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ANALYSIS OF DIVERGENCES IN A NEUTRAL SPIN 1 MESON THEORYWITH PARITY NON-CONSERVING INTERACTIONS +)

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A B S T R A C T

The unrenormalizable theory of the neutral spin 1 meson with a parity non-conserving interaction is analyzed. The formal series expansion of any transition amplitude in powers of the coupling constant g can be written as :

$$\text{lowest order expression} \times \left\{ 1 + \sum A_n (g^2 \Lambda^2)^n + \sum B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1} + \sum C_n g^2 (g^2 \Lambda^2)^{n-1} + \dots \right\}$$

where Λ is the four-momentum cut-off parameter. It is shown that these singular sums can be carried out for arbitrary processes. The most singular sum $\sum A_n (g^2 \Lambda^2)^n$ of all processes can be completely removed by a single mass renormalization. The next and the third most singular sums,

$$\sum B_n g^2 (\ln \Lambda^2) (g^2 \Lambda^2)^{n-1} \quad \text{and} \quad \sum C_n (g^2 \Lambda^2)^{n-1},$$

of different processes are also related ; these can be removed by introducing further renormalization constants.

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I. INTRODUCTION

It is well known that the theory of spin 1 mesons interacting with a non-conserved current is not renormalizable, and this has been the main difficulty which prevents one from evaluating high order weak interaction effects ^{1),2)} in the usual intermediate boson theory. In such a theory, the formal power series expansion of any transition amplitude in terms of the square of the coupling constant g^2 leads to an expression in which every higher order term in g^2 becomes increasingly more singular. It has been suggested that if one could sum over the most divergent part in the power series ³⁾ and then the next most divergent part, and so on, sensible high order weak interaction effects might emerge; that is, these higher order weak interaction effects would become finite and remain small.

In this paper, we will examine a special mathematical model of a neutral spin 1 field $W_\mu(x)$ interacting with a spin $\frac{1}{2}$ field $N(x)$ through a parity non-conserving interaction

$$g W_\mu j_\mu \quad (1)$$

where g corresponds to the dimensionless weak interaction coupling constant,

$$j_\mu = i N^\dagger \gamma_4 \gamma_\mu (a + b \gamma_5) N, \quad (2)$$

and a and b can be any numbers, but not zero; otherwise, parity would be conserved. The masses of both fields are not zero. The requirements $b \neq 0$ and the mass $m_N \neq 0$ imply that

$$\frac{\partial j_\mu}{\partial x_\mu} \neq 0, \quad (3)$$

and therefore the theory is non-renormalizable. As we shall see, in a power series expansion, the degrees of divergence in this example are just like those in the realistic case (in which W_μ is a charged field). Yet,

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explicit sums of the most singular part, and the next most singular part,... can be carried out for any transition amplitude. Rules can then be devised which enable one to obtain high order radiative corrections to any given order in g^2 . These higher order corrections are finite and remain small.

It will be shown in the next Section that the formal series expansion of any transition amplitude in powers of g^2 can be written in the form

$$\begin{aligned} & (\text{lowest order expression}) \times \left\{ 1 + \sum_1^{\infty} A_n (g^2 \Lambda^2)^n \right. \\ & + \sum_1^{\infty} B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1} + \sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1} \\ & \left. + \dots \right\} \quad (4) \end{aligned}$$

where the lowest order expression is in general finite, Λ is the usual cut-off parameter in four-momentum, and the coefficients $A_n, B_n, C_n, \dots$ are finite functions depending on the external momenta of the particular transition. To the order g^{2n} , the term $A_n (g^2 \Lambda^2)^n$ represents the most singular part, $B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1}$ the next most singular part, etc. A systematic method will be developed which allows one to carry out these sums

$$\sum A_n (g^2 \Lambda^2)^n, \quad \sum B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1},$$

etc., for arbitrary processes. It turns out that the most singular sums $\sum A_n (g^2 \Lambda^2)^n$ in different processes are simply related; they can be all removed by a single mass renormalization counter term

$$- \delta m_N N^\dagger \gamma_4 N \quad (5)$$

in the Lagrangian, where δm_N is given by

$$\delta m_N = m_N [\exp(2G\Lambda^2) - 1], \quad (6)$$

m_N is the physical mass of N , and G , similar to the usual Fermi constant in the weak interaction, is related to the boson mass m_W by

$$G = (g b / m_W)^2 \quad (7)$$

II. A MODEL OF WEAK INTERACTION

In order to see how these singular sums can be extracted from different processes in the model, it is most convenient to adopt the Stueckelberg formalism ⁴⁾ of the spin 1 field. We write for the W meson field

$$W_\mu = \phi_\mu + m_W^{-1} \frac{\partial \theta}{\partial x_\mu} \quad (10)$$

where ϕ_μ ($\mu = 1, 2, 3, 4$) and θ denote five independent canonical fields. Both the θ meson and the spin 1 part of ϕ_μ are of positive metric, but the spin 0 part of ϕ_μ is of negative metric. The interactions are constructed [see (12)-(14) below] such that

$$\frac{\partial \phi_\mu}{\partial x_\mu} + m_W \theta \quad (11)$$

always satisfies the free particle equation. Thus, all radiation processes involving the unphysical spin 0 ϕ_μ mesons are completely compensated by the corresponding processes involving θ mesons. If one confines only to reactions in which both the initial and the final states do not contain any spin 0 part of ϕ_μ meson, nor any θ meson, the resulting S matrix remains a unitary one.

