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A RELATIVISTIC FORMULATION OF THE QUARK MODEL FOR MESONS ^{*)}

C.H. Llewellyn Smith

CERN - Geneva

A B S T R A C T

Mesons are treated as bound states in quark-antiquark scattering using the Bethe-Salpeter equation (this does not necessarily imply that the properties of mesons are dominated by the $Q-\bar{Q}$ state). It is shown that the existence of very tightly bound states with a non-relativistic spinor structure (such as are considered in the usual non-relativistic quark model) requires a Dirac scalar potential (or sum of irreducible graphs) as well as small internal three momentum ($\vec{q}^2 \ll M_Q^2$). The assumption that the pseudo-scalar mesons have a particular simple relativistic spinor structure leads naturally to the correct value of $\Gamma(\pi \rightarrow \rho \nu) / \Gamma(K \rightarrow \rho \nu)$ (in contrast to the usual model) and gives the prediction

$$\gamma_\rho^2 : \gamma_\phi^2 : \gamma_\omega^2 = M_\rho^2 : \frac{3M_\phi^2}{\cos^2 \theta} : \frac{3M_\omega^2}{\sin^2 \theta}$$

(where γ_V are the usual $V-\gamma$ coupling constants and θ is the $\omega-\phi$ mixing angle) which is compatible with the data. The assumption that the internal momentum is small and the potential is spin-independent can lead to approximate $SU(6)$ symmetry, as in the usual model, although the spinor structure is relativistic. Using the value of the weak axial vector coupling constant for quarks given by the non-relativistic quark model for baryons, these assumptions lead to the Kawarabayashi-Suzuki relation $2 \gamma_\rho f_\pi = M_\rho$ which agrees with experiment. The states which have $L = 1$ in the usual model are discussed. The model leads naturally to quadratic mass formulae for mesons. The processes $\pi^0 \rightarrow \gamma\gamma$ and $\eta^0 \rightarrow \gamma\gamma$ are discussed using the relativistic formalism.

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1. - INTRODUCTION *)

The non-relativistic quark model has been very successful in classifying hadrons and describing many of their properties ³⁾. The basis of this model is the fact, pointed out by Morpurgo ⁴⁾, that very tightly bound particles (for which the bound state mass $M_B \ll$ mass of the constituent particles M_Q) may have small relative momentum $\vec{q}$ (i.e., $\vec{q}^2 \ll M_Q^2$) ; in this case it is usually assumed that non-relativistic concepts are applicable. Using a non-relativistic description and a few simple assumptions, it is possible to derive many interesting results - particularly an understanding of the successes of SU(6) symmetry - although some of the assumptions are difficult to justify. If, however, the ratio

$$R = \frac{\Gamma(\pi \rightarrow \mu \nu)}{\Gamma(K \rightarrow \mu \nu)} \quad (1.1)$$

is considered in the framework of this model, it leads to the result

$$\frac{\Phi^\pi(0)}{\Phi^K(0)} = \sqrt{\frac{M_\pi}{M_K}} \approx \frac{1}{1.9} \quad (1.2)$$

where $\Phi(\vec{r})$ is the internal wave function of the π or K meson which is normalized to one ^{5),6)}. This result is hard to reconcile with the assumption of approximate invariance under unitary transformations between quarks, which is the basis of the quark model and the fundamental assumption of all other quark model calculations.

Thus Eq. (1.2) presents a difficulty for the naïve non-relativistic quark model. To discover whether this difficulty is a special feature of the non-relativistic model or whether the value of R [Eq. (1.1)] requires large symmetry breaking in all quark models, it is necessary to use a relativistic formalism. In this paper we discuss the quark model for mesons in terms of the fermion-antifermion Bethe-Salpeter (B.S.) equation ^{7),8)}.

*) A more primitive version of some of this work was given in Ref. ¹⁾. Some of the most important results have been briefly summarized in Ref. ²⁾.

In Section 2, we collect some necessary formalism ; particular emphasis is placed on the normalization condition for the bound state B.S. wave function because it is a vital ingredient of any practical calculation. In Section 3, we discuss the assumptions which we make about quarks.

In Section 4, we discuss the different structures which the wave functions of the pseudoscalar mesons can have. It turns out that the assumption $\vec{q}^2 \ll M_Q^2$ never necessarily leads to a non-relativistic spinor structure, as has previously been assumed ; in fact a non-relativistic spinor structure can only occur if the potential (sum of irreducible graphs) is a Dirac scalar. In Section 5, we show that the value of R [Eq. (1.1)] is simply explained if the wave functions of the pseudoscalar mesons belong to a particular (relativistic) class. In Section 6, assuming that the wave functions of the vector mesons have a similar relativistic structure, we calculate the ratio of the coupling constants in the virtual processes $\rho \rightarrow \gamma$, $\omega \rightarrow \gamma$ and $\phi \rightarrow \gamma$; this provides a test of the model.

Does this relativistic model retain the successful features of the non-relativistic model ? The most attractive of these was the simple explanation of the successes of $SU(6)$ symmetry ; in Section 7 we show that the usual assumptions that $\vec{q}^2 \ll M_Q^2$ and that the potential is spin independent, can lead to the results of approximate $SU(6)$ for mesons - even if the states have a relativistic spinor structure. If these assumptions are satisfied it is possible to derive a relation between $\Gamma(\rho \rightarrow \gamma)$ and $\Gamma(\pi \rightarrow \mu \nu)$ which agrees with experiment (the same relation was derived by Kawarabayashi and Suzuki ⁹⁾ using current algebra and PCAC ; however, their derivation is ambiguous ¹⁰⁾).

In fact the relativistic model advocated leads naturally to all the results for meson annihilation which were previously obtained in the non-relativistic model using the symmetry breaking prescription ^{5),6)} :

$$\Phi^m(0) = C \sqrt{M_m} \quad (1.3)$$

where C is a constant for all mesons and M_m is the mass of the meson m considered. In connection with the success of Eq. (1.3), Weisskopf and Van Royen comment that : "There must be a more natural reason for it, which may be connected with the limitations of the model" ⁶⁾. We believe that the limitation was the non-relativistic assumption.

In Section 8 the mesons which have $L = 1$ in the non-relativistic model are discussed. In Section 9 it is pointed out that the model discussed here leads naturally to quadratic mass formulae. The treatment of other decays using the relativistic formalism is discussed in Section 10.

The implications of this work are discussed in Section 11.

Some relevant equations are collected in Appendix A. A soluble model using a factorizing potential is presented in Appendix B; this provides an explicit illustration of some of the general conclusions in the rest of the paper. In Appendix C, we show that the numerical solution of the B.S. equation (in a simple approximation) carried out by Narayanaswamy and Pagnamenta gives a ground state which almost certainly has the relativistic structure which we favour. Appendix D is concerned with the B.S. equation for zero mass bound states which has sometimes been wrongly formulated in the past.

2. - NECESSARY FORMALISM

Our discussion of mesonic states in the quark model will be based on the Bethe-Salpeter (B.S.) equation for fermions. To introduce the notation, we begin by outlining the derivation of the bound state B.S. equation ^{7),8)}.

The inhomogeneous B.S. equation for the two body propagator (represented by $\propto$) is :

$$\propto = \text{---} + \text{---} \boxed{V} \propto = \text{---} + \propto \boxed{V} \text{---} \quad (2.1)$$

The square box ($\boxed{V}$) represents the sum of all irreducible graphs and will be referred to as the potential [recall that iteration of Eq. (2.1) gives all the terms in perturbation theory ; this equation also arises in a general Green's function treatment of quantum field theory ¹¹⁾]. If the scattered particles can form a bound state, then the two-body propagator must contain a pole at the bound state mass. In the momentum representation :

$$\begin{array}{c} \text{P+q} \\ \nearrow \\ \text{---} \bigcirc \text{---} \\ \nwarrow \\ \text{P-k} \end{array} \quad \begin{array}{c} \text{P+k} \\ \nearrow \\ \text{---} \end{array} = \begin{array}{c} \text{---} \bigcirc \text{---} \end{array} + \text{regular terms} \quad (2.2)$$

$$= \frac{i}{(2\pi)^4 (K^2 - M_B^2)} + \text{regular terms} \quad (2.3)$$

where $K = 2P$ is the total four momentum and M_B is the bound state mass. Equation (2.3) expresses the fact that the residue at the pole necessarily factorizes into a part which depends only on the spins and momenta of the incoming particles and a part which depends only on the spins and momenta of the outgoing particles. Substituting Eq. (2.3) into Eq. (2.1) and equating the residues at the poles, we obtain the bound state, or homogeneous, B.S. equation :

$$\begin{array}{c} \diagup \end{array} = \begin{array}{c} \diagup \end{array} \begin{array}{c} \boxed{V} \end{array} \begin{array}{c} \diagdown \end{array} \quad (2.4)$$

$$, (K^2 = M_B^2)$$

$$\begin{array}{c} \diagdown \end{array} = \begin{array}{c} \diagdown \end{array} \begin{array}{c} \boxed{V} \end{array} \begin{array}{c} \diagup \end{array} \quad (2.5)$$

These are eigenvalue equations for the bound state ; the quantity $\begin{array}{c} \diagup \end{array}$ is referred to as the B.S. wave function and $\begin{array}{c} \diagdown \end{array}$ as its conjugate.

The wave function necessarily satisfies a normalization condition which follows from the definition in Eq. (2,3) ^{*) 12)}. To derive such a condition it is necessary to consider a bilinear equation ; an obvious choice, which follows immediately from Eq. (2.1), is

$$\begin{array}{c} \diagdown \end{array} \left[(=)^{-1} - \boxed{V} \right] \begin{array}{c} \diagup \end{array} = \begin{array}{c} \diagdown \end{array} \begin{array}{c} \diagup \end{array} \quad (2.6)$$

where

$$(=)^{-1} (=) = (=)(=)^{-1} = 1 \quad (2.7)$$

Equating the residues at the pole in the variable K_0 in Eq. (2.6) [i.e., equating the coefficients of $1/(K_0 \pm \sqrt{K^2 + M_B^2})$] and using the homogeneous equation in the form:

^{*)} We could, in principle, solve Eq. (2.1) and find the residues at any poles. Therefore the normalization is determined by the inhomogeneous term in Eq. (2.1) and the factors in Eq. (2.3). The first derivation of the normalization condition known to us which is based on this observation is in Ref. ¹²⁾ ; the method outlined here is also given in Ref. ¹³⁾.

$$\alpha \left[(=)^{-1} - \square \right] = \left[(=)^{-1} - \square \right] \triangleright = 0, \quad (k^2 = M_B^2) \quad (2.8)$$

we obtain the normalization condition

$$\frac{i}{(2\pi)^4} \alpha \left(\frac{\partial}{\partial k_0} \left[(=)^{-1} - \square \right] \right) \triangleright = 2K_0, \quad (k^2 = M_B^2). \quad (2.9)$$

Or, in covariant form :

$$\frac{i}{(2\pi)^4} \alpha \left(\frac{\partial}{\partial k_\mu} \left[(=)^{-1} - \square \right] \right) \triangleright = 2K_\mu, \quad (k^2 = M_B^2). \quad (2.10)$$

A form of this condition was originally derived by Mandelstam¹⁴⁾ and by Nishijima¹⁵⁾ who considered systems having conserved currents ; for a system of two equally charged particles it is clear from the Ward identity that Eq. (2.10) can be interpreted as a statement of current conservation - i.e., Eq. (2.10) is given by equating the current obtained treating the bound state as a composite particle to the current obtained treating it as an elementary particle. To construct matrix elements between bound states we use the graphical interpretation of the wave function and Feynman's rules ; the prescription is^{14),15)} to put in between α and $\triangleright$ all graphs which are irreducible from the point of view of the two lines attached to α and $\triangleright$ (this is explicitly illustrated in Ref.¹⁴⁾).

Since Ξ is the sum of all Feynman diagrams, it is apparent that the wave function defined according to Eq. (2.3) is the correct quantity to use in the construction of invariant amplitudes^{*) 13)}. [This is supported by the covariance of Eq. (2.11).]

We will denote the wave function and its conjugate by χ and $\bar{\chi}$ respectively. Then Eq. (2.4), for example, may be written :

$$\chi(p, q) = S_1(p+q, M_1) S_2(p-q, M_2) \int V(p, q, k) \chi(p, k) d^4k \quad (2.11)$$

*) Note that some authors, e.g., Mandelstam¹⁴⁾, use a wave function which is equal to ours divided by $\sqrt{2K_0}$. In this case the wave function gives non-invariant amplitudes and the right-hand side of Eq. (2.9) is equal to one.

Here S_1 and S_2 are the one-body propagators (two point functions) for the two particles and $\hat{V}$ is the potential. For fermion systems, χ has 16 components, corresponding to the spinor indices of the two particles, and it transforms like the outer product of two spinors; for a fermion-antifermion system it transforms like the outer product of a spinor and a conjugate spinor. We shall use a matrix representation for χ and $\bar{\chi}$ ^{16) *)}, i.e., in the lowest approximation for the propagators, for a fermion and antifermion of (renormalized) masses M_1 and M_2 respectively, Eq. (2.11) becomes ^{**)}

$$(\not{q} + \not{p} - M_1) \chi(p, q) (\not{q} - \not{p} - M_2) = - \int \hat{V}(p, q, k) \chi(p, k) \frac{d^4 k}{(2\pi)^4} \quad (2.12)$$

If we define four two-by-two matrices χ_{ij} by :

$$\chi(\vec{p}=0, p_0 = \frac{M_1}{2}, q) = \begin{pmatrix} \chi_{+-} & \chi_{++} \\ \chi_{--} & \chi_{-+} \end{pmatrix} \quad (2.13)$$

then in the non-relativistic weak binding limit :

$$\begin{aligned} \chi_{+-} &\sim \chi_{-+} \sim \frac{\vec{\sigma} \cdot \vec{q}}{M} \chi_{++} \\ \chi_{--} &\sim \frac{\vec{q}^2}{M^2} \chi_{++} \end{aligned} \quad (2.14)$$

In the naïve non-relativistic quark model it is, in effect, assumed that these relations are also true for tightly bound states.

