Nonperturbative SUSY Correlators at Finite Temperature

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ABSTRACT

We calculate finite temperature effects on a correlation function in the two dimensional supersymmetric nonlinear O(3) sigma model. The correlation function violates chiral symmetry and at zero temperature it has been shown to be a constant, which gives rise to a double–valued condensate. Within the bilinear approximation we find an exact result in a one–instanton background at finite temperature. In contrast to the result at zero temperature we find that the correlation function decays exponentially at large distances.
1. Introduction

During the last few years there has been a lot of progress in understanding the dynamics of supersymmetric (SUSY) theories. Although the main achievements have been obtained in $N = 2$ supersymmetry, where e.g. the low energy effective action can be calculated exactly [1], there also exist some remarkable results in $N = 1$. Among these are the dual descriptions of supersymmetric QCD, originally proposed by Seiberg [2].

Related to this last issue is the calculation of certain instanton induced correlation functions in $N = 1$ SUSY Yang–Mills theory [3, 4]. The action for this theory possesses a global $R$–symmetry that is conserved classically, but broken by quantum effects. Even though the symmetry is anomalous, there still remains a nonanomalous discrete subgroup. Some specific correlation functions that violate chiral symmetry, and therefore vanish to all orders in perturbation theory, can be calculated in an instanton background for small distances and are found to have a constant value, independent of the actual distance. The reason for this nonvanishing value is of course that the instantons change the chirality by a definite value. Furthermore, by using supersymmetric arguments it was shown that this result actually is exact, for all distances. By the cluster decomposition this implies a nonvanishing value of the gluino condensate, $\langle \lambda \lambda \rangle$, and the nonvanishing value spontaneously breaks the discrete symmetry down to $Z_2$. As has recently been noted, this condensate gives rise to domain wall solutions [5].

It is a well known feature that if a symmetry is spontaneously broken at zero temperature, it is often restored above some, possibly high, temperature. This is believed to be the case e.g. for the Higgs mechanism in the Electroweak Standard Model [6] and for the spontaneously broken chiral symmetry in QCD [7]. In view of this, it is clearly interesting to study also the finite temperature effects on the spontaneously broken discrete symmetries in supersymmetry, especially when one considers the solid results that already exist at $T = 0$.

Since the exact results at zero temperature depend crucially on specific properties of the supersymmetric theory, and since these relations in general are not valid at finite
temperature, one should not expect the $T > 0$ results to be constrained in the same precise manner as at $T = 0$. However, the underlying structure of these nontrivial theories may still be simple enough in order to obtain interesting results.

In the SUSY SU(2) Yang–Mills theory at finite temperature, some preliminary results were recently obtained [8], but no definite, conclusive arguments could be given. In such a case, it may prove worthwhile to study the possible features in a somewhat simpler context. For this purpose we will use the two dimensional supersymmetric O(3) $\sigma$–model as a toy model.

The SUSY $\sigma$–model shares many of the properties of the SUSY Yang–Mills theory. Apart from being supersymmetric, they also have asymptotic freedom and instanton solutions in common [9]. Furthermore, in this toy model there also exists an exact calculation of a condensate [10], that spontaneously breaks the nonanomalous discrete chiral symmetry $Z_4 \to Z_2$ (for a comprehensive review of the SUSY $\sigma$–model, see e.g. [11]). Some precaution in the analogy at finite temperature is needed however, since by Peierls’ argument one would expect a theory in only one spatial dimension to restore a broken discrete symmetry at any nonzero temperature. Nevertheless, our hope is that this model still could serve as a mathematical laboratory and provide some new insights to the possible scenarios these discrete symmetries may undergo at finite temperature, e.g. in the SUSY Yang–Mills theory.