The total Lagrangian for the model weak interaction is given by ⁵⁾

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{wk}} , \quad (12)$$

$$\begin{aligned} \mathcal{L}_{\text{free}} = & -\frac{1}{2} \left[\left(\frac{\partial \phi_\mu}{\partial x_\nu} \right)^2 + \left(\frac{\partial \theta}{\partial x_\nu} \right)^2 \right] - \frac{1}{2} m_W^2 (\phi_\mu^2 + \theta^2) \\ & - N^\dagger \gamma_4 \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_N \right) N \end{aligned} \quad (13)$$

In a similar but somewhat more complicated way, the other singular sums

$$\sum_1^{\infty} B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1} \quad \text{and} \quad \sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1}$$

in different processes are also all related: they can be removed by introducing further counter terms, corresponding to the mass renormalization of W , the various wave function renormalizations and coupling constant renormalizations. The details are given in Section III.

The weak interaction Lagrangian (1) and the free Lagrangian are clearly invariant under the U_1 transformation

$$N(x) \rightarrow e^{i\alpha} N(x) \quad (8)$$

provided α is a constant. Except for the mass m_N , it is also invariant under the chiral $U_1 \times U_1$ transformation

$$N(x) \rightarrow \exp(i\alpha + i\alpha' \gamma_5) N(x) \quad (9)$$

where α and α' are both constants.

It will be shown that the problem of high order weak interaction radiative corrections takes a particularly simple form if one assumes that $N(x)$ has, in addition to the weak interaction (1), also a parity conserving strong interaction which is invariant (except for some additional mass terms) under the same chiral $U_1 \times U_1$ transformation. In this case, both the strong and the weak interaction Lagrangians are non-renormalizable. A sharp distinction between the strong and the weak interactions lies in the former being parity conserving and the latter parity violating. It turns out to be much simpler to combine the parity conserving part of the weak interaction together with the strong interaction; their most divergent part, and the next most divergent part, ... can again be explicitly summed. The remaining parity violating weak interaction amplitude corresponds in fact to that of a usual renormalizable theory. The details are given in Section IV.

The general systematics of the present method is discussed in Section V.

It is most instructive to compare the Feynman graphs in terms of W_μ by using the original weak interaction Lagrangian (14) and those in terms of ϕ_μ and Θ by using the transformed Lagrangian (18). Although the N and ψ propagators generated by these two Lagrangians are not the same, they must have the same pole, and therefore the same mass shift. This is because these two Lagrangians being connected by a canonical transformation must give the same S matrix. The relationship between these different sets of graphs for the same mass shift is illustrated in Fig. 1. The left-hand side (i.e., the first row of graphs) is in terms of the W propagator by using (14) and (15); the most divergent terms are all due to the $(q_\mu q_\nu)$ term in the W propagator. The right-hand side (i.e., the second row, etc.) is in terms of Θ and ϕ_μ propagators by using the Lagrangian (18); the most divergent parts are completely due to closed loops of Θ lines which begin and end at the same space time point. The sum over all such loops is given by

$$m_N \langle \text{vac} / \exp(2iG^{\frac{1}{2}}\gamma_5\Theta) / \text{vac} \rangle = m_N \exp(-2G\Lambda^2) \quad (19)$$

where, for definiteness, the cut-off parameter Λ^2 is chosen to be related to an arbitrarily regularized Feynman function $D_F^\Lambda(x)$ by

$$D_F^\Lambda(0) = \Lambda^2. \quad (20)$$

To eliminate such singular sums, we introduce a counter term

$$-\delta m_N N^\dagger \gamma_4 N \quad (21)$$

in the original weak interaction Lagrangian (14). In terms of ψ , this counter term becomes

$$-\delta m_N \psi^\dagger \gamma_4 \exp(2iG^{\frac{1}{2}}\gamma_5\Theta) \psi$$

which, similarly to (19), gives an additional contribution

$$\delta m_N \exp(-2G\Lambda^2)$$

and

$$\mathcal{L}_{wk} = i g W_\mu N^\dagger \gamma_4 \gamma_\mu (a + b \gamma_5) N + \text{counter terms} \quad (14)$$

where all subscripts μ and ν are summed over from 1 to 4, m_W , m_N and g all refer to the renormalized quantities. The counter terms are introduced only for renormalization purposes; their precise forms will be determined later. We note that while the free propagator of

$$W_\mu = \phi_\mu + m_W^{-1} \frac{\partial \theta}{\partial x_\mu}$$

is given by the usual expression

$$-i (q^2 + m_W^2 - i\epsilon)^{-1} (\delta_{\mu\nu} + m_W^{-2} q_\mu q_\nu) \quad (15)$$

the free propagators of ϕ_μ and θ are given by

$$-i (q^2 + m_W^2 - i\epsilon)^{-1} \quad (16)$$

multiplied by $\delta_{\mu\nu}$ and 1 respectively. In terms of ϕ_μ and θ the non-renormalizable character of the theory rests entirely on the derivative coupling between $\theta(x)$ and $N(x)$.