*) Explicitly, in the fermion-antifermion case, we take :

$$\begin{aligned} \chi_{\alpha\beta}(p, q) &= \int e^{i p \cdot (x_1 + x_2) + i q \cdot (x_1 - x_2)} \langle 0 | T(\psi'_\alpha(x_1) \bar{\psi}_\beta^2(x_2)) | B \rangle \frac{d^4 x}{(2\pi)^4} \\ \bar{\chi}_{\alpha\beta}(p, q) &= \int e^{-i p \cdot (x_1 + x_2) - i q \cdot (x_1 - x_2)} \langle B | T(\bar{\psi}'_\beta(x_1) \psi_\alpha^2(x_2)) | 0 \rangle \frac{d^4 x}{(2\pi)^4} \end{aligned}$$

where $\chi = x_1 - x_2$.

**) Our metric is such that $a \cdot b = a_0 b_0 - \vec{a} \cdot \vec{b}$.

We use γ matrices : $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $\not{a} = a_0 \gamma_0 - \vec{a} \cdot \vec{\gamma}$.

We now consider the construction of eigenstates of parity and charge conjugation for fermion-antifermion systems. It is easy to show that for a bound state with parity Π_B ¹⁷⁾

$$\chi(p, q) = \Pi_B \gamma_0 \chi(\tilde{p}, \tilde{q}) \gamma_0 \quad (2.15)$$

where

$$\tilde{q}_0 = q_0, \quad \tilde{\vec{q}} = -\vec{q} \quad \text{etc.} \quad (2.16)$$

[for bound states with spin, Eq. (2.15) is strictly only true in the rest frame]. For a state with charge conjugation parity $\mathcal{C}_B$:

$$\chi(p, q) = \mathcal{C}_B C \chi^T(p, -q) C^{-1} \quad (2.17)$$

where T denotes transpose and C is a matrix such that :

$$C \gamma_\mu^T C^{-1} = -\gamma_\mu. \quad (2.18)$$

For a pseudoscalar bound state, the most general form of χ is therefore :

$$\chi = \gamma_5 (A + q \cdot P B \not{q} + C \not{P} + D(q \not{P} - P \not{q})) \quad (2.19)$$

If the bound state is even under charge conjugation, then A , B , C and D are even functions of $q \cdot P$.

For a vector particle, χ and $\bar{\chi}$ depend on a Lorentz index as well as 16 spinor indices ; (χ is the Fourier transform of $\langle 0 | T(\Psi_\alpha^1(x_1) \bar{\Psi}_\beta^2(x_2)) | B_\mu \rangle$). To ensure that the bound state has spin one, we impose the usual subsidiary condition :

$$P_\mu \chi_\mu = 0 \quad (2.20)$$

which must hold on the mass shell. χ_μ must therefore depend on the functions :

$$T_\mu = \gamma_\mu - \frac{P_\mu \not{P}}{\mu^2} \quad (2.21)$$

and

$$Q_\mu = q_\mu - \frac{P_\mu (q \cdot P)}{\mu^2} \quad (2.22)$$

where $2 \mu = M_B$ ($P^2 = \mu^2$ on the mass shell) is introduced as a convenient variable.

Under charge conjugation, useful relations are :

$$\begin{aligned}
 T_\mu &\rightarrow -T_\mu \\
 Q_\mu &\rightarrow -Q_\mu \\
 T_\mu \not{x} &\rightarrow -T_\mu \not{x} \\
 T_\mu \not{x} &\rightarrow T_\mu \not{x} - 2Q_\mu \\
 T_\mu (\not{x} \not{x} - \not{x} \not{x}) &\rightarrow -T_\mu (\not{x} \not{x} - \not{x} \not{x}) + 4Q_\mu \not{x}
 \end{aligned} \tag{2.23}$$

The most general form on the wave function for a vector fermion-antifermion bound state odd under charge conjugation is therefore :

$$\begin{aligned}
 \chi_\mu = & T_\mu (A + q \cdot P B \not{x} + C \not{x} + D(\not{x} \not{x} - \not{x} \not{x})) \\
 & - Q_\mu (q \cdot P B + 2 \not{x} D) \\
 & + Q_\mu (E + F \not{x} + q \cdot P G \not{x} + H(\not{x} \not{x} - \not{x} \not{x}))
 \end{aligned} \tag{2.24}$$

where A, B, C, D, E, F, G and H are even functions of (q.P).

Similarly one can construct the general form of the wave function for all possible states (see Section 8 for some examples).

Having constructed the ansatz for the wave function of a state with a particular J, P, C we can always obtain the ansatz for a state with J, P, -C by multiplying by q.P. Non-relativistically C is already determined by L, S and P ; evidently there are extra states relativistically - these states have no non-relativistic limit since $(P \pm q)^2 = M^2$ and $q \cdot P = 0$ when the external particles are on the mass shell. Therefore such bound states are not poles of the on-mass shell scattering amplitude but they can occur in off mass shell matrix elements. Although the author believes that such solutions may be unavoidable ^{*)} we ignore them in the present paper leaving them

^{*)} These solutions do not appear to violate any general principles since we can write Lagrangians exhibiting allied phenomena: e.g., $\mathcal{L}_{int} = ig \partial_\mu A(x) (B(x) \partial_\mu B^\dagger(x) - \partial_\mu B(x) B^\dagger(x))$. This couples a C = -1 particle to an L = 0 state of particle-antiparticle which is forbidden by the usual selection rules for free particles by C conservation. $\mathcal{L}_{int}$, however, is C invariant and indeed in momentum space $\mathcal{L}_{int} \sim q \cdot P$ which is zero if B and $\bar{B}$ are on the mass shell. (I am indebted to J.S. Bell for this observation.)

for a future investigation. States with these "anomalous" quantum numbers are required by a Regge theory with parallel daughter trajectories. (These states are just the states with the "wrong time parity" investigated by several authors ¹⁸⁾). The "time parity" is a symmetry of the B.S. equation for two particles, or particle-antiparticle, with the same mass and certain symmetries in the potential - for identical particles it is related to the exchange symmetry, or to C, and "wrong time parity" simply means that the state is forbidden by the selection rules for on mass shell particles. In certain approximations these solutions seem to violate the normalization condition ¹⁸⁾).

To close this Section we illustrate our matrix notation by writing out the normalization condition, Eq. (2.11), explicitly in the case of a particle "1" and an antiparticle "2". The most general form of the one particle propagator is :

$$S(q, M) = \frac{i}{(2\pi)^4 (f(q^2)\not{q} - g(q^2)M)} , \quad (2.25)$$

where

$$f(M^2) = g(M^2) = 1. \quad (2.26)$$

We introduce the notation

$$f^\pm = f((P \pm q)^2) , \quad g^\pm = g((P \pm q)^2) \quad (2.27)$$

Then the normalization condition may be written :

$$\begin{aligned} -i \text{Trace} \int \frac{d^4 q}{(2\pi)^4} \left\{ (2\pi)^8 \bar{\chi}(P, q) \frac{\partial}{\partial K_\nu} (f_1^+(q+P) - g_1^+ M_1) \chi(P, q) (f_2^-(q-P) - g_2^- M_2) \right. \\ + (2\pi)^8 \bar{\chi}(P, q) (f_1^+(q+P) - g_1^+ M_1) \chi(P, q) \frac{\partial}{\partial K_\nu} (f_2^-(q-P) - g_2^- M_2) \\ \left. + \int \bar{\chi}(P, q) \frac{\hat{V}}{\partial K_\nu}(P, q, k) \chi(P, k) d^4 k \right\} = 2 K_\nu . \end{aligned} \quad (2.28)$$

If the interaction is instantaneous in the rest frame, the dependence of $\chi(P, q)$ on q_0 is entirely determined by the one-particle propagators. Mandelstam ¹⁴⁾, and Klein and Zemach ¹⁹⁾, have given the normalization condition for two fermions in this case, making the simplest approximation

$f=g=1$ and neglecting $\partial \hat{V} / \partial K_p$ ^{*)}. Adapting their remarks to the case of fermion-antifermion, we find :

$$\text{Trace} \int d^3 q \left(\Lambda_-^2(-\vec{q}) \phi^\dagger(\vec{q}) \Lambda_+^1(\vec{q}) \phi(\vec{q}) - \Lambda_+^2(\vec{q}) \phi^\dagger(\vec{q}) \Lambda_-^1(\vec{q}) \phi(\vec{q}) \right) = \frac{2 M_B}{(2\pi)^3} \quad (2.29)$$

where

$$\phi(\vec{q}) = \int \mathcal{X}(\vec{p}=0, p_0, q) dq_0. \quad (2.30)$$

$$\Lambda_\pm^i(\vec{q}) = \frac{\sqrt{\vec{q}^2 + M_i^2} \pm (\vec{\alpha} \cdot \vec{q} + \gamma_0 M_i)}{2\sqrt{\vec{q}^2 + M_i^2}} \quad (2.31)$$

In order to evaluate integrals such as those occurring in Eq. (2.28) we need the relation of $\bar{\mathcal{X}}$ to $\mathcal{X}$. Inserting a complete set of states in the definition of $\mathcal{X}$ and $\bar{\mathcal{X}}$ in terms of ψ and $\bar{\psi}$ we obtain a spectral representation which defines an analytic continuation in q_0 . For complex q_0 we find $\bar{\mathcal{X}}(P, q) = \gamma_0 \mathcal{X}^\dagger(P, q) \gamma_0$. [Here and below we use the possibly unusual definition of a complex conjugate function $f^*(z) = (f(z^*))^*$.] For real q_0 , $\mathcal{X}$ and $\bar{\mathcal{X}}$ are then defined as the limit of the continued functions evaluated at $q_0 + i\epsilon$ as $\epsilon \rightarrow 0^+$ on the right-hand cut and $\epsilon \rightarrow 0^-$ on the left-hand cut (both $\mathcal{X}$ and $\bar{\mathcal{X}}$ satisfy Feynman boundary conditions). Below we denote by $\bar{A}$, $\bar{B}$, etc., functions which satisfy $\bar{A} = A^*$, $\bar{B} = B^*$, etc., for complex q_0 and are defined by this limiting procedure for real q_0 .

3. - THE QUARK MODEL

The basic hypothesis of the single triplet quark model ³⁾ considered in this paper is that there exist three quarks of baryon number $\frac{1}{3}$ (denoted by p , n and λ) and a corresponding triplet or antiquarks (denoted by $\bar{p}$, $\bar{n}$ and $\bar{\lambda}$) which are the fundamental constituents of hadrons. We do not attempt to discuss the origin of the super-strong forces between these particles, which are supposed to bind them together and might be due to some undiscovered fundamental meson or a non-linear contact interaction, for example.

*) Instantaneous is not an invariant concept and $\hat{V}$ necessarily depends on K ; hence this is not really consistent.

Except in a few instances explicitly indicated, we do not make the usual assumption that particular hadron states are "dominated" by particular simple quark states. Nevertheless hadrons will appear as poles in the scattering amplitudes of the simplest combination of quarks and antiquarks which have the same quantum numbers. For example, the π^+ meson must occur as a pole in $p\bar{n}$ scattering and the B.S. equation for this channel can therefore be used to investigate the system; this does not mean, however, that the π^+ meson is necessarily dominated by the $p\bar{n}$ channel - it may contain an arbitrary number of quark-antiquark pairs (which can occur in the potential).

The simplest interpretation of the fact that quarks have not been directly observed, is that they are very heavy objects; we will assume $M_Q \gg 5 \text{ GeV}$. Therefore, the lightest mesonic states (pseudoscalar and vector mesons) have very large binding energies $\sim 2 M_Q$. In comparison, the mass splitting between members of the same unitary multiplet is very small, e.g., $(M_K - M_\pi) \ll 2 M_Q$. It is therefore usually supposed that there is approximate invariance under unitary transformations between quarks and that the breaking of this invariance is characterized by a small parameter of the order $(M_K - M_\pi)/2M_Q \ll \frac{1}{25}$.

This argument is obviously naïve. We treat the pseudoscalar mesons, for example, as poles in $Q\bar{Q}$ scattering and although these poles are $\sim 2 M_Q$ from the $Q\bar{Q}$ threshold there are many other thresholds (e.g., nucleon-antinucleon, which occurs in the $3Q-3\bar{Q}$ channel) which are comparatively very close. Our formalism is quite general and these channels can, in principle, easily be included in the potential and may well be important.^{*)} However, the positions of the thresholds of these "nearby channels" breaks

*) Of course, it is possible simply to assume that the effect of these channels is negligible, as is done in the naïve quark model. This involves the assumption that the ρ meson, for example, is "made up" of $Q\bar{Q}$ rather than $\pi\pi$. Squires and Watson have pointed out the difficulties of this assumption²⁰⁾; unfortunately their model calculation neglects spin and hence the angular momentum barrier in the $\pi\pi$ channel, which could be important.

the symmetry ; we must therefore assume that the exact position of these thresholds is unimportant in order that the potential is approximately SU(3) invariant. The nature of this assumption is unclear and merits further investigation *).