The calculation is also interesting from a more technical point of view, since we find that the quadratic fluctuations around the instanton solution cancel between the bosons and the fermions, at all temperatures. This is a rather surprising result, since the different boundary conditions imposed on the bosonic and fermionic field at finite temperature naively seems to destroy such a “supersymmetric” cancellation. Although this behavior seems to be a specific property in two dimensions (in contrast to e.g. SUSY SU(2) Yang–Mills in four dimensions [8]), it supports the reasoning that the SUSY theories provide simple and constrained, but still nontrivial, examples of doable models at finite temperature.
The paper is organized as follows. In the next section we set up the model and recall some of the main results at zero temperature. In section 3 we generalize the necessary ingredients to finite temperature, and section 4 is devoted to the calculation of the finite–$T$ correlator, with some of the details given in the appendices. Finally, in section 5 we give our conclusions, and comment on the results.

2. Supersymmetric O(3) $\sigma$-model at zero temperature.

The Euclidean action of the supersymmetric O(3) $\sigma$-model in two dimensions is defined as \[ S = \frac{1}{2g^2} \int d^2x \, d^2\theta \, \varepsilon^{\alpha\beta} D_\alpha \Phi_a D_\beta \Phi_a , \] (2.1)
where $a = 1, 2, 3$ is the internal isospin index, $g$ the coupling constant, $D_\alpha$ the supercovariant derivative and $\Phi_a$ a real superfield,

$$\Phi_a(\vec{x}, \theta) = \varphi_a(\vec{x}) + \bar{\theta} \Psi_a(\vec{x}) + \frac{1}{2} \theta^2 F_a(\vec{x}) ,$$ (2.2)

satisfying the constraint $\sum_a \Phi_a(\vec{x}, \theta) \Phi_a(\vec{x}, \theta) = 1$. This model generalizes the ordinary, non–supersymmetric $\sigma$–model.

Instead of three variables and one constraint, it is convenient to use a stereographic projection and trade the original fields for a complex-valued unconstrained field,

$$\Theta = \Phi_1 + i\Phi_2 \quad \frac{1}{1 + \Phi_3} .$$ (2.3)

The dynamical component fields will then be a complex scalar $\phi$ and its fermionic counterpart $\psi$. The original fields $\varphi_a$ and $\Psi_a$ transform as vectors under O(3)–rotations, and based on these transformation properties one can find the corresponding action on $\phi$ and $\psi$. Using the three Euler angles $0 \leq \alpha, \gamma \leq 2\pi$, $0 \leq \beta \leq \pi$ and defining $\lambda = \tan(\beta/2)$, we have \[ \phi \rightarrow e^{i\gamma} \frac{\phi e^{i\alpha} - \lambda}{1 + \lambda\phi e^{i\alpha}} \quad \psi \rightarrow (1 + \lambda^2)\psi e^{i(\alpha + \gamma)} \quad \frac{1}{(1 + \lambda\phi e^{i\alpha})} . \] (2.4)
In order to regulate the infrared divergencies it is assumed that the Euclidean space is restricted to a sphere of radius $R$, and the original flat metric then becomes the conformally flat one, $g_{\mu\nu} = \Omega_0^2 \delta_{\mu\nu}$, with $\Omega_0 = (1 + (\vec{x}^2/4R^2))^{-1}$. At the end of the calculations the flat space limit $R \to \infty$ is understood to be taken. With this modification the action (2.1) becomes

$$S = \frac{2}{g^2} \int d^2x \Omega_0^2 \chi^{-2} \left[ \Omega_0^{-2} \partial_\mu \phi^* \partial_\mu \phi + \frac{i}{2} \Omega_0^{-3/2} \left( \overline{\psi} \gamma_\mu \partial_\mu \Omega_0^{1/2} \psi - (\partial_\mu \Omega_0^{1/2} \psi) \gamma_\mu \psi \right) \right] - i \Omega_0^{-1} \overline{\psi} \gamma_\mu \psi (\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) + \frac{1}{2} (\chi^2)^{-1} (\overline{\psi} \psi)(\psi \psi) ,$$

(2.5)

where $\chi = 1 + \phi^* \phi$ and $\gamma_\mu = \sigma_\mu$, $\mu = 1, 2$ ($\sigma$ being the Pauli matrices).