Following Dyson ⁶⁾, we define

$$\psi(x) = \exp[-i g (a + b \gamma_5) \theta / m_W] N(x) \quad (17)$$

Upon substituting (17) into (13) and (14), one finds that the Lagrangian (12) becomes

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \left[\left(\frac{\partial \phi_\mu}{\partial x_\nu} \right)^2 + \left(\frac{\partial \theta}{\partial x_\mu} \right)^2 \right] - \frac{1}{2} m_W^2 (\phi_\mu^2 + \theta^2) \\ & - \psi^\dagger \gamma_4 \gamma_\mu \left[\frac{\partial}{\partial x_\mu} - i g (a + b \gamma_5) \phi_\mu \right] \psi \\ & - m_N \psi^\dagger \gamma_4 \exp(2i G^{\frac{1}{2}} \gamma_5 \theta) \psi + \text{counter terms} \end{aligned} \quad (18)$$

where G is given by (7).

III. FURTHER ANALYSIS OF DIVERGENCES

In this Section, we will be concerned only with graphs which do not contain any closed loops of θ lines that begin and end at the same space time point; these have already been removed by the mass renormalization term (21). As we shall see, the removal of these closed θ loops also reduces the next most singular sums

$$\sum_1^{\infty} B_n g^2 \ln \Lambda^2 (g^2 \Lambda^2)^{n-1}$$

to simple $g^2 \ln \Lambda^2$ divergences. After the mass renormalization of m_N , the corresponding series expansion (4) of any transition amplitude becomes then

$$(\text{lowest order expression}) \times \left\{ 1 + B g^2 \ln \Lambda^2 + \sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1} + \dots \right\}. \quad (25)$$

Both $B g^2 \ln \Lambda^2$ and $\sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1}$ will be studied in this Section. We note that in the sum

$$\sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1}$$

the first term $C_1 g^2$ is always finite; this term will become the $O(g^2)$ radiative correction to the final renormalized amplitude. The other divergent terms, $B g^2 \ln \Lambda^2$ and $C_n g^2 (g^2 \Lambda^2)^{n-1}$ for $n > 1$, are all linear functions of external momenta; they contribute only to the appropriate renormalization constants. The details will be given in the following. Throughout this Section, only the Lagrangian given by (18) will be used.

to the mass term in the N propagator. The renormalized mass m_N is, then, related to the unrenormalized mass $(m_N + \delta m_N)$ by

$$m_N = (m_N + \delta m_N) \exp(-2G\Lambda^2). \quad (22)$$

A remarkable fact is that by introducing this mass renormalization counter term (21), one eliminates not only the most singular part in the N propagator, but also the most singular sum $\sum_n A_n (g^2 \Lambda^2)^n$ for all processes. This can be seen by noting that the most singular part for any process can be obtained from its lowest order graphs by attaching to each Θ vertex an arbitrary additional number of closed loops of Θ lines which all begin and end at the same vertex point. If in the lowest order graph this Θ vertex is represented by

$$2G^{\frac{1}{2}} (m_N + \delta m_N) \gamma_5 \Theta, \quad (23)$$

the summing over all such closed Θ loops merely gives a multiplicative factor $\exp(-2G\Lambda^2)$, changing the Θ vertex from (23) to

$$2G^{\frac{1}{2}} m_N \gamma_5 \Theta \quad (24)$$

where m_N is the renormalized mass.

10.

and

$$\sum_{\theta}(p) = (4\pi)^{-2} G m_N^2 (\ln \Lambda^2) (2\not{p} - 4m_N) + g^2 O(1) + \sum_{\theta}^{ren}(p) \quad (30)$$

where $\sum_{\phi}^{ren}(p)$ and $\sum_{\theta}^{ren}(p)$ are both finite as $\Lambda^2 \rightarrow \infty$. The $g^2 O(1)$ terms denote the appropriate finite linear functions of $\not{p}$; they are chosen such that, as $\not{p} \rightarrow m_N$, both $\sum_{\phi}^{ren}(p)$ and $\sum_{\theta}^{ren}(p)$ are proportional to $(\not{p} - m_N)^2$.

The summation over all graphs in the first three rows of Fig. 2 is straightforward but tedious. We give only the final result. Let $N_{ren}(x)$ and $S_{ren}(p)$ be, respectively, the renormalized field operator and the renormalized N propagator, related to the corresponding unrenormalized field operator $N(x)$ and the unrenormalized N propagator $S(p)$ by

$$N(x) = Z_N^{\frac{1}{2}} \exp(-\lambda \gamma_5) N_{ren}(x) \quad (31)$$

and

$$S(p) = Z_N \exp(-\lambda \gamma_5) S_{ren}(p) \exp(\lambda \gamma_5) \quad (32)$$

where

$$S_{ren}^{-1}(p) = -i \left[(\not{p} - m_N) + \sum_{\phi}^{ren}(p) + \sum_{\theta}^{ren}(p) \right] \quad (33)$$