The general structure of the potential is complicated since γ_μ , q_μ , k_μ , P_μ and γ_5 are available for its construction. Typical possible terms are :

$$\hat{V} X = V_1 X + V_2 \gamma_5 X \gamma_5 + V_3 \gamma_\mu X \gamma_\mu + V_4 q X q + V_5 k X k + V_6 q \cdot P (X X + X X) + V_8 X X X + \dots \quad (3.1)$$

We have put in coefficients $q \cdot P$ so that the V_i are even functions of P for eigenstates of C . We assume generally that the V_i constructed thus are approximately SU(3) invariant (they must depend weakly on P and hence on M_B) and that the V_i which occur multiplied by P are less than or of the order of the other V_i in units with the quark mass = 1 in the regions of the variables which are important in the solution [this ensures that the potential is approximately SU(3) invariant in the rest frame. Since the terms which involve P explicitly presumably arise from propagators in intermediate states in the potential this is a rather plausible assumption].

We also assume that the three quarks are approximately degenerate and the functions f and g [Eq. (2.25)] are approximately the same for all quarks. Furthermore we assume that $f((P+q)^2) \simeq f((P-q)^2)$ in the rest frame, and a similar relation for g , in the region of the variable which is important in the solution [$(P+q)^2 = q^2 + 2Pq_0 + P^2$ so this is quite plausible as P is small and the wave function has poles at $(P+q)^2 = M_Q^2$].

*) There is a direct analogy with the Oakes-Yang problem ²¹⁾ : why should the baryon decuplet display any symmetry given that the thresholds of the communicating channels deviate so much from the symmetry limit ? It turns out that if the criterion $ka \ll L$ is satisfied in a given channel (where k is the channel momentum, L the angular momentum and a is the range of the forces) then the position of the threshold is unimportant ²²⁾. The situation with respect to S wave channels (e.g., the $N\bar{N}$ contribution to the potential which binds the pion in the quark model) is unclear.

These assumptions allow us to relate χ^π to χ^K , etc., in a way which we shall now investigate.

4. -

THE PSEUDOSCALAR MESONS

We now consider the equations for a very tightly bound state with quantum numbers $J^{PC} = 0^{-+}$. We write the homogeneous B.S. equation in this case as :

$$\begin{aligned} (f^+(q+P) - g^+M)\chi(f^-(q-P) - g^-M) &= - \int \hat{V} \chi \frac{d^4k}{(2\pi)^4} \\ &= \gamma_5 (\alpha + \beta \not{q} + \gamma \not{P} + \delta (\not{q}\not{P} - \not{P}\not{q})) \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \alpha &= A[f^+f^-(\mu^2 - q^2) + M^2g^+g^-] + BMq \cdot P[q^2(f^+g^- - f^-g^+) \\ &+ q \cdot P(f^-g^+ + f^+g^-)] + CM[\mu^2(f^-g^+ + f^+g^-) + q \cdot P(f^+g^- - f^-g^+)] \\ &+ 4Df^+f^-(\mu^2q^2 - (q \cdot P)^2) \end{aligned} \quad (4.2)$$

$$\begin{aligned} \beta &= AM[f^+g^- - g^+f^-] + Bq \cdot P[M^2g^+g^- - (q^2 + \mu^2)f^+f^-] \\ &- 2Cq \cdot Pf^+f^- - 2DM[\mu^2(g^+f^- - g^-f^+) + q \cdot P(f^+g^- + g^+f^-)] \end{aligned} \quad (4.3)$$

$$\begin{aligned} \gamma &= AM(f^+g^- + f^-g^+) + 2(q \cdot P)^2f^+f^- + C[f^-f^+(q^2 + \mu^2) + M^2g^+g^-] \\ &+ 2DM[q^2(f^+g^- + f^-g^+) + q \cdot P(f^+g^- - g^+f^-)] \end{aligned} \quad (4.4)$$

$$\begin{aligned} \delta &= f^-f^+A + Bq \cdot PM(g^+f^- - f^+g^-)/2 + CM(f^-g^+ + f^+g^-)/2 \\ &+ D[f^-f^+(q^2 - \mu^2) + M^2g^+g^-] \end{aligned} \quad (4.5)$$

We can separate (4.1) into four scalar equations :

$$4\alpha = - \text{Trace } \gamma_5 \int \hat{V} \chi \frac{d^4k}{(2\pi)^4} \quad (4.6)$$

$$4(q^2 \beta + q \cdot \mathcal{P} \gamma) = - \text{Trace } q \gamma_5 \int \hat{V} \mathcal{X} \frac{d^4 k}{(2\pi)^8} \quad (4.7)$$

$$4(q \cdot \mathcal{P} \beta + \mu^2 \gamma) = - \text{Trace } \mathcal{P} \gamma_5 \int \hat{V} \mathcal{X} \frac{d^4 k}{(2\pi)^8} \quad (4.8)$$

$$4((q \cdot \mathcal{P})^2 - \mu^2 q^2) \delta = - \text{Trace } (q \mathcal{P} - \mathcal{P} q) \gamma_5 \int \hat{V} \mathcal{X} \frac{d^4 k}{(2\pi)^8} \quad (4.9)$$

These equations are exact. We now make the assumption $f^+ \simeq f^- = f$ and $g^+ \simeq g^- = g$, discussed in the previous Section, and write three possible terms in the potential explicitly for the purpose of illustration :

$$\frac{\hat{V} \mathcal{X}}{(2\pi)^8} = V_1 \mathcal{X} + V_2 \gamma_5 \mathcal{X} \gamma_5 + V_3 \gamma_\mu \mathcal{X} \gamma_\mu + \dots \quad (4.10)$$

Equations (4.6)-(4.9) then become, in the rest frame :

$$\begin{aligned} A(g^2 M^2 + (\mu^2 - q^2) f^2) + 2 M f g \mu^2 (B q_0^2 + C) - 4 f^2 \mu^2 \vec{q}^2 D \\ = - \int (V_1 + V_2 - 4 V_3) A d^4 k + \dots \end{aligned} \quad (4.11)$$

$$\begin{aligned} q_0 \{ 2 M f g A + B [q^2 (M^2 g^2 - f^2 (q^2 + \mu^2)) + 2 q_0^2 \mu^2 f] \\ + C (M^2 g^2 + f^2 (\mu^2 - q^2)) \} = - \int (V_1 - V_2 + 2 V_3) (q \cdot k k_0 B + q_0 C) d^4 k + \dots \end{aligned} \quad (4.12)$$

$$\begin{aligned} 2 M f g A + q_0^2 (g^2 M^2 - f^2 (q^2 - \mu^2)) B + (M^2 g^2 + f^2 (q^2 - 2 q_0^2)) C \\ - 4 M f g \vec{q}^2 D = - \int (V_1 - V_2 + 2 V_3) (k_0^2 B + C) d^4 k + \dots \end{aligned} \quad (4.13)$$

$$\begin{aligned} \vec{q}^2 \{ f^2 A + f g M C + (M^2 g^2 + f^2 (\vec{q}^2 - \mu^2)) D \} \\ = - \int (V_1 + V_2 - 2 V_3) \vec{q} \cdot \vec{k} D d^4 k + \dots \end{aligned} \quad (4.14)$$

Since $\mu^2 \ll M^2$ it is plausible that $\mu^2 f^2$ can be neglected compared to $f^2 q^2$ and $g^2 M^2$ in Eqs. (4.11)-(4.14), but further simplification can only occur if we already know the relative magnitude of A, B, C and D. For example, in the usual non-relativistic quark model it is assumed that $\vec{q}^2 \ll M^2$, which is taken to imply $\mathcal{X}_{--} \ll \mathcal{X}_{++}$ and $\mathcal{X}_{\pm-} \ll \mathcal{X}_{++}$. Since :

$$\chi_{\pm\pm} = A \mp \mu (Bq_0^2 + C) \quad (4.15)$$

$$\chi_{\pm\mp} = \vec{\sigma} \cdot \vec{q} (2\mu D \pm Bq_0\mu) \quad (4.16)$$

this requires

$$A \simeq -\mu (Bq_0^2 + C) \quad (4.17)$$

and

$$\mu |\vec{q}| D \ll A \quad (4.18)$$

Hence, Eq. (4.14) would decouple and determine D. Equation (4.11) becomes :

$$A (g^2 M^2 - f^2 q_0^2) \simeq - \int (V_1 + V_2 - 4V_3) A d^4k + \dots \quad (4.19)$$

and Eqs. (4.12) and (4.13) become :

$$(Bq_0^2 + C) (g^2 M^2 - f^2 q_0^2) \simeq - \int (V_1 - V_2 + 2V_3) (Bk_0^2 + C) d^4k + \dots \quad (4.20)$$

Unless some "accidental" cancellation occurs due to other terms in the potential, Eqs. (4.17), (4.19) and (4.20) are only compatible if the potential is dominated by V_1 and/or the conspiracy $V_2 \simeq 3V_3$ occurs (i.e., $|V_1 - V_3| \gg |V_2 - 3V_3|$ is required)*). The usual quark model can therefore only occur in these conditions. Actually if $\vec{q}^2 \ll M^2$ and $|V_1 - V_2| \gg |V_1 - 2V_3|$ the approximate equations for χ_{++} and χ_{--} decouple [see Eqs. (7.1) and (7.2) and Appendix A] - any ratio of χ_{++} to χ_{--} may occur and there is no particular reason to expect $\chi_{--} \ll \chi_{++}$.

*) The fact that the vector nonet is so nearly "ideal" and the pseudoscalar nonet so far from ideal³⁾ suggests that the fundamental interaction of quarks has a pseudoscalar, SU(3) scalar nature. Such interactions allow annihilation graphs in the potential for pseudoscalar mesons in the singlet state but not in the octet state - thus separating these states. In the approximation that SU(3) is exact, the potential will be the same for octet and singlet states for all other J^P and "ideal" mixing ($\tan \theta = 1/\sqrt{2}$, $M_8 = M_1$) is predicted; this prediction is in quite good agreement with the data according to the analysis of Dalitz³⁾.

In general each of the Eqs. (4.6) to (4.9) is either odd or even when $P \rightarrow -P$ and hence only contains M_B^2 in the rest frame ; we recall that A, B, C and D are functions of M_B^2 (and not of M_B). If they are sufficiently smooth functions so that they can be expanded in powers of M_B^2 for fixed q^2 and q_0^2 , then it follows from Eq. (4.15) that when $M_B \rightarrow 0$, $\chi_{++} \rightarrow \pm \chi_{--}$ [the conventional model, which requires Eq. (4.17) to hold wherever the wave function is appreciable, is therefore incompatible with "smoothness" as $M_B \rightarrow 0$; for finite M_B it could hold numerically for some mass (e.g., M_π) but, given smoothness, it would then fail for others (e.g., M_K) and the symmetry would be buried].

"Smoothness" is not established. However, in the case of a factorizing potential which is given in Appendix B, and is exactly soluble, it turns out that :

- 1) - as $\mu \rightarrow 0$, $\chi_{++} \rightarrow \pm \chi_{--}$;
- 2) - provided the conditions $|\vec{q}^2| \ll M^2$ and $|V_1 - V_2| \gg |V_2 - 3V_3|$ are not satisfied : $\chi_{++} = \pm \chi_{--} + O(\mu)$, for finite $\mu \ll M$, where the coefficient of μ is of the order of χ_{++} ;
- 3) - if $|\vec{q}^2| \ll M^2$ and $|V_1 - V_2| \gg |V_2 - 3V_3|$ still $\chi_{++} \rightarrow \pm \chi_{--}$ as $\mu \rightarrow 0$, but if $|\vec{q}^2| \ll \mu M$ then either $\chi_{++} \ll \chi_{--}$ or $\chi_{++} \gg \chi_{--}$; if $|\vec{q}^2| \gg \mu M$ the ratio depends critically on the details of the model and on μ .

Below we shall consider models with $\chi_{++} \simeq \pm \chi_{--}$ in which the wave functions of different mesons are related due to the (assumed) approximate SU(3) invariance of the potential ^{*)}. To formulate these relations precisely we must return to the normalization condition since it depends on M_B , in which the symmetry is badly broken. The normalization condition has the structure :

$$\int \bar{\chi} \hat{O}_\mu \chi = \mathcal{P}_\mu \quad (4.21)$$

*) We consider the symmetry breaking terms to be small perturbations in the B.S. equation (cf. Section 3); we have not written them explicitly since their presence is manifested by the changes in M_B^2 which they produce. These changes then play an essential role since A, B, C and D may differ by orders of magnitude of M_B and the relation of χ to A, B, C and D involves M_B .