As is well known, the $\sigma$-model possesses nontrivial field configurations that extremize the action, the instanton solutions [9]. Introducing complex coordinates $z = x_1 + ix_2$ instead of the Euclidean coordinates $x_1$ and $x_2$, any instanton solution $\phi_{\text{inst}}$ is characterized by an integer $k$, the topological charge, given by

$$k = \frac{1}{4\pi} \int d^2x \frac{|\partial_z \phi_{\text{inst}}|^2 - |\partial_\tau \phi_{\text{inst}}|^2}{(1 + \phi_{\text{inst}}^* \phi_{\text{inst}}/4)^2} ,$$

(2.6)

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

For $k \geq 0$, the minimal value of the action is obtained for holomorphic fields, satisfying $\partial_z \phi = 0$, and to this solution corresponds $4k + 2$ real-valued bosonic zero modes and $4k$ fermionic ones. Expanding around this instanton solution,

$$\phi = \phi_{\text{inst}}(z) + \frac{g}{\sqrt{2}} \phi_q , \quad \psi = \psi_{\text{cl}} + \frac{g}{\sqrt{2}} \psi_q ,$$

the action (2.5) becomes in the bilinear approximation,

$$S = S_0 + \int d^2x \Omega_0^2 \left[ \phi_q \left( -4\Omega_0^{-2} \frac{\partial}{\partial z} \chi_0^{-2} \frac{\partial}{\partial \tau} \right) \phi_q + 2i\Omega_0^{-3/2} \overline{\psi_q} \left( 0 \chi_0^{-2} \frac{\partial}{\partial \tau} \right) \phi_q \right] .$$

(2.7)

where $S_0 = 4\pi k/g^2$ is the classical action and $\chi_0 = 1 + \phi_{\text{inst}}^* \phi_{\text{inst}}$.

Specializing to $k = 1$, the most general instanton solution is

$$\phi_{\text{inst}}(z) = \frac{y}{z - z_0} + c ,$$

(2.8)
where \(|y| = \rho\) corresponds to the size of the instanton, \(z_0\) to its position and \(c\), together with the phase of \(y\), to a rotation of the instanton solution in the original internal isotopic space. These three parameters are all complex, giving six real collective coordinates and hence six bosonic zero modes. There are four real-valued fermionic zero modes,

\[
\psi_0^{(i)\alpha}(z) = \theta^{(i)} \delta^{\alpha 1} \frac{y}{(z - z_0)^i},
\]

with \(\theta^{(i)}\) \((i = 1, 2)\) a Grassmann parameter and \(\alpha\) the spinor index. Note that the fermionic zero modes, as they stand, are not normalized.

Now consider the correlation function,

\[
\Pi^{(n)}(x_1, \ldots, x_n) = \langle 0 | T\{O(x_1) \ldots O(x_n)}|0 \rangle ,
\]

where \(O = \chi^{-2}\bar{\psi}(1 + \gamma_5)\psi\), \(T\) stands for time ordering and \(\gamma_5 = \sigma_3\). Due to conservation of chirality, it is clear that \(\Pi^{(n)}\) receives no perturbative contributions. However, similarly to e.g. QCD, there exists an axial–vector current that is conserved classically but broken by quantum effects, by a “diangle” anomaly [11]. The change in the axial charge \(Q_5\) is given by

\[
\Delta Q_5 = 4k ,
\]

implying the chiral selection rule \(n = 2k\) for the correlator. Hence, in a given topological sector there is only one correlation function that does not vanish trivially, but can get contributions from nonperturbative, instanton effects.