is finite,

$$Z_N = \left[(1 + F_1^0)^2 - (F_2^0)^2 \right]^{-\frac{1}{2}}, \quad (34)$$

$$\sinh 2\lambda = Z_N F_2^0 \quad (35)$$

and

$$\cosh 2\lambda = Z_N (1 + F_1^0). \quad (36)$$

1. N propagator

The unrenormalized N propagator can be written as

$$S^{-1}(p) = -i \left[\not{p} - (m_N + \delta m_N) \exp(-2G\Lambda^2) + \Sigma(p) \right] \quad (26)$$

where

$$\not{p} = -i \gamma_\mu p_\mu,$$

$(m_N + \delta m_N) \exp(-2G\Lambda^2)$ contains the sum over the most singular diagrams given previously in Fig. 1, and $\Sigma(p)$ refers to the sum over the less singular diagrams given in Fig. 2. It is useful to exhibit the spinor dependence of $\Sigma(p)$:

$$\Sigma(p) = \not{p} F_1(p^2) + \not{p} \gamma_5 F_2(p^2) - F_3(p^2). \quad (27)$$

By definition, $S(p)$ has a pole at $p^2 = -m_N^2$. Thus, the renormalized mass m_N is given by

$$m_N = \left[(m_N + \delta m_N) \exp(-2G\Lambda^2) + F_3^0 \right] \left[(1 + F_1^0)^2 - (F_2^0)^2 \right]^{-\frac{1}{2}} \quad (28)$$

where F_i^0 ($i=1, 2$ and 3) denotes the value of $F_i(p^2)$ at $p^2 = -m_N^2$. Equation (28) reduces to the previous expression (22), if only the most singular diagrams (given in Fig. 1) are kept, in which case $F_i^0 = 0$.

To evaluate the $Bg^2 \ln \Lambda^2$ term and the sum $\sum_1^\infty C_n g^2 (g^2 \Lambda^2)^{n-1}$ in $\Sigma(p)$, we need only to consider the graphs listed in the first three rows of Fig. 2. Among these only the first two graphs contribute to $Bg^2 \ln \Lambda^2$ and $C_1 g^2$. These two graphs are respectively given by

$$\begin{aligned} \Sigma_\phi(p) = & (4\pi)^{-2} g^2 (\ln \Lambda^2) \left[(a^2 + b^2) \not{p} + 2ab \not{p} \gamma_5 - 4m_N (a^2 - b^2) \right] \\ & + g^2 O(1) + \sum_\phi^{ren} (p) \end{aligned} \quad (29)$$

In deriving these expressions for F_1^0 and F_2^0 we only need to sum over the first row of graphs in Fig. 2. Those in the second and third rows only contribute to the $g^4 \Lambda^2$, $g^6 \Lambda^4$, ..., terms in F_3^0 . Furthermore, as noted before, the finite g^2 corrections, $\sum_{\phi}^{\text{ren}}(p)$ and $\sum_{\theta}^{\text{ren}}(p)$, are due entirely to the first two graphs in Fig. 2; they are simply the $C_1 g^2$ term in the sum $\sum_1^{\infty} C_n g^2 (g^2 \Lambda^2)^{n-1}$.

2. W propagator

Let $D_{\mu\nu}^{\phi}(k)$ be the sum of all W propagator graphs in which both the initial and the final states are ϕ_{μ} (not θ). We write

$$[D^{\phi}(k)]_{\mu\nu}^{-1} = i \left[(k^2 + m_W^2 + \delta m_W^2) \delta_{\mu\nu} + \pi_{\mu\nu} \right] \quad (41)$$

where m_W is the physical W mass, $(m_W^2 + \delta m_W^2)$ is the square of the unrenormalized W mass and $\pi_{\mu\nu}$ denotes the sum of all graphs given in Fig. 3.

The first graph in Fig. 3 is closely related to the lowest order vacuum polarization diagram in the usual electrodynamics. It can be readily evaluated in the same way, and it is given by

$$g^2 (4\pi^2)^{-1} \ln \Lambda^2 \left[\frac{1}{3} (a^2 + b^2) (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) - 2 m_N^2 b^2 \delta_{\mu\nu} \right] + g^2 O(1) + \pi_{\mu\nu}^{\text{ren}}(k) \quad (42)$$

where $\pi_{\mu\nu}^{\text{ren}}(k)$ is finite as $\Lambda^2 \rightarrow \infty$, and $g^2 O(1)$ denotes a finite linear function in k^2 , consisting of two terms, a constant times $(k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu})$ and another constant times $\delta_{\mu\nu}$. These constants in $g^2 O(1)$ are so chosen such that as $k^2 \rightarrow -m_W^2$, the $\delta_{\mu\nu}$ part of $\pi_{\mu\nu}^{\text{ren}}(k)$ is proportional to $(k^2 + m_W^2)^2$.