Dividing χ into parts which are odd or even under $P \rightarrow -P$, so that in the rest frame $\chi = \chi_e(p^2) + p\chi_o(p^2)$, our assumptions about the one particle propagators and the potential (Section 3) give (symbolically) :

$$\int (\alpha \bar{\chi}_e \chi_e + \beta \bar{\chi}_o \chi_e + \gamma \bar{\chi}_e \chi_o + \delta p^2 \bar{\chi}_o \chi_o) + O(p^4) = 1. \quad (4.22)$$

or

$$\begin{aligned} & \int d^4q d^4k [f_1 \bar{A} A + \bar{A} (f_2 B + f_3 C + f_4 D) + (f_5 \bar{B} + f_6 \bar{C} + f_7 \bar{D}) A \\ & + M_B^2 (f_8 \bar{B} B + f_9 \bar{C} C + f_{10} \bar{D} D + f_{11} \bar{B} C + f_{12} \bar{C} B + f_{13} \bar{B} D + f_{14} \bar{D} B \\ & + f_{15} \bar{C} D + f_{16} \bar{D} C)] + O(M_B^2) = 1. \end{aligned} \quad (4.23)$$

where the f_i are independent of M_B^2 and A, B, C and D are taken in the rest frame.

Using Eq. (4.23), the models which we shall consider below can now be formulated :

Model 1 $\chi_{++} = \chi_{--} + O(M_B)$, with coefficient of M_B of order χ_{++} .

$$\chi_{++}^\pi \sim \chi_{++}^K \quad \text{etc.}$$

$$\frac{\chi_{++}^\pi - \chi_{--}^\pi}{M_\pi} \sim \frac{\chi_{++}^K - \chi_{--}^K}{M_K} \quad \text{etc.}$$

i.e., we take $A \sim B \sim C \sim D$ as suggested by letting $p^2 \rightarrow 0$ in Eqs. (4.11)-(4.14) ^{*}). [This model may only make sense if the potential does not have an attractive singularity as strong as that due to single particle exchange at the origin in co-ordinate space. This condition is in any case probably necessary if we require $\frac{1}{q^2} \ll M^2$ so that SU(6) can hold and high spin mesons can be interpreted as orbital excitations. This is discussed at the end of Appendix D.]

^{*}) In order of magnitude relations we take units in which the quark mass $M = 1$ here and in the rest of the paper.

Model 2 $\chi_{++} = -\chi_{--} + O(M_B)$, with coefficient of M_B or order χ_{++} .

$$\chi_{++}^{\pi} \simeq \chi_{--}^K \quad \text{etc.}$$

$$\chi_{++}^{\pi} - \chi_{--}^{\pi} \simeq \chi_{++}^K - \chi_{--}^K \quad \text{etc.}$$

i.e., we take $A \sim \mu^2_B \sim \mu^2_C \sim \mu^2_D$ as suggested by assuming $A \sim \mu^2_B$ and then letting $\mu^2 \rightarrow 0$ in Eqs. (4.11)-(4.14) *).

Model 3 Although we consider it rather implausible, we shall also consider the usual model [in which Eqs. (4.17) and (4.18) hold] for the sake of comparison. In this model :

$$\chi_{--} \ll \chi_{++}$$

$$\frac{\chi_{++}^{\pi}}{\sqrt{M_{\pi}}} \simeq \frac{\chi_{++}^K}{\sqrt{M_K}}$$

where we have assumed that the symmetry holds approximately in the functional dependence of χ_{++} , the factors $\sqrt{M_B}$ arising from the normalization condition [cf. Eqs. (4.17), (4.18) and (4.23)]. (The most precise formulation of the usual model using a relativistic formalism is to assume that the potential is instantaneous, $f = g = 1$ in the one-particle propagators, and $\partial V / \partial P_{\mu}$ is negligible in the normalization condition. This model then gives precisely the results of the non-relativistic formalism - see the end of Section 5.)

In the next Section, we shall see that only Model 1 is compatible with experiment.

*) This model is not just an absurd mathematical possibility - it does occur with a factorizing potential (see Appendix B).

5. - THE PROCESSES $\pi \rightarrow \mu \nu$ AND $K \rightarrow \mu \nu$ AS TESTS OF THE RELATIVISTIC MODELS

We assume that the weak interactions of quarks have a current-current form :

$$\mathcal{L}_{weak} = \frac{G}{\sqrt{2}} (J_\mu + \ell_\mu)(J_\mu + \ell_\mu)^\dagger \quad (5.1)$$

where ℓ_μ is the well-known lepton current and G is the weak coupling constant. It is natural to write the quark current in the form

$$J_\mu = \bar{\Psi}_p \gamma_\mu (1 + R \gamma_5) (\cos \theta \Psi_n + \sin \theta \Psi_\lambda) \quad (5.2)$$

where Ψ_p , Ψ_n and Ψ_λ are the quark fields, R is a constant and θ is the Cabibbo angle for quarks. (We could have put a different angle for the vector and axial-vector currents - we return to this point later.) The naïve quark model for baryons gives

$$R = \frac{3 G_A}{5 G_V} \approx 0.7 \approx \frac{1}{\sqrt{2}} \quad (5.3)$$

(The value of R is actually irrelevant in the present Section. The naïve quark model also gives $\alpha = D/D+F = 0.6$ in remarkable agreement with the latest one angle fit to hyperon decays, which gives 0.611 ± 0.014 ²³⁾.)

The process $\pi \rightarrow \mu \nu$ depends on the current

$$\begin{aligned} \langle 0 | J_\mu(0) | \pi^+ \rangle &= \langle 0 | T(\bar{\Psi}_p(0) \gamma_\mu \gamma_5 R \cos \theta \Psi_n(0)) | \pi^+ \rangle \\ &= R \cos \theta \text{Trace} \int \chi(p, q) \gamma_\mu \gamma_5 d^4 k \\ &= 2 R \cos \theta K_\mu^\pi \int (B^\pi(\vec{p}=0) q_0^2 + C^\pi(\vec{p}=0)) d^4 k \\ &= R \cos \theta K_\mu^\pi \text{Trace} \int \frac{\chi_{++}^\pi - \chi_{--}^\pi}{M_\pi} d^4 k. \end{aligned} \quad (5.4)$$

This equation is exact to all orders in the strong interactions. Diagrammatically the amplitude is just

Conventionally the currents in π and K decay are parametrized :

$$\langle 0 | J_\mu(0) | \pi^+ \rangle = f_\pi \cos \theta_A K_\mu^\pi \quad (5.5)$$

$$\langle 0 | J_\mu(0) | K^+ \rangle = f_K \sin \theta_A K_\mu^K \quad (5.6)$$

So that, assuming $\theta \simeq \theta_A$, we obtain :

$$\begin{aligned} \frac{f_\pi}{f_K} &= \frac{\int (B^\pi(\vec{p}=0) q_\mu^2 + C^\pi(\vec{p}=0)) d^4k}{\int (B^K(\vec{p}=0) q_\mu^2 + C^K(\vec{p}=0)) d^4k} \\ &= \frac{M_K \text{Trace} \int (\chi_{++}^\pi - \chi_{--}^\pi) d^4k}{M_\pi \text{Trace} \int (\chi_{++}^K - \chi_{--}^K) d^4k} \end{aligned} \quad (5.7)$$

The three models therefore give the following results :

<u>Model 1</u>	$(\chi_{++} \simeq \chi_{--})$	$f_\pi \simeq f_K$
<u>Model 2</u>	$(\chi_{++} \simeq -\chi_{--})$	$M_\pi f_\pi \simeq M_K f_K$
<u>Model 3</u>	$(\chi_{--} \ll \chi_{++})$	$\sqrt{M_\pi} f_\pi \simeq \sqrt{M_K} f_K$

Using the value of θ_A determined in hyperon decays, experiment gives $f_\pi \simeq f_K$. Since $(M_K/M_\pi) \sim 3.5$ this clearly shows that Model 1 is the only model compatible with experiment and our assumptions. The strongest of these assumptions is that there is one Cabibbo angle for quarks and that this is approximately equal to θ_A determined in hyperon decays ; the success of fits to hyperon decays with one angle ($\sin \theta_A = \sin \theta_V = 0.228 \pm 0.006$ with 62% confidence according to the very latest data ²³⁾) and the remarkable equality of this angle with that determined in K decay [$\sin \theta_V = 0.220 \pm 0.003$ from K_{e3}^+ and $\sin \theta_V = 0.201 \pm 0.004$ from K_{e3}^0 , assuming constant form factors with their SU(3) values ²⁴⁾] virtually forces us to accept

this assumption as approximately true unless we regard the success of Cabibbo's elegant parametrization as accidental ^{*)}.

Model 1 is therefore the only model with small symmetry breaking in the wave function which can explain the result $f_\pi \simeq f_K$ in a natural way. Symmetry breaking is required to explain this result which must be of order M_K/M_π in Model 2 and of order $\sqrt{M_K/M_\pi}$ in Model 1 - this is the "Weisskopf-Van Royen paradox" in the latter case ^{5),6)}. To illustrate this explicitly define :

$$\psi^{K,\pi}(x) = \int \frac{\text{Trace } \chi_{++}^{K,\pi}(q) e^{iq \cdot x} d^4 q}{\sqrt{M_{K,\pi}}} \quad (5.8)$$

Neglecting $\partial V / \partial P_\nu$ and putting $f = g = 1$ in the one-particle propagators, the normalization condition gives

$$\int |\psi^K(x)|^2 d^4 x \simeq \int |\psi^\pi(x)|^2 d^4 x \quad (5.9)$$

for Model 3. However, Eqs. (5.9) and (5.8) give :

$$\frac{\psi^\pi(0)}{\psi^K(0)} = \sqrt{\frac{M_\pi}{M_K}} \frac{f_\pi}{f_K} \simeq \sqrt{\frac{M_\pi}{M_K}} \quad (5.10)$$

Comparison of (5.9) and (5.10) shows that the usual model requires symmetry breaking of order (M_K/M_π) at $x = 0$. If the system is non-relativistic in the sense that $\vec{q}^2 \ll M^2$ and the interaction is instantaneous, then defining :

$$\Phi^{K,\pi}(\vec{r}) = \frac{1}{2\sqrt{M_{K,\pi}}} \int \text{Trace } \phi_{++}^{K,\pi}(\vec{q}) e^{i\vec{q} \cdot \vec{r}} d^3 \vec{q} \quad (5.11)$$

*) If we take $(f_\pi/f_K) \tan \theta_A$ [Eqs. (5.5) and (5.6)] from experiment, we can calculate the axial Cabibbo angle for quarks θ [Eq. (5.2)] in the different models. With this inverse procedure, Model 1 gives $\sin \theta \simeq 0.27$, Model 2 gives $\sin \theta \simeq 0.72$, and Model 3 gives $\sin \theta \simeq 0.47$. According to the latest two-angle fit to hyperon decay ²³⁾ $\sin \theta_A = 0.224^{+0.023}_{-0.019}$; (we quote the two-angle fit here since the error is more realistic) - in Models 2 and 3 huge symmetry breaking is required to explain this result and suppress strangeness changing decays of baryons.

where $\phi_{++}(\vec{q})$ is defined by Eq. (2.30), Eq. (2.29) gives :

$$\int |\Phi(\vec{r})|^2 d^3\vec{r} \simeq \int |\Phi^\pi(\vec{r})|^2 d^3\vec{r}. \quad (5.12)$$

Using the non-relativistic result $R \simeq 1/\sqrt{2}$ [Eq. (5.3)] and the experimental fact $f_\pi \simeq f_K \simeq M_\pi$ we get

$$\frac{\Phi^\pi(0)}{\Phi^K(0)} \simeq \sqrt{\frac{M_\pi}{M_K}} \quad (5.13)$$

$$|\Phi^\pi(0)|^2 \simeq \frac{M_\pi^3}{2} \quad (5.14)$$

These are exactly the results found by Matveev, Struminski and Tavekhelidze ⁵⁾ and by Weisskopf and Van Royen ⁶⁾. Equation (5.13) was a serious difficulty for the naïve quark model. We propose Model 1 for pseudo-scalar mesons ; it surmounts the previous difficulties and is, in the author's opinion, a priori more appealing. (Model 1 is also the only model in which particle physics is not drastically changed as $M_\pi \rightarrow 0$; see Ref. ²⁾).

6. - VECTOR MESONS

In terms of the ansatz of Eq. (2.24), the vector meson wave functions have the structure :

$$\vec{\chi}_{++} = \vec{\sigma} (A - \nu(Bq_0^2 + C)) + \vec{q} \cdot \vec{\sigma} \cdot \vec{q} (2\nu H - F) \quad (6.1)$$

$$\vec{\chi}_{--} = -\vec{\sigma} (A + \nu(Bq_0^2 + C)) + \vec{q} \cdot \vec{\sigma} \cdot \vec{q} (2\nu H + F) \quad (6.2)$$

For eigenstates of charge parity A, B, etc. are functions of μ^2 ; if they are "smooth" functions, we see that $\vec{\alpha}_{++} \rightarrow \pm \vec{\alpha}_{--}$ as $\mu \rightarrow 0$ exactly as for pseudoscalar mesons. Once again we consider three models ^{*)}:

Model 1 $\vec{\alpha}_{++} \simeq \vec{\alpha}_{--} + O(M_B)$, with coefficient of M_B of order $\vec{\alpha}_{++}$.

$$\vec{\alpha}_{++}^\rho \simeq \vec{\alpha}_{++}^\phi \quad \text{etc.}$$

$$\frac{\vec{\alpha}_{++}^\rho - \vec{\alpha}_{--}^\rho}{M_\rho} \simeq \frac{\vec{\alpha}_{++}^\phi - \vec{\alpha}_{--}^\phi}{M_\phi} \quad \text{etc.}$$

Model 2 $\vec{\alpha}_{++} = -\vec{\alpha}_{--} + O(M_B)$, with coefficient of M_B of order $\vec{\alpha}_{++}$.

$$\vec{\alpha}_{++}^\rho \simeq \vec{\alpha}_{++}^\phi \quad \text{etc.}$$

$$\vec{\alpha}_{++}^\rho - \vec{\alpha}_{--}^\rho \simeq \vec{\alpha}_{++}^\phi - \vec{\alpha}_{--}^\phi \quad \text{etc.}$$

Model 3 $\vec{\alpha}_{--} \ll \vec{\alpha}_{++}$

$$\frac{\vec{\alpha}_{++}^\rho}{\sqrt{M_\rho}} \simeq \frac{\vec{\alpha}_{++}^\phi}{\sqrt{M_\phi}} \quad \text{etc.}$$

^{*)} The relationship of $\vec{\alpha}_{++}^\rho$ to $\vec{\alpha}_{++}^\phi$ etc. follows from the normalization condition if the relative orders of magnitude of A, B...H are known. For pseudoscalar mesons we stated our assumptions about these quantities explicitly in each model; the assumptions followed rather directly from the assumed relation of α_{++} to α_{--} and Eqs. (4.11)-(4.14). In the present case the possibilities are far more numerous since there are eight equations for eight functions; we simply assume the necessary order of magnitude relations for each model, which we then test. (If we first assume that the 3D_1 part of α is negligible so that $E = F = G = H = 0$ the relative orders of magnitude required here occur rather naturally - cf. Section 7.)

