Eqs. (24) and (28), we arrive at the desired representation of $G(s, \alpha^2)$:

$$G(s, \alpha^2) = \frac{\beta}{8} e^{\beta s/4} \left[\gamma + \sum_{n=1}^{\infty} \frac{1}{n(n!)} \left(-\frac{\beta s}{4}\right)^n + \log(\beta/4) \right] \\ + \frac{1}{-s} + \frac{\beta}{8} \int_0^{\infty} \frac{ds'}{s'-s} \left[\frac{s+1}{s'+1} e^{\beta s'/4} - e^{-\beta s'/4} \right] \quad (29)$$

The asymptotic behaviour of $G(s, \alpha^2)$ as $|s| \rightarrow \infty$ can best be obtained by writing

$$G(s, \alpha^2) = \frac{1}{-s} - \frac{\beta}{8} \left\{ \int_0^{\infty} \frac{ds'}{s'-s} e^{-\beta s'/4} \right. \\ \left. + e^{\beta s/4} [-E_1(\beta s/4) \mp i\pi] \right\} \text{ for } \operatorname{Im} s \gtrless 0 \quad (30)$$

and noting the asymptotic behaviour of $E_1(x)$:

$$E_1(x) \rightarrow x^{-1} e^{-x}$$

as $|x| \rightarrow \infty$, with $|\arg x| < 3\pi/2$. We find that

$$\lim_{|s| \rightarrow \infty} G(s, \alpha^2) \rightarrow \frac{\text{const}}{s}, \quad \operatorname{Re} s < 0 \\ \rightarrow \pm i\pi \frac{\beta}{8} e^{\beta s/4}, \quad \operatorname{Re} s > 0, \operatorname{Im} s \gtrless 0 \quad (31)$$

The analytic properties of $G(s, \alpha^2)$ in α^2 can be seen most readily if we write Eq. (30) in the form

$$G(s, \alpha^2) = \frac{1}{-s} - \frac{\beta}{8} \left\{ e^{-\beta s/4} E_1\left(-\frac{\beta s}{4}\right) + e^{\beta s/4} [-E_1\left(\frac{\beta s}{4}\right) \mp i\pi] \right\} \quad (32)$$

and use Eq. (27) for an explicit representation of the exponential integral functions. One obtains

$$\begin{aligned}
 G(s, \alpha^2) = & \frac{1}{-s} - \frac{\beta}{4} \sinh(\beta s/4) \left[\gamma + \log(-s/4) + \frac{1}{2} \ln \beta^2 \right] \\
 & - \frac{\beta}{4} \sinh(\beta s/4) \sum_{\substack{n=2 \\ \text{even}}} \frac{1}{n \cdot n!} \left(\frac{\beta s}{4} \right)^n \\
 & - \frac{1}{4} \cosh(\beta s/4) \sum_{\substack{n=1 \\ \text{odd}}} \frac{1}{n \cdot n!} \left(\frac{s}{4} \right)^n \beta^{n+1}
 \end{aligned} \tag{33}$$

Thus, $G(s, \alpha^2)$ is an analytic function in $\alpha^2 = (4\pi^2)^2 \beta^2$, with a cut from $\beta^2 = 0$ to $-\infty$.

5. - DISCUSSIONS

To understand physically the preceding discussions, let us consider the imaginary part of $G_N(s)$ for the zero mass case :

$$\text{Im } G_N(s) = \pi \delta(s) + \pi \sum_{l=1}^N (2l)! \alpha^{2l} \rho_{2l+1}(s) \quad (34)$$

where $\rho_N(s)$ is the N particle (zero mass) phase space volume :

$$\rho_N(s) = \frac{1}{(N-2)!(N-1)!} \frac{1}{s} \left(\frac{s}{16\pi^2} \right)^{N-1} \theta(s)$$

The sum of Eq. (23) is well-defined even in the limit of $N \rightarrow \infty$:

$$\frac{1}{\pi} \text{Im } G_\infty(s) = \delta(s) + \frac{\beta}{4} \sinh(\beta s/4) \theta(s) \quad (35)$$

$$\beta = \alpha/4\pi^2$$

However, this quantity increases exponentially as $s \rightarrow \infty$, reflecting the fact that, as $s \rightarrow \infty$, the non-linear interaction allows multi-particle productions to fill the ever-increasing phase volume. That one cannot write down a dispersion relation with a finite number of subtractions means simply that the theory is non-renormalizable in the usual sense. Of course one can write down a spectral representation of the form

$$G_\infty(s) = a(s) + \frac{b(s)}{\pi} \int_0^\infty \frac{ds'}{s' - s} \frac{\text{Im } G_\infty(s')}{b(s')} \quad (36)$$

where $a(s)$ and $b(s)$ are entire functions, but this requires specifying an infinite number of coefficients. Clearly the function $G(s, \alpha^2; b)$ derived in the last Section has a generalized spectral representation of the form of Eq. (36). In this example, the arbitrariness in the choice of the entire function $a(s)$ is reflected by the arbitrary real parameter b . While the particular entire function $-bD(s, \alpha^2)$ arose naturally in our construction of $G(s, \alpha^2; b)$ from the regularized perturbation series, there does not seem to be any reason to prefer this particular, additive, entire function over others.

The function $G(s, \alpha^2; b)$ may have advantages over some others in that it has the asymptotic behaviour for large $|s|$, given by

$$\lim_{\substack{|s| \rightarrow \infty \\ \operatorname{Re} s < 0}} G(s, \alpha^2; b) = \frac{\text{const.}}{s}$$

$$\lim_{\substack{|s| \rightarrow \infty \\ \operatorname{Re} s \geq 0}} G(s, \alpha^2; b) = \text{const.} e^{\beta s/4}$$

i.e., $G(s, \alpha^2; b)$ behaves, for large $|s|$, like a free propagator in the left half s plane, while in the right half plane and on the imaginary axis, it increases just as much as the imaginary part of $G(s, \alpha^2; b)$ does. [If $b = 0$, then it is only the analytic continuation of the imaginary part that diverges as $|s| \rightarrow \infty$, $\operatorname{Re} s \geq 0$.] Thus, if higher order Feynman diagrams are formed from this propagator and the Feynman integrals are carried out after the Wick rotation of integration contours, then the integrals may again be finite. However, we do not know, for example, whether such a prescription will yield amplitudes which are unitary when external momenta are set to physical values, nor whether the consistency of the entire theory in all orders of perturbation theory will restrict the choice of the entire functions $a(s)$.

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- 9) This can be derived from the integral representation of $K_\nu(z)$:

$$2\left(\frac{z}{\zeta}\right)^{1/2} K_\nu(2\sqrt{z\zeta}) = \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-(zt + \zeta t^{-1})}$$

See, e.g. :
W. Magnus, F. Oberhettinger and R.P. Soni - "Formulas and Theorems for the Special Functions of Mathematical Physics", Springer Verlag, Berlin, p. 85 (1966).
- 10) See, for example :
E.T. Whittaker and G.N. Watson - "Modern Analysis", Cambridge University Press, p. 154 (1952).

- 11) See, for example :
W. Magnus, F. Oberhettinger and R.P. Soni - loc.cit. p. 342 et seq.
- 12) G.V. Efimov - Soviet Phys. JETP 17, 1417 (1963).
- 13) A similar prescription is given for the case in which $U[\phi(x)]$
is an exponential function of the field by M.K. Volkof,
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- 14) W. Magnus, F. Oberhettinger and R.P. Soni - loc.cit. p. 93.