The constants F_1^0 and F_2^0 are given by

$$F_1^0 = \int i d^4x D_N \left\{ g^2 (a^2 + b^2) D_W + m_N^2 (4G D_W)^{-1} [\exp(4G D_W) - (4G D_W) - 1] \right\} \quad (37)$$

and

$$F_2^0 = 2 a b g^2 \int i D_N D_W d^4x \quad (38)$$

where $D_N(x)$ is the usual Feynman function, i.e.,

$$D_N(x) = -i(2\pi)^{-4} \int (q^2 + m_N^2 - i\epsilon)^{-1} \exp(iq \cdot x) d^4q, \quad (39)$$

and $D_W(x)$ is a regularized Feynman function given by

$$D_W(x) = -i(2\pi)^{-4} \int (q^2 + m_N^2 - i\epsilon)^{-1} [q^2 + (4\pi\Lambda)^2 - i\epsilon]^{-2} \cdot (4\pi\Lambda)^4 \exp(iq \cdot x) d^4q. \quad (40)$$

At the origin $x_\mu = 0$,

$$D_W(0) = \Lambda^2.$$

Both constants F_1^0 and F_2^0 are real and positive; they both diverge as $\Lambda^2 \rightarrow \infty$, since for large Λ^2 ,

$$\int i D_N D_W d^4x \rightarrow (4\pi)^{-2} \ln \Lambda^2$$

and

$$\int i D_N D_W^\ell d^4x \rightarrow O(\Lambda^{2\ell-2})$$

for $\ell \geq 2$.

$$- g \gamma_\mu (a + b \gamma_5) \quad (47)$$

between W_μ and N . In order that (ga) and (gb) refer to the renormalized coupling constants, we must add to the original weak interaction Lagrangian (14) a counter term

$$i W_\mu N^\dagger \gamma_4 \gamma_\mu [\delta(ga) + \gamma_5 \delta(gb)] N. \quad (48)$$

Let Γ_μ be the sum of all vertex graphs given in Fig. 4 in which the external meson refers to ϕ_μ , the spin 1 part of W_μ . The first graph is the lowest order term in the series expansion (25). The second and the third graphs contribute to $Bg^2 \ln \Lambda^2$ and $C_1 g^2$; these two graphs are respectively given by

$$(\Gamma_\phi)_\mu = -i g^3 \gamma_\mu [a^3 + 3ab^2 + (3a^2b + b^3) \gamma_5] \int D_N D_W d^4x + (\Gamma_\phi^{\text{ren}})_\mu \quad (49)$$

and

$$(\Gamma_\theta)_\mu = -2i g m_N^2 G \gamma_\mu (a - b \gamma_5) \int D_N D_W d^4x + (\Gamma_\theta^{\text{ren}})_\mu \quad (50)$$

where G is given by (7), D_N and D_W are given by (39) and (40), and $(\Gamma_\phi^{\text{ren}})_\mu$ and $(\Gamma_\theta^{\text{ren}})_\mu$ are both finite.

The other graphs in Fig. 4 give the rest of the sum over $C_n g^2 (g^2 \Lambda^2)^{n-1}$ for $n > 1$. Their total contribution is

$$-\frac{1}{2} i g m_N^2 \gamma_\mu (a - b \gamma_5) \int D_N [\exp(4GD_W) - (4GD_W) - 1] d^4x. \quad (51)$$

The other diagrams in Fig. 3 are proportional to $g^4 \Lambda^2$, $g^6 \Lambda^4$, etc., as $\Lambda^2 \rightarrow \infty$. These singular terms are independent of the external momentum k , and therefore contribute only to the mass renormalization δm_W .

Let $[D_{\mu\nu}^\phi(k)]_{\text{ren}}$ be the renormalized propagator, related to the unrenormalized propagator $D_{\mu\nu}^\phi(k)$ by

$$D_{\mu\nu}^\phi(k) = Z_\phi [D_{\mu\nu}^\phi(k)]_{\text{ren}}. \quad (43)$$

We find [accurate only up to $O(g^2 \ln \Lambda^2)$]

$$Z_\phi^{-1} = 1 + \frac{1}{3} g^2 (4\pi^2)^{-1} (a^2 + b^2) \ln(\Lambda^2/m_N^2) \quad (44)$$

The renormalized W propagator is finite, and is given by

$$[D_{\text{ren}}^\phi(k)]_{\mu\nu}^{-1} = i [(k^2 + m_W^2) \delta_{\mu\nu} + \pi_{\mu\nu}^{\text{ren}}] \quad (45)$$

There are other W propagators in which either the initial meson, or the final meson (or both), is Θ . The singular parts of these propagators can be similarly summed over. Their explicit expressions are rather complicated, and will not be given here.

3. $W \rightarrow N + \bar{N}$

Assuming that

$$m_W > 2 m_N,$$

the physical spin 1 meson can decay into a fermion pair

$$W \rightarrow N + \bar{N} \quad (46)$$

It is convenient to use the amplitude of this physical process to define the renormalized coupling

$$\begin{aligned}
N + W &\rightarrow N + W, \\
N + W &\rightarrow N + W + W, \\
N + W &\rightarrow N + W + W + W,
\end{aligned}
\tag{54}$$

etc., are now finite; i.e., the apparent divergent expressions, up to and including the third most singular sum: $\sum C_n g^2 (g^2 \Lambda^2)^{n-1}$, are completely removed by the set of renormalization constant introduced above. In (54), all W bosons are the physical ones, and therefore of spin 1.