To test the models we once again consider annihilation processes. These virtual processes $\rho \rightarrow \gamma$, etc., are given by the diagram

Or equivalently in terms of fields by the current

$$\begin{aligned} \langle 0 | J_\rho(0) | \rho \rangle &= \langle 0 | \left(\frac{2e}{3} \bar{\psi}_\rho(0) \gamma_\mu \psi_\rho(0) - \frac{e}{3} \bar{\psi}_\eta(0) \gamma_\mu \psi_\eta(0) - \frac{e}{3} \bar{\psi}_\lambda(0) \gamma_\mu \psi_\lambda(0) \right) | \rho \rangle \\ &= \sum_i g_i e_i \text{Trace} \int \chi_\rho^\mu(p, q) \gamma_\mu d^4 q \\ &= (g_{\rho\eta} - \frac{p_\mu p_\eta}{\mu^2}) \sum_i g_i e_i 4 \int (A^\rho g_{\eta\mu} + F q_\eta q_\mu) d^4 q. \end{aligned} \quad (6.3)$$

where g_i are the Clebsch-Gordan coefficients for the coupling $Q\bar{Q}-\rho$ and e_i are the charges of the quarks. This current is conventionally parametrized

$$\langle 0 | J_\rho(0) | \rho \rangle = \frac{e M_\rho^2}{2 \gamma_\rho} (g_{\rho\rho} - \frac{p_\mu p_\mu}{\mu^2}) \quad (6.4)$$

Working in the rest frame we find :

$$\begin{aligned} \delta_{jk} \frac{M_\rho^2}{2 \gamma_\rho} &= \delta_{jk} \sum_i g_i \frac{e_i}{e} 4 \int (A^\rho - \frac{\vec{q}^2 F^\rho}{3}) d^4 q \\ &= \sum_i g_i \frac{e_i}{e} \text{Trace} \sigma_j \int (\chi_{++}^\rho - \chi_{--}^\rho) d^4 q. \end{aligned} \quad (6.5)$$

Comparing this expression for the different neutral vector mesons, the models give :

Model 1 $(\vec{\chi}_{++} \approx \vec{\chi}_{--})$

$$\gamma_\rho^2 : \gamma_\phi^2 : \gamma_\omega^2 = M_\rho^2 : \frac{3 M_\phi^2}{\cos^2 \theta} : \frac{3 M_\omega^2}{\sin^2 \theta} = 1 : 9.2 : 7.8 \quad *)$$

*) This result has been obtained in the usual quark model assuming large symmetry breaking according to the prescription $\sqrt{M_\phi} \Phi^\rho(\vec{r}=0) = \sqrt{M_\rho} \Phi^\rho(\vec{r}=0)$, etc. 5), 6).

Model 2 ($\vec{\chi}_{++} \approx -\vec{\chi}_{--}$)

$$\gamma_\rho^2 : \gamma_\phi^2 : \gamma_\omega^2 = M_\rho^4 : \frac{3M_\phi^4}{\cos^2\theta} : \frac{3M_\omega^4}{\sin^2\theta} = 1 : 16.6 : 8.3$$

Model 3 ($\vec{\chi}_{..} \ll \vec{\chi}_{++}$)

$$\gamma_\rho^2 : \gamma_\phi^2 : \gamma_\omega^2 = M_\rho^3 : \frac{3M_\phi^3}{\cos^2\theta} : \frac{3M_\omega^3}{\sin^2\theta} = 1 : 12.4 : 8.1$$

where θ is the ω/ϕ mixing angle and we have used the value given by the Gell-Mann - Okubo mass formula for the masses squared: $\theta = 39.9^\circ$ (using the "ideal" mixing angle $\tan\theta = 1/\sqrt{2}$ and $M_\rho \simeq M_\omega$ the three models give $1:8:9$, $1:14:9$ and $1:10.6:9$ respectively).

The world average of all experiments ²⁵⁾ gives $\gamma_\rho^2 : \gamma_\phi^2 : \gamma_\omega^2 = 1 : 5.85_{-1.8}^{+3.1} : 9.0_{-2.4}^{+3.9}$. Therefore Models 2 and 3 are untenable.

We propose Model 1 for vector mesons.

7. - APPROXIMATE SU(6) SYMMETRY

The most compelling reason for assuming that quarks could be treated non-relativistically was that it gave an immediate explanation of the success of SU(6) symmetry ³⁾. We have argued above that a non-relativistic description is not necessarily applicable to tightly bound systems and that a relativistic description is more plausible and in better agreement with experiment. Nevertheless approximate SU(6) symmetry can hold.

If two states of different J belong mainly to the same angular momentum state L , then the χ_{++} and χ_{--} which describe them differ only in their spin structure. If $\vec{q}^2 \ll M^2$ and if the potential is approximately spin independent (i.e., does not contain $\vec{\gamma}$ in the rest frame) then all the terms in the B.S. equation will be approximately scalar in the two-by-two spin subspaces of χ_{++} , etc. If, moreover, the solution is such that $\chi_{\pm\mp} \approx |\vec{q}| \chi_{\pm\pm}$ then the B.S. equation reduces essentially to two coupled equations involving only χ_{++} and χ_{--} (the mathematical details of these arguments are given in Appendix A). In this case, the states of the same L and different J will be approximately degenerate (on the scale of the quark mass). Thus $SU(6)$ can hold for exactly the same reasons that it is supposed to hold in the usual model. (This is explicitly illustrated in Appendix B for a factorizing potential.)

In the naïve non-relativistic model, however, it is in effect assumed that $\chi_{\pm\mp} \sim |\vec{q}| \chi_{\pm\pm}$. For tightly bound states, this not necessarily the case [cf., Eqs. (8.7) and (8.8)], so that the assumption of spin independence and that $\vec{q}^2 \ll M^2$ do not necessarily lead to $SU(6)$ for all multiplets (nevertheless a smaller multiplet structure may remain - see Section 8); in this Section we investigate the consequences of these assumptions for the pseudoscalar and vector mesons for which we are already committed to models which imply $\chi_{\pm\mp} \approx |\vec{q}| \chi_{\pm\pm}$, and hence approximate $SU(6)$.

It is clear from Eqs. (6.1) and (6.2) that the part of the wave function which depends on H and F corresponds to a 3D_1 state non-relativistically. We now assume that the vector mesons are mainly 3S_1 states and neglect the functions E, F, G and H in Eq. (2.24). Assuming $\vec{q}^2 \ll M^2$, Eqs. (4.11) and (4.13) may be written as follows :

$$(\chi_{++} + \chi_{--})(g^2 M^2 - f^2 q^2) + 2Mfg\mu(\chi_{--} - \chi_{++}) - 8f^2\mu^2 \vec{q}^2 \mathcal{D} \approx - \int (V_1 + V_2 - 4V_3)(\chi_{++} + \chi_{--})d^4k + \dots \quad (7.1)$$

$$2M\mu fg(\chi_{++} + \chi_{--}) + (\chi_{--} - \chi_{++})(M^2 g^2 - f^2 q^2) - 4\mu f^2 \vec{q}^2 \mathcal{C} - 8\mu Mfg\vec{q}^2 \mathcal{D} \approx - \int (V_1 - V_2 + 2V_3)(\chi_{--} - \chi_{++})d^4k + \dots \quad (7.2)$$

*) This assumption may be hard to reconcile with the assumption that the position of the nearby thresholds is unimportant if the latter requires that the important forces have a short range (cf., Section 3). Note that the interpretation of high spin mesons as $Q - \bar{Q}$ excitations probably require: $\vec{q}^2 \ll M^2$ (cf. the end of Appendix D).

The equivalent equations for vector mesons are :

$$(x_{++}^v + x_{--}^v)(M^2 g^2 - f^2 q^2) + 2Mfg\mu^v(x_{--}^v - x_{++}^v) + \frac{32}{3}\bar{q}^2 Mfg\mu^v D^v - \frac{16}{3}f\bar{q}^2\mu^v C^v \simeq - \int (V_1 + V_2 + \frac{2V_3}{3})(x_{++}^v + x_{--}^v) d^4k + \dots \quad (7.3)$$

$$2M\mu^v fg(x_{++}^v + x_{--}^v) + (x_{--}^v - x_{++}^v)(M^2 g^2 - f^2 q^2) + \frac{16}{3}f^2\mu^v \bar{q}^2 D^v + \frac{2}{3}\bar{q}^2 f^2(x_{++}^v - x_{--}^v) \simeq - \int (V_1 - V_2 - 2V_3)(x_{--}^v - x_{++}^v) d^4k + \dots \quad (7.4)$$

where the superscript v denotes vector meson and we have introduced the notation

$$\vec{x}_{\pm\pm} = \vec{\sigma} x_{\pm\pm}^v$$

(where $x_{\pm\pm}^v$ are two-by-two matrices proportional to the unit matrix). We have already assumed $D \sim C \sim x_{++}$ and $D^v \sim C^v \sim x_{++}^v$. [Once we have assumed that the vector mesons are 3S_1 states, this follows rather naturally from the assumption $x_{++}^v \simeq x_{--}^v$ and the B.S. equation, just as $D \sim C \sim x_{++}$ follows from Eqs. (4.11)-(4.14) for pseudoscalar mesons.] Comparing Eqs. (7.1)-(7.4), we see that in Model 1 :

$$x_{++} \simeq x_{++}^v \quad (7.5)$$

and also

$$\frac{x_{++} - x_{--}}{\mu} \simeq \frac{x_{++}^v - x_{--}^v}{\mu^v} \quad (7.6)$$

Actually Eqs. (7.5) and (7.6) only follow from equations above as proportionalities. Taking into account the conditions $\vec{q}^2 \ll M^2$ and that $\hat{V}$ is spin independent, however, the normalization condition does give :

$$x_{++}^\pi \simeq x_{++}^\rho \text{ etc.}$$

$$\frac{x_{++}^\pi - x_{--}^\pi}{M_\pi} \simeq \frac{x_{++}^\rho - x_{--}^\rho}{M_\rho} \text{ etc.} \quad (7.7)$$

Comparing Eqs. (5.4), (5.5), (6.3) and (7.7), we find :

$$\begin{aligned}
 \frac{1}{2 \gamma_\rho} &= \frac{\sum_i g_i e_i}{M_\rho^2 e} \text{Trace} \int (\chi_{++}^\rho - \chi_{--}^\rho) d^4 q. \\
 &= \frac{1}{\sqrt{2} M_\rho} \text{Trace} \int \frac{(\chi_{++}^\pi - \chi_{--}^\pi)}{M_\pi} d^4 q. \quad (7.8) \\
 &= \frac{f_\pi}{\sqrt{2} R M_\rho}
 \end{aligned}$$

Using the non-relativistic result $R = 1/\sqrt{2}$ [Eq. (5.3)] this becomes the famous Kawarabayashi-Suzuki relation ⁹⁾ which gives $(\gamma_\rho^2/4\pi) = 0.61$ ^{*)} in reasonable agreement with the world average value of $0.52^{+0.07}_{-0.06}$ ²⁵⁾. This result was previously derived in the non-relativistic quark model by making the essentially ad hoc assumption

$$\sqrt{M_\pi} \Phi^\rho(0) \approx \sqrt{M_\rho} \Phi^\pi(0)$$

where the $\Phi(\vec{r})$ are the non-relativistic wave functions [as in Equation (5.12) ^{5), 6), **,)}]. This calculation relies on the result of a non-relativistic calculation of R . Therefore, it cannot be regarded as a straightforward result of our model (cf., remarks in Section 11).

8. - HIGHER MESONIC STATES

We have seen that $SU(6)$ can apply to the $L=0$ states in the sense that the existence of a tightly bound pseudoscalar state implies the

*) Here we have assumed $f_\pi \equiv f_K$ and then used the value of f_π given by $\Gamma(\pi \rightarrow \rho \nu) / \Gamma(K \rightarrow \rho \nu)$.