There are other processes, such as N - N scattering and W - W scattering, for which additional renormalizations are required. A full analysis lies outside the scope of the present paper. In the following, we will only give some brief discussions. The scattering

$$N + N \rightarrow N + N$$

consists of two parts, a parity conserving amplitude and a parity violating amplitude. It can be shown that the singular part

$$g^2 \left\{ B g^2 \ln \Lambda^2 + \sum C_n (g^2 \Lambda^2)^{n-1} g^2 \right\} \tag{55}$$

in the parity violating amplitude is also completely removed by the same renormalization constants that have already been introduced. There is, however, a remaining singular part in the parity conserving amplitude, which can only be removed by introducing an additional counter term

Combining these results together with the appropriate factors $Z_{\phi}^{\frac{1}{2}}$ and $Z_N^{\frac{1}{2}} \exp(-\lambda \gamma_5)$ for the external meson ⁷⁾ and nucleon lines, we find the renormalized coupling constants g_a and g_b are related to the corresponding unrenormalized coupling constants $g_a + \delta(g_a)$ and $g_b + \delta(g_b)$ by

$$\delta(g_a) = g_a \left\{ (6\pi)^{-1} g^2 (4\pi^2)^{-1} (a^2 + b^2) \ln(\Lambda^2/m_N^2) - \frac{i}{2} m_N^2 \int D_N (4GD_W)^{-1} \left[(4GD_W - 2) \exp(4GD_W) + 4GD_W + 2 \right] d^4x \right\} \quad (52)$$

and

$$\delta(g_b) = g_b \left\{ (6\pi)^{-1} g^2 (4\pi^2)^{-1} (a^2 + b^2) \ln(\Lambda^2/m_N^2) + \frac{i}{2} m_N^2 \int D_N (4GD_W)^{-1} \left[(4GD_W + 2) \exp(4GD_W) - 12GD_W - 2 \right] d^4x \right\}. \quad (53)$$

These expressions are only accurate to their singular terms $g^3 \ln \Lambda^2$, $g^5 \Lambda^2$, $g^7 \Lambda^4$, etc.

4. Other results

We note that from the view point of the S matrix, all above renormalizations are merely definitions. Physical masses can only be defined by examining poles in the relevant one-particle propagators. Physical coupling constants can only be defined by considering some definite physical processes. With the above definitions, it can be readily verified that a large class of reactions such as

IV. CHIRAL $U_1 \times U_1$ INVARIANCE

The Lagrangian (18) is invariant under the U_1 transformation

$$\psi(x) \rightarrow e^{i\alpha} \psi(x) \quad (58)$$

where α is a constant. Except for the mass term $-\frac{1}{2}m_W^2\theta^2$, it is also invariant under

$$\psi(x) \rightarrow e^{i\alpha'\gamma_5} \psi(x) \quad (59)$$

and

$$\theta(x) \rightarrow \theta(x) - \frac{1}{2}\alpha' \quad (60)$$

provided α' is a constant.

It is of interest to consider a model of strong interaction which, except for some other mass terms, is also invariant under a chiral $U_1 \times U_1$ transformation

$$N(x) \rightarrow e^{i(\beta + \beta'\gamma_5)} N(x) \quad (61)$$

where β and β' are arbitrary constants. The simplest model is to assume an additional Hermitian pseudoscalar field $\pi(x)$ which transforms as

$$\pi(x) \rightarrow \pi(x) - f^{-1}\beta' \quad (62)$$

under the chiral $U_1 \times U_1$ transformation. The strong interaction is given by

$$m_N N^\dagger \gamma_4 \exp(2if\gamma_5\pi) N \quad (63)$$

$$\delta G_1 (N^\dagger \gamma_4 N)(N^\dagger \gamma_4 N) + \delta G_2 (N^\dagger \gamma_4 \gamma_5 N)(N^\dagger \gamma_4 \gamma_5 N) \quad (56)$$

in the Lagrangian. This counter term is adjusted to give only a final g^4 correction to the N-N scattering amplitude. The $O(g^2)$ N-N scattering amplitude is completely given by the usual finite lowest order Feynman diagram. The radiative correction is finite, $O(g^4)$, and therefore remains small at least at low energy. Furthermore, the infinities induced by the new contact term (56) are less singular than those that have already been summed up.

The scattering

$$W + W \rightarrow W + W \quad (57)$$

contains, as usual, additional divergent terms which can be removed by introducing further renormalization counter terms. Again, we require that the resulting amplitude for W-W scattering consists of a finite g^4 term, plus a finite radiative correction term proportional to g^6 .

V. REMARKS

Throughout our analysis for the model weak interaction, the existence of a well-defined fundamental theory is implicitly assumed. If higher order radiative corrections are neglected, then the entire S matrix of this well-defined fundamental theory should be identical with that obtained by using the lowest order expressions of the Lagrangian (12). Except for W - W scatterings, these lowest order expressions are the same as "tree approximations". To any given order of radiative corrections, the entire S matrix can be evaluated with a certain set of renormalized parameters. These higher order radiative corrections are finite; in addition, they remain small, at least at low energy. This is reflected by our choice of the appropriate renormalization constants. For example, the counter term (56) is chosen to make only a $O(g^4)$ correction to the N - N scattering amplitude. The corresponding $O(g^2)$ amplitude is completely determined by the lowest order diagram. It is clear that beyond the lowest order expression, with each increasing order of accuracy in g^2 , the method requires a larger set of parameters, and therefore it gives a smaller set of results. These parameters should be calculable in a fundamental theory, but not by the present method. Yet, to any given order of accuracy g^2 , the present method could, nevertheless, reproduce the correct results with the aid of these renormalization constants. Throughout our analysis, we do not assume any relations between the fictitious momentum cut-off parameter Λ and the Fermi constant G . In a well-defined theory, Λ should be infinite and G finite. [See footnote 3).] These renormalization constants are used to express that sector of the theory which is insensitive to how Λ approaches infinity.

In the model, if $b=0$, then the Lagrangian (12) becomes renormalizable in the conventional sense ⁹⁾. In this case

$$A_n = 0 \quad \text{for} \quad n \geq 1$$

and

$$B_n = C_n = 0 \quad \text{for} \quad n \geq 2.$$

The analysis given in Sections II and III reduces to the usual renormalization method.

where f is the strong interaction coupling constant, This new chiral transformation is broken only by the mass of π field. These two chiral $U_1 \times U_1$ transformations can be trivially multiplied together, and the product is

$$\psi(x) \rightarrow \exp[i(\alpha + \beta) + i(\alpha' + \beta') \gamma_5] \psi(x). \quad (64)$$

It is then natural to combine the strong interaction (63) with the θ dependent part of the weak interaction. By using (17), (63) becomes

$$m_N \bar{\psi}(x) \gamma_4 \exp[2i(f\pi + G^{\frac{1}{2}}\theta) \gamma_5] \psi(x) \quad (65)$$

which is parity conserving. The remaining part of the weak interaction

$$ig \bar{\psi} \gamma_4 \gamma_\mu (a + b \gamma_5) \psi \phi_\mu \quad (66)$$

is parity violating; it is renormalizable in the usual sense.

The parity conserving Lagrangian (63), or (65), may serve as a simple model of the various strong interaction phenomenological Lagrangians⁸⁾ that have been extensively discussed in the literature. It is unrenormalizable, but its infinities can be analyzed by the techniques developed in the previous Sections. In a limited sense, the method of summing up the most divergent series, then the next most divergent series, and so on, can give some meanings to such a strong interaction Lagrangian beyond the conventional tree approximations.

The present method can be regarded as a generalization of the conventional renormalization procedures, applicable for an unrenormalizable theory; it also serves as a generalization of the usual phenomenological Lagrangian approaches, but beyond the "tree approximations".

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A P P E N D I X

The free Lagrangian (13) can be generalized to

$$\begin{aligned} \mathcal{L}_{\text{free}} = & -\frac{1}{4} \left(\frac{\partial \phi_\mu}{\partial x_\nu} - \frac{\partial \phi_\nu}{\partial x_\mu} \right)^2 - \frac{1}{2} \xi \left(\frac{\partial \phi_\nu}{\partial x_\nu} \right)^2 - \frac{1}{2} m_W^2 \phi_\nu^2 \\ & - \frac{1}{2} \left(\frac{\partial \theta}{\partial x_\nu} \right)^2 - \frac{1}{2} M^2 \theta^2 + \lambda \phi_\nu \frac{\partial \theta}{\partial x_\nu} \end{aligned} \quad (\text{A.1})$$

The total Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{free}} + g j_\mu \left(\phi_\mu + \frac{1}{m_W} \frac{\partial \theta}{\partial x_\nu} \right) \quad (\text{A.2})$$

In (A.1), ξ is positive, and λ is chosen to be

$$\lambda = \xi^{\frac{1}{2}} M - m_W \quad (\text{A.3})$$

From (A.1) and (A.2), it follows that

$$\begin{aligned} \left[\frac{\partial^2}{\partial x_\nu^2} - \xi^{-1} (m_W^2 - \lambda^2) \right] \frac{\partial \phi_\nu}{\partial x_\nu} + \xi^{-1} \lambda M^2 \theta \\ = -g \xi^{-1} (1 + m_W^{-1} \lambda) \frac{\partial j_\nu}{\partial x_\nu} \end{aligned}$$

and

$$\left[\frac{\partial^2}{\partial x_\nu^2} - M^2 \right] \theta - \lambda \frac{\partial \phi_\nu}{\partial x_\nu} = g m_W^{-1} \frac{\partial j_\nu}{\partial x_\nu}$$

Thus, independently of $(\partial j_\nu / \partial x_\nu)$,

$$\frac{\partial \phi_\nu}{\partial x_\nu} + \xi^{-\frac{1}{2}} M \theta \quad (\text{A.4})$$

satisfies the free particle equation

$$\left(\frac{\partial^2}{\partial x_\lambda^2} - \mu^2 \right) \left(\frac{\partial \phi_\nu}{\partial x_\nu} + \xi^{-\frac{1}{2}} M \theta \right) = 0 \quad (\text{A.5})$$

where

$$\mu^2 = (m_W M / \xi^{\frac{1}{2}}) \quad (\text{A.6})$$

If $j_\nu = 0$, the mass of the spin 1 part of ϕ_μ is m_W ; because of the constraint (A.3), both the mass of θ and that of the spin 0 part of ϕ_μ equal to μ . According to (A.6), μ may be different from m_W .