**) Dar states that in the three quark model of Han and Nambu ²⁶⁾ the value of γ_ρ^{-1} calculated thus is increased by $\sqrt{3}$ ²⁷⁾. It seems to us, however, that the relation of f_π to χ also changes by $\sqrt{3}$ and the prediction for $\gamma_\rho f_\pi$ is unchanged.

existence of a tightly bound vector state with a closely related wave function if $\vec{q}^2 \ll M^2$ and $\hat{V}$ is spin independent. The investigation of the mass splitting between states of different J^P is evidently much more complicated than in the naïve model (since we must treat coupled equations) and we have not succeeded in making a general analysis of the spectrum such as has been made in the naïve model ³⁾. Nevertheless, it is interesting to consider the $L=1$ states in our model as it turns out that they may split into submultiplets in a way which has no parallel in the naïve model.

The states which have $L=1$ in the naïve quark model have B.S. wave functions of the form :

$$\chi(J^{PC}=0^{++}) = A + Bq + Cq \cdot P \not{P} + D(q \not{P} - \not{P} q) \quad (8.1)$$

$$\begin{aligned} \chi(J^{PC}=1^{++}) = & \gamma_5 T_\mu (A + Bq + Cq \cdot P \not{P} + D(q \not{P} - \not{P} q)) \\ & - Q_\mu \gamma_5 (B + 2 \not{P} D) \end{aligned} \quad (8.2)$$

$$\begin{aligned} \chi(J^{PC}=2^{++}) = & M_{\mu\nu} (A + Bq \cdot P \not{q} + C \not{P} + D(q \not{P} - \not{P} q)) \\ & - N_{\mu\nu} (2q \cdot P B + 4 D \not{P}) \end{aligned} \quad (8.3)$$

$$\chi(J^{PC}=1^{+-}) = \gamma_5 Q_\mu (A + Bq \cdot P \not{q} + C \not{P} + D(q \not{P} - \not{P} q)) \quad (8.4)$$

where :

$$M_{\mu\nu} = T_\mu Q_\nu + Q_\mu T_\nu - \frac{2}{3} \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right) (q - \frac{P q \cdot P}{P^2}) \quad (8.5)$$

$$N_{\mu\nu} = Q_\mu Q_\nu - \frac{1}{3} \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right) (q^2 - \frac{(q \cdot P)^2}{P^2}) \quad (8.6)$$

In these equations A, B , etc., are again functions of $(q \cdot P)^2$ for eigenstates of C and we have neglected a 3F_1 contribution to Eq. (8.3) (constructed from $N_{\mu\nu}$) and a part with no non-relativistic limit (as discussed on p.8) in Eq. (8.2). In the rest frame :

$$\chi(J^{PC} = 0^{++}, 3P_0) = \begin{pmatrix} A + Bq_0 + Cq_0 \mu^2, \vec{\sigma} \cdot \vec{q} (-B + 2\mu D) \\ \vec{\sigma} \cdot \vec{q} (B + 2\mu D), A - Bq_0 - Cq_0 \mu^2 \end{pmatrix} \quad (8.7)$$

$$\chi_i(J^{PC} = 1^{++}, 3P_1) = \begin{pmatrix} -\sigma_i (A + Bq_0 + Cq_0 \mu^2), i\epsilon_{ijk} \sigma_j q_k (B - 2\mu D) \\ i\epsilon_{ijk} \sigma_j q_k (B + 2\mu D), \sigma_i (A - Bq_0 - Cq_0 \mu^2) \end{pmatrix} \quad (8.8)$$

$$\chi_{ij}(J^{PC} = 2^{++}, 3P_2) = \begin{pmatrix} (T_{ij} \vec{\sigma} \cdot \vec{q} - 2Q_{ij})(B\mu q_0 + 2\mu D), T_{ij}(A - B\mu q_0^2 - \mu C) \\ T_{ij}(-A - B\mu q_0^2 - \mu C), (T_{ij} \vec{\sigma} \cdot \vec{q} - 2Q_{ij})(B\mu q_0 + 2\mu D) \end{pmatrix} \quad (8.9)$$

$$\chi_i(J^{PC} = 1^{+-}, 1P_1) = \begin{pmatrix} q_i \vec{\sigma} \cdot \vec{q} (B\mu q_0 + 2\mu D), q_i (A - \mu B q_0^2 - \mu C) \\ q_i (A + \mu B q_0^2 + \mu C), q_i \vec{\sigma} \cdot \vec{q} (-B\mu q_0 + 2\mu D) \end{pmatrix} \quad (8.10)$$

$$T_{ij} = \sigma_i q_j + \sigma_j q_i - \frac{2}{3} \delta_{ij} \vec{\sigma} \cdot \vec{q} \quad (8.11)$$

$$Q_{ij} = q_i q_j - \frac{\delta_{ij}}{3} \vec{q}^2 \quad (8.12)$$

The functions $\chi_{++}(q)$ and $\chi_{--}(q)$ describing these states differ only in their spin structure (as we expect since they have the same L) and the states will be approximately degenerate, according to our general argument, provided $\vec{q}^2 \ll M^2$, $\hat{v}$ is spin independent and $\chi_{\pm\mp} \sim |\vec{q}| \chi_{\pm\pm}$. Hence approximate $SU(6)$ can hold.

Note, however, that although Eqs. (8.7) - (8.10) suggest $\chi_{\pm\mp} \sim |\vec{q}| \chi_{\pm\pm}$ for the 3P_2 and 1P_1 states, they suggest $\chi_{\pm\pm} \sim |\vec{q}| \chi_{\pm\mp}$ for the 3P_2 and 3P_1 states. With $\vec{q}^2 \ll M^2$ and $\hat{v}$ spin independent, the 3P_2 and 1P_1 states would still form a doublet in this case. The 3P_0 and 3P_1 states would be essentially described by the equations for their $\chi_{\pm\mp}$ (cf. Appendix A). They would not form a doublet, however, since these $\chi_{\pm\mp}$ differ in more than their spin structure; for 3P_0 the combination $\chi_{+-} + \chi_{-+}$ is even in q_0 while $\chi_{+-} - \chi_{-+}$ is odd - for 3P_1 the opposite situation holds.

With the substitution ${}^1P_1, {}^3P_2, {}^3P_1, {}^3P_0 \rightarrow {}^1L_L, {}^3L_{L+1}, {}^3L_L, {}^3L_{L-1}$ the remarks above actually apply to states of arbitrary $L > 0$.

We therefore see that $SU(6)$ does not necessarily follow directly from the assumption of small internal momentum and spin independence in the relativistic model; the existence of ${}^3L_{L+1}$ and 1L_L doublets might be the only symmetry higher than $SU(3)$ for multiplets with $L > 0$.

9. -

MASS FORMULAE

Quadratic mass formulae for mesons were originally introduced empirically in order to make the pseudoscalar mesons fit the Gell-Mann-Okubo mass formula. If, however, $\eta - \eta'$ mixing is allowed²⁸⁾, this motivation is no longer valid; there is therefore no experimental reason to choose quadratic rather than linear formulae [$\Gamma(\eta \rightarrow 2\gamma)$ is actually easier to understand using a linear formula, cf., Section 10] and it is very interesting to see which form the B.S. equation suggests. Our model in which the solutions are smooth functions of p^2 strongly suggests quadratic mass formulae; this is not the case in the usual non-relativistic model.

10. - OTHER DECAY PROCESSES

To investigate processes other than meson annihilation, it is necessary :

- a) to assume that some particular diagram dominates the process (as is usual but a priori quite unjustifiable);
- b) to have a more detailed knowledge of the wave function if absolute rates are required.

These facts preclude us from performing any other reliable calculations. Nevertheless, it is interesting to consider how simple models can be used to investigate the effect of mass splitting on $SU(3)$ relationships. For example, it is well known that $SU(3)$ gives the result

$$\langle \eta | 2\gamma \rangle = \frac{1}{\sqrt{3}} \langle \pi^0 | 2\gamma \rangle \quad (10.1)$$

but it is unclear how this relation should be interpreted - it might refer to the invariant amplitudes or to the non-invariant matrix elements. In fact, it is usually assumed that if we write the interaction as

$$\mathcal{L}_{m \rightarrow 2\gamma}^{\text{eff}} = g_{m\gamma\gamma} \phi_m \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \quad (10.2)$$

then

$$g_{\eta\gamma\gamma} = \frac{1}{\sqrt{3}} g_{\pi^0\gamma\gamma} \quad (10.3)$$

To investigate these possibilities, we make the questionable assumption that the process is dominated by the graphs

so that the invariant amplitude is :

$$A(m \rightarrow 2\gamma) = \text{Trace} \sum_i g_i e_i^2 \left(\int \mathcal{X}^k \gamma_\mu \frac{1}{k-M} \gamma_\nu A'_\mu A'_\nu \frac{d^4 k}{(2\pi)^4} + \int \mathcal{X}^m \gamma_\mu \frac{1}{k'-M} \gamma_\nu A'_\mu A'_\nu \frac{d^4 k'}{(2\pi)^4} \right) \quad (10.4)$$

Then, in the model we have advocated, we obtain :

$$g_{\eta\gamma\gamma} \approx \frac{(\cos\theta - 2\sqrt{2}\sin\theta)}{\sqrt{3}} g_{\pi\gamma\gamma} \quad (10.5)$$

where we have taken into account the possibility of $\eta - \eta'$ mixing with a mixing angle θ ²⁸⁾. [This is the result usually assumed, except that the factor $2\sqrt{2}$ has previously been derived from other assumptions, e.g., $\tilde{U}(12)$ symmetry or vector dominance.] In contrast, the usual quark model, in which $\chi_{--} \ll \chi_{++}$ and (if there is small symmetry breaking in the form of the wave function) $\sqrt{M_\eta} \chi_{++}^\pi = \sqrt{M_\pi} \chi_{++}^\eta$, gives an extra factor $\sqrt{M_\pi/M_\eta}$ on the right-hand side of Eq. (8.7). Using $\Gamma(\pi^0 \rightarrow 2\gamma) = 7.4 \pm 1.9$ eV, and $\theta = -10.8^\circ$ ^{*}), our model gives $\Gamma(\eta \rightarrow 2\gamma) = 374 \pm 100$ eV, and the usual model gives $\Gamma(\eta \rightarrow 2\gamma) = 94 \pm 25$ eV. Neither result agrees with the measured value $\Gamma(\eta \rightarrow 2\gamma) = 875 \pm 230$ eV ²⁹⁾, but there is no reason to assume that the diagram considered is dominant. However, this calculation provides an excellent illustration of the very different effects of mass splitting in different models.

A better assumption might be that $\eta \rightarrow 2\gamma$ and $\pi^0 \rightarrow 2\gamma$ are given by the vector dominance model. Writing

$$\mathcal{L}_{\eta \rightarrow \gamma\gamma} = g_{\eta\gamma\gamma} \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha V_\beta \partial_\gamma V'_\delta \phi_\eta \quad (10.6)$$

and taking a simple quark-antiquark loop model for this interaction our model gives :

$$g_{\eta\pi^0\omega} \approx \frac{\sqrt{3} g_{\eta\rho\rho}}{\cos\theta + \sqrt{2}\sin\theta} \approx \frac{\sqrt{3} g_{\eta\omega\omega}}{\cos\theta + \sqrt{2}\sin\theta} \approx \left(\frac{M_\phi}{M_\rho}\right)^2 \frac{\sqrt{3} g_{\eta\phi\phi}}{(-2\cos\theta + \sqrt{2}\sin\theta)} \quad (10.7)$$

where we have put $M_\omega = M_\rho$ and assumed "ideal" ω/ϕ mixing; the other relevant couplings are zero. Using the $V \rightarrow \gamma$ couplings given by Model 1 in Section 6, we obtain :

^{*}) This is the value given by a quadratic mass formula with the sign given by the naïve quark model ²⁸⁾. A linear mass formula gives $\theta = -23.3^\circ$, in which case our model gives $\Gamma(\eta \rightarrow 2\gamma) = 685 \pm 175$ eV.

$$g_{\eta\gamma\gamma} = \frac{g_{\pi\gamma\gamma}}{\sqrt{3}} \left[\frac{\cos \theta}{3} \left(5 - 2 \left(\frac{M_\rho}{M_\phi} \right)^4 \right) - \frac{\sin \theta}{3} \sqrt{2} \left(5 + \left(\frac{M_\rho}{M_\phi} \right)^4 \right) \right] \quad (10.8)$$

With $\theta = -10.8^\circ$ this gives $\Gamma(\eta \rightarrow 2\gamma) = 595 \pm 150$ eV. (Note that in the symmetry limit $M_\rho = M_\phi$ and we recover the previous result.) Again this is an almost totally unreliable calculation but it illustrates the extreme sensitivity to the symmetry breaking [Weisskopf and Van Royen ⁶⁾ discuss yet another model.]

Another example of using these techniques to suggest the form of symmetry breaking is given in a footnote to Ref. 3).

11. - CONCLUSIONS AND IMPLICATIONS

The power of the naïve quark model is that it works in so many cases although the assumptions (dominance of simple quark states, additivity, etc.) cannot be justified a priori. It could be that this reflects some property of hadronic matter other than the existence of quarks (except possibly as excitations or quasi-particles without corresponding asymptotic states). If quarks exist and are the fundamental constituents of hadrons, then the annihilation of mesons is determined by their B.S. wave function. It must be emphasized, however, that unless the properties of mesons are dominated by simple quark states, the B.S. wave function may be almost irrelevant to all other processes, although they can be described in terms of it in principle. Correspondingly, the potential in the B.S. equation may have important contributions from immensely complicated intermediate states; this does not necessarily mean that the equation is irrelevant to meson spectroscopy as the potential might still be approximated by a simple function which we could treat phenomenologically without expecting to "derive" it from first principles.