The special case $m_W = \mu$ can be obtained if $\xi = 1$ and $M = m_W$ (therefore, $\lambda = 0$). In this case, (A.4) reduces to (11).

REFERENCES AND FOOTNOTES

- 1) Higher order weak interaction effects have been studied in the uncrossed ladder approximation by G. Feinberg and A. Pais, Phys.Rev. 131, 2724 (1963) and Phys.Rev. 133, B477 (1964). For the model weak interaction discussed in this paper, the most divergent part of every uncrossed ladder diagram is cancelled by that of the corresponding crossed diagrams.
- 2) For more recent works on the systematics of divergences in the intermediate boson weak interaction theory, see M. Gell-Mann et al. (preprint); B.L. Ioffe and E.B. Shabalin, J.Nuclear Phys. (USSR) 6, 828 (1967); R.N. Mohapatra et al., Phys.Rev.Letters 20, 1081 (1968); C. Bouchiat et al. (CERN preprint).
- 3) It may be instructive to recall some physical problems in which the technique of summing up the most divergent parts, and then the next most divergent parts, and so on, has been successful. For example, the ground state energy E of a system of N Bose hard spheres of diameter a in a box of volume L^3 can be expressed as a formal power series in a . As $L \rightarrow \infty$, but keeping $\rho = N/L^3$ fixed, this power series becomes increasingly singular; its most divergent part is given by (in units of $\hbar = 2m = 1$)

$$(E/N) = 4\pi\rho a + (a^2 N/L^4) \left[A_0 + A_1 (aN/L) + A_2 (aN/L)^2 + \dots \right]$$

where $A_0, A_1, A_2, \dots$ are finite constants. Yet, the explicit sum of the series

$$F(aN/L) = \sum_0^{\infty} A_n \left(\frac{aN}{L} \right)^n$$

gives, as $(aN/L) \rightarrow \infty$, $F(aN/L) \rightarrow \text{constant} \times (aN/L)^{\frac{1}{2}}$ and, therefore,

as $L \rightarrow \infty$ at fixed ρ (but $\rho a^3 \ll 1$),

$$(E/N) = 4\pi\rho a \left[1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{\frac{1}{2}} \right]$$

For details see T.D. Lee, Kerson Huang and C.N. Yang, Phys.Rev. 106, 1135 (1957).

We emphasize that in order to obtain the correct asymptotic behaviour of $F(aN/L)$ as $L \rightarrow \infty$, at fixed ρ , one must use the correct series $\sum A_n (aN/L)^n$, and not tolerate any approximations on A_n . Another point is that one must not terminate the series at a finite n ; for example, if one stops at the first divergent term $A_1(aN/L)$ in the series and requires that its contribution to (E/N) should be smaller than the lowest order expression $4\pi\rho a$, then one might reach the ridiculous conclusion that the physical box dimension L cannot be bigger than $O(1/\rho a^2)$. Similarly, in the usual weak interaction theory, one must be careful in accepting results derived from summing over certain particular series of graphs, which are not the most divergent series, but are included only because of being summable. Also, it is totally unclear whether there is much meaning to the various estimations of the four-momentum cut-off parameter Λ in the weak interaction by including only the first divergent term $(G\Lambda^2)$ in the power series expansion.

- 4) E.C.G. Stueckelberg, Helv.Phys.Acta 11, 225 and 299 (1938).

The method used in the paper is a variant of the original Stueckelberg formalism. The spin 0 part of ϕ_μ is assumed to be of negative metric, but positive energy, instead of positive metric and negative energy. This is necessary in order to have $\epsilon = 0+$ in the ϕ_μ propagator $-i \delta_{\mu\nu} (q^2 + m_W^2 - i\epsilon)^{-1}$; otherwise, the spin 0 part would have $\epsilon = 0-$.

- 5) For simplicity, we assume in (13) the same mass for ϕ_μ and θ . Actually, in order to satisfy unitarity for the S matrix, it is only necessary to assume the mass of θ , denoted by μ , to be the same as that of the spin 0 part of ϕ_μ . The mass m_W of the spin 1 part of ϕ_μ can be different from μ . See Appendix for a discussion of the general case. It is important to note that if the renormalized masses m_W and μ are equal, then, in general, their unrenormalized masses would be different, and vice versa.
- 6) F.J. Dyson, Phys.Rev. 73, 929 (1948).
- 7) Here, we are only interested in the decay of the physical W into $N-\bar{N}$. The external boson must be of spin 1. Therefore, there are no contributions from graphs in which the external ϕ_μ is connected through vacuum polarization to a θ line, which is then attached to the $\bar{N}N\theta$ vertex.
- 8) J. Cronin, Phys.Rev. 161, 1483 (1967).
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- 9) See, R.J. Glauber, Prog.Theoret.Phys. 9, 295 (1953).

FIGURE CAPTIONS

Figure 1 : Feynman diagrams for δm_N .

Figure 2 : Feynman diagrams for N propagator.

Figure 3 : Feynman diagrams for ϕ propagator.

Figure 4 : Feynman diagrams for the vertex $W \rightarrow N\bar{N}$. (In these diagrams, the external ϕ refers to the physical spin 1 boson W.)

+ less singular diagrams.

FIG.1

FIG.2

FIG.3

FIG.4