Therefore the only part of this work which is on a really secure basis (given that quarks exist) is the treatment of annihilation processes. Here the quark model plays the essential role of providing a large parameter M , so that if $f_\pi = f_\pi(M_\pi/M)$ is analytic in its argument we can obtain

$f_{\pi} = f_K$, which Cabibbo assumed in his original analysis. [This is not a completely obvious result; we also exhibited models in which $M_{\pi} f_{\pi} = M_K f_K$ and $\sqrt{M_{\pi}} f_{\pi} = \sqrt{M_K} f_K$.] The calculation shows that if we do make the usual assumptions that simple quark states dominate other processes, then we should use a relativistic wave function ($\alpha_{++} \simeq \alpha_{--}$), at least for $L = 0$ mesons.

This work provides no justification for the many successes of the simple quark model but these can usually be obtained in other ways which are quite compatible with the model (e.g., from current algebra which follows from the quark model). The outstanding cases are the calculation of $\Gamma(\omega \rightarrow \pi^0 \gamma)$ in terms of the proton magnetic moment μ_p (which remains a mystery), the calculation of μ_p/μ_n which reflects the symmetry of the wave function [and can be derived from $SU(6)$ as an abstract symmetry] and the amazingly successful classification of resonances achieved by Dalitz³⁾. The following are therefore highly desirable :

1. a systematic investigation of meson resonances in the relativistic model. In particular the anomalous states (cf., p.8) are not fully understood (they seem to be absent experimentally).
2. a relativistic formulation of the quark model for baryons. An outstanding problem here is that Dalitz's successful analysis requires symmetric wave functions^{*)}. It is of great interest to see if the non-relativistic result $R = \frac{3}{5}(G_A/G_V)$ can also be derived in a relativistic model (the calculation necessarily involves some assumption about the intermediate states involved); note that our calculation of γp depends on R and can therefore not be regarded as a straightforward result of the model.

*) If we make the bold assumption that quark fields commute, it is interesting to observe that the non-leptonic part of the Lagrangian in Eq. (5.1) automatically has only a $\Delta I = \frac{1}{2}$ part for parity violating decays (and also for parity conserving decays if $R=1$) - this is obtained by making a Fierz transformation. This result has also been (implicitly) derived by Feldman and Matthews³⁰⁾ who have systematically exploited the Fierz transformation in an investigation of weak Lagrangians (I am indebted to Professor J. Prentki for bringing their work to my attention.)

The B.S. formalism is a powerful tool for investigating the quark model and it is interesting that Salpeter and Bethe suggested using it to investigate the Fermi-Yang model in their original paper. Ultimately, however, any model involving real quarks depends on the discovery of these elusive objects. The following report (taken from a letter to the Times of London, by Mr. Roger Robinson - 28/2/68) may be of interest to quark hunters :

"... I was lucky enough to see and hear Nycticorax Cyanocephalus Falklandicus, commonly known as the Quark ... only to be found in these remote islands whose inhabitants told me they derive their names from the strangely beautiful call they emit on being disturbed."

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A P P E N D I X - A

EXPLICIT EQUATIONS FOR THE χ_{ij}

It is sometimes convenient to use the explicit equations for the functions χ_{++} , etc., [Eq. (2.13)]. In the approximation of Eq. (2.12) these are :

$$\chi_{+-} = [(q_0 + M)^2 - \mu^2] W - \vec{\sigma} \cdot \vec{q} Z \vec{\sigma} \cdot \vec{q} - (q_0 - \mu + M) \vec{\sigma} \cdot \vec{q} Y + (q_0 + \mu + M) X \vec{\sigma} \cdot \vec{q} \quad (A.1)$$

$$\chi_{-+} = [(q_0 - M)^2 - \mu^2] Z - \vec{\sigma} \cdot \vec{q} W \vec{\sigma} \cdot \vec{q} + (q_0 + \mu - M) Y \vec{\sigma} \cdot \vec{q} + (q_0 - \mu + M) \vec{\sigma} \cdot \vec{q} X \quad (A.2)$$

$$\chi_{++} = [(M + \mu)^2 - q_0^2] X + \vec{\sigma} \cdot \vec{q} Y \vec{\sigma} \cdot \vec{q} - (q_0 + \mu - M) W \vec{\sigma} \cdot \vec{q} + (q_0 - \mu - M) \vec{\sigma} \cdot \vec{q} Z \quad (A.3)$$

$$\chi_{--} = [(M - \mu)^2 - q_0^2] Y + \vec{\sigma} \cdot \vec{q} X \vec{\sigma} \cdot \vec{q} + (q_0 - \mu + M) \vec{\sigma} \cdot \vec{q} W - (q_0 + \mu - M) Z \vec{\sigma} \cdot \vec{q} \quad (A.4)$$

where

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \frac{-1}{[(P+q)^2 - M^2][(P-q)^2 - M^2]} \int \hat{V} X \frac{d^4 k}{(2\pi)^4}, (\vec{p}=0). \quad (A.5)$$

Note that for a spin independent potential W and Z depend only on χ_{+-} and χ_{-+} and X and Y depend only on χ_{++} and χ_{--} . If also $|\vec{q}| \ll M$, then a solution satisfying the condition

$$\chi_{\pm\mp} \sim \frac{|\vec{q}|}{M} \chi_{\pm\pm}$$

may occur and it will be essentially given by solving Eqs. (A.3) and (A.4) for χ_{++} and χ_{--} , as they approximately decouple from (A.1) and (A.2). Such a solution will satisfy approximate $SU(6)$ symmetry since the functions $\chi_{\pm\pm}$ differ only in their spin structure for members of the same $SU(6)$ multiplet. Note that in these circumstances a solution satisfying the condition $\chi_{\pm\pm} \sim \frac{|\vec{q}|}{M} \chi_{\pm\mp}$ may also occur and it will be essentially given by

solving Eqs. (A.1) and (A.2) for χ_{+-} and χ_{-+} as they approximately decouple from (A.3) and (A.4). One might expect that such solutions satisfied a kind of SU(6) for states whose χ_{++} have the same L (and whose χ_{--} , by which they are classified, therefore have $L \pm 1$) ; this is not the case as these χ_{+-} differ not only in their spin structure but also in their dependence on q_0 (cf. Section 8).

Using the potential of Eq. (4.10), we find :

$$X = V_1 \chi_{++} + V_2 \chi_{--} - V_3 (\chi_{++} + \sigma_i \chi_{--} \sigma_i) \quad (A.6)$$

$$Y = V_1 \chi_{--} + V_2 \chi_{++} - V_3 (\chi_{--} + \sigma_i \chi_{++} \sigma_i) \quad (A.7)$$

Hence we see once again that the usual model ($\vec{q}^2 \ll M^2$, $p^2 \ll M^2$, $\chi_{+-} \sim |\vec{q}| \chi_{++}$, $\chi_{--} \ll \chi_{++}$) can only occur if V_2 and V_3 are negligible or conspire to cancel the contribution of χ_{++} in Eq. (A.7).

A P P E N D I X - E

FACTORIZING POTENTIAL MODEL

If we take the potential

$$- \frac{\hat{V}(p, q, k)}{(2\pi)^8} \chi(p, k) = W(p, q) W(p, k) g (V_1 \chi + V_2 \gamma_5 \chi \gamma_5 + V_3 \gamma_\mu \chi \gamma_\mu) \quad (B.1)$$

with a given function W and constants V_i , then the B.S. equation is exactly soluble if we use the simplest one-body propagators. For a $J^P = 0^-$ state :

$$(\not{q} + \not{p} - M) \chi(q, p - M) = \gamma_5 (a + c \not{p}) W(q, p) \quad (B.2)$$

where a and c are constants. The B.S. equation reduces to two algebraic equations which determine a/c and g (for given μ^2). We find

$$a = g (V_1 + V_2 - 4V_3) \left[(a(M^2 + \mu^2) - 2\mu^2 c) I_1 - a I_2 + a I_3 \right] \quad (B.3)$$

$$c = g (V_1 - V_2 + 2V_3) \left[(-2Ma + c(\mu^2 + M^2)) I_1 - c I_2 - c I_3 \right] \quad (B.4)$$

and hence

$$\frac{a}{c} = \frac{\alpha \pm \sqrt{\alpha^2 + \beta^2 z^2 \mu^2}}{\beta} \quad (B.5)$$

where

$$\alpha = [I_1 (M^2 + \mu^2) - I_2] (1 - z) - I_3 (1 + z)$$

$$\beta = 4 M I_1$$

$$z = \frac{V_1 + V_2 - 4 V_3}{V_1 - V_2 + 2 V_3}$$

$$\begin{aligned}
 I_1 &= \int \frac{W(k, P) d^4 k}{[(k+P)^2 - M^2][(k-P)^2 - M^2]}, \quad (\vec{P}=0). \\
 I_2 &= \int \frac{k_0^2 W(k, P) d^4 k}{[(k+P)^2 - M^2][(k-P)^2 - M^2]}, \quad (\vec{P}=0). \\
 I_3 &= \int \frac{\vec{k}^2 W(k, P) d^4 k}{[(k+P)^2 - M^2][(k-P)^2 - M^2]}, \quad (\vec{P}=0).
 \end{aligned} \tag{B.6}$$

In this model :

$$\chi_{\pm\pm} = \frac{[a(M^2 - q^2 + p^2) - 2Mp^2C \pm p(2q_0^2C + 2Ma - c(p^2 + q^2 + M^2))]W}{[(q+P)^2 - M^2][(q-P)^2 - M^2]} \tag{B.7}$$

For vanishingly small μ , a/c can be expanded in a power series in μ^2 giving either $a/c \sim M$ (which implies $\chi_{++} \simeq \chi_{--}$) or $a/c \sim \mu^2/M$ (which implies $\chi_{++} \simeq -\chi_{--}$). For small but finite $\mu^2 \ll M^2$ these expansions will still be valid, provided $\alpha \gg \beta z\mu$. Hence the expansion fails when $z=1$, which requires $|V_1 - V_2| \gg |V_2 - 3V_3|$, and $\langle \vec{k}^2 \rangle \ll \mu M$ (excluding conspiracies such as $I_1 M^2 = I_2$). These are the conditions in which we showed that the usual model can occur. If $\langle \vec{k}^2 \rangle \ll \mu M$ and $z=1$, so that $\alpha \ll \beta \mu z$, we find $a/c = \pm \mu$, so that $\chi_{++} \ll \chi_{--}$ or $\chi_{--} \ll \chi_{++}$.

We find that :

$$\text{if } \chi_{++} \simeq \chi_{--} : I \simeq g(V_1 + V_2 - 4V_3)[M^2 I_1 - I_2 + I_3] \tag{B.8}$$

$$\text{if } \chi_{++} \simeq -\chi_{--} : I \simeq g(V_1 - V_2 + 2V_3)[M^2 I_1 - I_2 + I_3] \tag{B.9}$$

Turning now to vector mesons, we have :

$$(q + P - M) \chi_\mu (q - P - M) = g W T_\mu (a' + c' P) \tag{B.10}$$

The B.S. equation gives :

$$a' = g(V_1 - V_2 - 2V_3) \left[(a'(M^2 + \mu^2) - 2M\mu^2 c') I_1 - a' I_2 + \frac{a' I_3}{3} \right] \quad (\text{B.11})$$

$$c' = g(V_1 + V_2) \left[(c'(M^2 + \mu^2) - 2Ma') I_1 - c' I_2 - \frac{5c' I_3}{3} \right] \quad (\text{B.12})$$

Hence

$$\frac{a'}{c'} = \frac{\eta \pm \sqrt{\eta^2 + \nu^2 \mu^2 z'}}{\nu} \quad (\text{B.13})$$

where

$$z' = \frac{V_1 - V_2 - 2V_3}{V_1 + V_2} \quad (\text{B.14})$$

$$\nu = 4MI_1$$

$$\eta = [I_1(M^2 + \mu^2) - I_2](1 - z) + \frac{(5 + z)}{3} I_3$$

In this model

$$\begin{aligned} \vec{x}_{\pm\pm} = & \left\{ \vec{\sigma} [2Ma'\mu + 2c'\mu q_0^2 - \mu c'(q^2 + M^2 + \mu^2) \pm (a'(M^2 + \mu^2 - q^2) - 2M\mu^2 c')] \right. \\ & \left. \mp 2a'\vec{q} \cdot \vec{\sigma} \cdot \vec{q} \right\} W / [(p+q)^2 - M^2][(p-q)^2 - M^2] \end{aligned} \quad (\text{B.15})$$

The same remarks apply to the approximation of a'/c' for small μ as apply to a/c above. Here, if $a' \sim c'$, we obtain $\chi_{++} \simeq -\chi_{--}$, and if $a' \sim \mu^2 c'$, we obtain $\chi_{++} \simeq \chi_{--}$. So that :

$$\text{if } \vec{x}_{++} \simeq \vec{x}_{--} : \quad 1 \simeq g(V_1 + V_2) \left[M^2 I_1 - I_2 - \frac{5I_3}{3} \right] \quad (\text{B.16})$$

$$\text{if } \vec{x}_{++} \simeq -\vec{x}_{--} : \quad 1 \simeq g(V_1 - V_2 - 2V_3) \left[M^2 I_1 - I_2 + \frac{I_3}{3} \right] \quad (\text{B.17})$$

If the potential is spin independent (V_3 negligible compared to V_1 and V_2) and $\vec{q}^2 \ll M^2$, so that I_3 is negligible compared to I_1 , we see that the equations (B.8) and (B.16) become approximately identical and so do (B.9) and (B.17). Thus we have approximate $SU(6)$; for a given g , the existence of a pseudoscalar meson with $\mu^2 \ll M^2$ and $\alpha_{++} \simeq \pm \alpha_{--}$ implies the existence of a vector with $\mu^2 \ll M^2$, $\bar{\alpha}_{++} \simeq \pm \bar{\alpha}_{--}$ and $\bar{\alpha}_{++} \sim \vec{\sigma} \alpha_{++}$.

A P P E N D I X - C

NUMERICAL SOLUTIONS OF THE FERMION B.S. EQUATION

Narayanaswamy and Pagnamenta (N.P.) have solved the fermion-antifermion B.S. equation for 0^- and 1^- states in momentum space in the approximation $f=g=1$ with a potential due to the exchange of a pseudo-scalar meson of mass m in ladder approximation with a cut-off at the quark mass M ³¹). They only found the eigenvalues of the coupling constant g and not the form of the wave functions. However, Holdsworth has solved the Goldstein equation (cf., Appendix D) for the same potential in co-ordinate space with a cut-off at M ³²). Comparing their spectra for $g^2/4\pi$ for 0^- states (with $m/M = 1/10$, $\mu = 0$) :

<u>Holdsworth</u>	<u>N.P.</u>
32	29
78	72
151	156

The similarity makes it reasonable to infer that N.P.'s solutions are the same as Holdsworth's, i.e., the ground state belongs to Model 1, in which the Goldstein equation gives the coupling constant if it is an eigenvalue equation (the small discrepancies are accounted for by the fact that the results are sensitive to the cut-off procedure which is different in co-ordinate space and in momentum space). For comparison with the ground state value $g^2/4\pi \simeq 32$, $g_{NN\pi}^2/4\pi \simeq 14$.

N.P. found that tightly bound 0^- states were lighter than 1^- states; taking $M_v/M_{ps} = 4$, gave M about two nucleon masses. These results are encouraging for the quark model.

A P P E N D I X - D

ZERO MASS BOUND STATES

There are in the literature many discussions of zero mass bound states $(14), (16), (17), (18), (31), (33), (34), (35)$. Many of these discussions treat fermion-fermion bound states (whose wave function we denote by χ_{ff}) and we begin by giving the relation with the fermion-antifermion problem (with wave function χ). In the simplest approximation (which we use for notational convenience in this Appendix), χ_{ff} satisfies :

$$(q + p - M)\chi_{ff}(p^T, q^T - M) = - \int V \chi_{ff} \frac{d^4 k}{(2\pi)^8} \quad (D.1)$$

Therefore

$$(q + p - M)\chi_{ff} C^{-1}(q - p - M) = - \int V \chi_{ff} \frac{d^4 k}{(2\pi)^8} \quad (D.2)$$

and χ and $\chi_{ff} C^{-1}$ satisfy essentially the same equation. For bound states with parity π_B [cf., Eq. (2.15)] :

$$\chi_{ff}(p, q) C^{-1} = - \pi_B \gamma_0 \chi_{ff}(\tilde{p}, \tilde{q}) \gamma_0 \quad (D.3)$$

For identical fermions, the exchange antisymmetry gives :

$$\chi_{ff}(p, q) C^{-1} = C (\chi_{ff}(p, q) C^{-1})^T C^{-1} \quad (D.4)$$

Hence $\chi_{ff} C^{-1}$ for a state with parity π_B is subject to the same constraints as χ for a state with parity $-\pi_B$ and $\epsilon_B = +1$ [apart from some sign changes in the potential since $(V\chi)C^{-1} \neq V(\chi C^{-1})$] and the discussion of a $J^{PC} = 0^{-+}$ state below also applies to a state of two identical fermions with $J^P = 0^{+}$.

The equations describing a zero mass $J^{PC} = 0^{-+}$ bound state are given by putting $P^2 = \mu^2 = 0$ in Eqs. (4.6) to (4.9); putting $f = g = 1$ and using the potential of Eq. (4.10), we obtain :

$$A(M^2 - q^2) + 2MB(q \cdot p)^2 - 4D(q \cdot p)^2 = - \int (V_1 + V_2 - 4V_3) A d^4 k. \quad (D.5)$$

$$(q \cdot P) \{ B q^2 (M^2 - q^2) + 2B(q \cdot P)^2 + C(M^2 - q^2) + 2MA \} \\ = - \int (V_1 - V_2 - 2V_3) (q \cdot k k \cdot P B + q \cdot P C) d^4 k. \quad (D.6)$$

$$(q \cdot P)^2 \{ B(M^2 - q^2) - 2C - 4MD \} = - \int (V_1 - V_2 + 2V_3) B(k \cdot P)^2 d^4 k \quad (D.7)$$

$$(q \cdot P) \{ A + MC + D(q^2 + M^2) \} = - \int (V_1 + V_2 - 2V_3) k \cdot P D d^4 k \quad (D.8)$$

When the variable $q \cdot P = 0$, Eq. (D.5) reduces to the notorious Goldstone equation; with a potential due to zero mass exchange this equation has solutions for all values of the coupling constant - this is the famous Goldstone paradox. In fact, it is clear from Eqs. (B.10) to (B.13) that Goldstone's equation is not the correct eigenvalue equation for a zero mass particle; in general, we are faced with the problem of solving four coupled equations in two variables - Goldstone's equation is only one of these if $B = 2D$ in which case three coupled equations for three functions remain (if $B = D = 0$ then $A = C = 0$ and the solution is trivial). Whether the coupled equations have a discrete spectrum is another problem.

The zero mass case has usually been treated by letting $P \rightarrow 0$ in the B.S. equation in matrix form. In this way, Eqs. (B.11) to (B.13) are not derived (giving the erroneous impression that a solution with $A \neq 0$, $B = C = D = 0$ is possible). In fact, the point $P = 0$ is physically uninteresting as it cannot be boosted to any other point by a Lorentz transformation. [Unfortunately the analogy with the boson B.S. equation is misleading; in that case there is only one integral equation which does not vanish at $P = 0$ and hence there are solutions which are only functions of q^2 in contrast to the present case.]

In general Mandelstam has shown ¹⁴⁾ that the Goldstone equation is an eigenvalue equation provided the potential is less singular than the potential due to one-particle exchange at the origin in co-ordinate space. In this case the spectrum of the Goldstone equation (regarded as an eigenvalue equation for the coupling constant) certainly includes the spectrum of the coupled zero mass equations with $(q \cdot P) \neq 0$. Recall that our Model 1 is suggested by the

limit $\mu^2 \rightarrow 0$ in Eqs. (4.11) - (4.14); that is, we considered all terms involving μ^2 as perturbations in these equations. However, in the simplest approximation for the one-body propagators $f=g=1$, Eq. (4.11) becomes the Goldstone equation as $\mu^2 \rightarrow 0$ [this is to be expected as we have effectively taken the limit $P \rightarrow 0$; note that Eqs. (4.12) - (4.14) do not become the correct zero mass equations if we let $\mu^2 \rightarrow 0$ - this is because they have been divided by μ]. In order that perturbation theory in μ^2 makes sense, we therefore require that the Goldstone equation is an eigenvalue equation (or else that the functions f and g provided an effective cut-off).

Model 1 may therefore restrict the form of the potential at the origin. Greenberg has pointed out ³⁶⁾ that the same restriction is needed in a Schrödinger equation quark model if $\vec{q}^2 \ll M^2$ is required; the same conclusion was reached by Domokos and Palmer, using the N/D method ³⁷⁾. In the course of detailed numerical investigations of the B.S. equation for bosons, Holdsworth ³²⁾ and Pagnamenta ³⁸⁾ also found this restriction to be necessary if $\vec{q}^2 \ll M^2$ is required. They found that it was only in this case that the mass of the orbitally excited states was of the order of magnitude required by the usual quark model interpretation. Narayanaswamy and Pagnamenta state ³¹⁾ that orbitally excited eigenstates of the fermion-antifermion B.S. equation with $f=g=1$ and a potential due to one-particle exchange also turn out to be far too massive for the usual interpretation. Therefore, the usual quark model requires the condition $\vec{q}^2 \ll M^2$ not only in order that $SU(6)$ can hold but probably also if we wish to interpret observed high spin mesons as systems with $L \gg 1$, rather than s wave states of $nQ - n\bar{Q}$ with $n \gg 2$ as discussed by Squires ²⁰⁾ (this is, perhaps, obvious). Furthermore, in all models which have been investigated, $\vec{q}^2 \ll M^2$ can only occur if the allowed singularities of the potential are restricted. This restriction is just such that the Goldstone equation is an eigenvalue equation. Therefore Model 1 is quite consistent with our interpretation of the meson spectrum.

REFERENCES

- 1) C.H. Llewellyn Smith, Oxford thesis, chapter 4 (September, 1967);
Oxford preprint (1967).
- 2) C.H. Llewellyn Smith, to be published in Physics Letters.
- 3) An excellent introduction is :
R.H. Dalitz, Lectures contained in "High Energy Physics", Gordon and
Breach, Publ. (1965).
The analysis of resonances in the quark model is discussed in :
R.H. Dalitz, "Excited nucleons and baryonic supermultiplets", talk
at the Topical Conference on πN Scattering, Irvine, Calif.,
December 1967, (Oxford preprint).
R.H. Dalitz, "Mesonic resonance states", paper presented at the Topical
Conference on Meson Spectroscopy, Philadelphia, Pa., May 1968,
(Oxford preprint).
The quark model for high energy scattering is discussed in the
following recent review by its originators :
E.M. Levin and L.I. Frankfurt, Uspekhi Fisicheskikh Nauk 94, 243 (1968).
- 4) C. Morpurgo, Physics 2, 95 (1965).
- 5) V.A. Matveev, B.V. Struminski and A.N. Tavkhelidze, Dubna report P2524
(1965).
- 6) R.P. Van Royen and V.F. Weisskopf, Nuovo Cimento 50A, 617 (1967).
- 7) E.E. Salpeter and H.A. Bethe, Phys.Rev. 82, 1232 (1951).
- 8) M. Gell-Mann and F.E. Low, Phys.Rev. 82, 350 (1951).
- 9) K. Kawarabayashi and M. Suzuki, Phys.Rev.Letters 16, 255 (1966).
- 10) D. Geffen, Phys.Rev.Letters 19, 770 (1967).
- 11) J.G. Taylor, Nuovo Cimento Suppl. 1, 857 (1963).
- 12) D. Lurié, A.J. Macfarlane and Y. Takahashi, Phys.Rev. 140, 1091 (1965).
- 13) C.H. Llewellyn Smith, CERN preprint TH.946 (1968) - to be published in
Nuovo Cimento.
- 14) S. Mandelstam, Proc.Roy.Soc. A233, 248 (1955).
- 15) K. Nishijima, Prog.Theoret.Phys. 10, 549 (1953); 12, 279 (1954);
13, 305 (1955).
- 16) J. Goldstein, Phys.Rev. 91, 1516 (1953).
- 17) See, e.g., J. Harte, Nuovo Cimento 45, 179 (1966).

- 18) The literature can be traced from :
M. Ciafaloni, Nuovo Cimento 51A, 1090 (1967).
- 19) A. Klein and C. Zemach, Phys.Rev. 108, 126 (1951).
- 20) E.J. Squires, in "Particle Interactions at High Energies", Oliver and Boyd (1967).
- 21) R.J. Oakes and C.N. Yang, Phys.Rev.Letters 11, 174 (1963).
- 22) R.H. Dalitz, Proc.Roy.Soc. A288, 183 (1965).
- 23) M. Roos, (private communication). This fit takes into account all the data presented at the Vienna Conference.
- 24) N. Brene, M. Roos and A. Sirlin, Nuclear Phys. B6, 255 (1968).
- 25) S.C.C. Ting, Proceedings of the Vienna Conference, published by CERN (1968).
- 26) M.Y. Han and Y. Nambu, Phys.Rev. 139, B1006 (1965).
- 27) A. Dar, "Quark model predictions for hadron decay - II", M.I.T. preprint (1967).
- 28) R.H. Dalitz and D.G. Sutherland, Nuovo Cimento 37, 1777 (1965).
- 29) Taken from the particle data group compilation, UCRL 8030 (August, 1968)-unpublished.
- 30) G. Feldman and P.T. Matthews, Nuovo Cimento 50A, 349 (1967).
- 31) P. Narayanaswamy and A. Pagnamenta, Nuovo Cimento 53A, 635 (1968).
- 32) D. Holdsworth, Oxford thesis (1968). I am grateful to Dr. Holdsworth for permission to quote these results.
- 33) S. Okubo and G. Feldman, Phys.Rev. 117, 279 (1960).
- 34) W. Kummer, Nuovo Cimento 31, 219 (1964).
- 35) A. Bastai, L. Bertocchi, G. Furlan and M. Tonin, Nuovo Cimento 30, 1532 (1963).
- 36) O.W. Greenberg, Phys.Rev. 147, 1077 (1966).
- 37) G. Domokos and W. Palmer, Phys.Rev. 149, 1152 (1966).
- 38) A. Pagnamenta, Nuovo Cimento 53A, 30 (1968).