Using the chiral selection rule, the only correlator receiving a one–instanton contribution is \(\Pi^{(2)}\), and this is the correlation function we will consider. When the action (2.7) (for \(k = 1\)) is used in the path integral, the integrations over the bosonic zero modes are traded for the collective coordinates \(c\), \(y\) and \(z_0\), together with the appropriate Jacobian. Similarly there is also a Jacobian associated with the fermionic zero modes. For correlation functions like \(\Pi^{(2)}\), the entire contribution is saturated by the zero modes in this approximation. Therefore we can integrate out the non–zero modes to get determinants of the bosonic and fermionic non–zero eigenvalues. The measure for the integral over the
collective coordinates is then given by

\[
I = e^{-4\pi/g_0^2} d^2x_0 d^2c d^2y d\theta^{(1)} d\theta^{(1)} d\theta^{(2)} d\theta^{(2)} (\text{Det}' D_B)^{-1} (\text{Det}' D_F) J ,
\]

where \( D_B \) and \( D_F \) are the operators in the quadratic fluctuation (2.7) for the bosonic and fermionic part respectively, with the prime indicating that only the non-zero modes should be taken into account, and \( J \) is the combined fermion and boson Jacobian for the transformation to the collective coordinates. The subscript on \( g_0 \) indicates that this is the bare coupling constant which will be renormalized. Without the zero modes, the supersymmetric pairing of bosonic and fermionic degrees of freedom and the degeneracy of non-zero eigenvalues (that still holds in the presence of the instanton [13]) normally imply that the boson and fermion determinants cancel. However, in the presence of an infrared regularization this is not necessarily the case. The determinants are also ultraviolet divergent, and an UV–regularization is therefore also needed. Now, by using the transformation rules (2.4) it is easily seen that \( \Pi^{(2)} \) is \( O(3) \)-invariant, and since the correlator is saturated by the zero mode solutions, whose product is invariant under \( O(3) \) transformations, the collective coordinate integration measure \( I \) has to share this invariance as well. Actually the \( O(3) \) invariance requires the contribution from the determinants and the Jacobian to depend on the collective coordinates as (in the limit \( R \to \infty \))

\[
(\text{Det}' D_B)^{-1} (\text{Det}' D_F) J \propto |y|^{-2} \left( 1 + |c|^2 \right)^{-2} ,
\]

and since the integration measure has to be dimensionless, one finds by dimensional arguments

\[
I = KM^2 e^{-4\pi/g_0^2} d^2x_0 \frac{d^2c}{(1 + |c|^2)^2} \frac{d^2y}{|y|^2} d\theta^{(1)} d\theta^{(1)} d\theta^{(2)} d\theta^{(2)} ,
\]

with \( K \) a constant and \( M^2 \) an UV cut-off. Of course, to find the correct numerical factor \( K \) one has to perform the explicit calculation.

The calculation of the correlation function then gives

\[
\Pi^{(2)}(\vec{x}_1, \vec{x}_2) = N \Lambda^2 ,
\]
where $N$ is a numerical factor and $\Lambda$ is the scale parameter in analogy with QCD,

$$\Lambda^2 = M^2 e^{-4\pi/g_0^2}.$$  \hfill (2.16)

This result for the correlation function is \textit{a priori} reliable only at small distances, $\Delta x = |\vec{x}_1 - \vec{x}_2| \to 0$, where $\Pi^{(2)}$ is dominated by small-size instantons, but supersymmetric arguments show that neither does there exist any multiloop corrections [15], nor can it ever be $x$-dependent [11]. So the result (2.15) is actually exact, for all values of $\Delta x$, and by the cluster decomposition this implies a double-valued vacuum condensate,

$$\langle O \rangle \propto \pm \Lambda,$$  \hfill (2.17)

leaving a discrete $Z_2$ invariance for the condensate.

3. Preliminaries at finite temperature.

In this section we generalize the ingredients necessary for a calculation of the correlation function $\Pi^{(2)}$ at finite temperature.

At finite temperature, the Euclidean space is restricted to the strip $\mathcal{M}$ defined by $\text{Re}(z) = x_1 \in \mathbb{R}$ and $0 \leq \text{Im}(z) = x_2 \leq \beta = 1/T$, where $T$ is the temperature. Although the temporal component of the Euclidean space is compact at finite temperature, the spatial part still needs to be regulated. Moreover, any infra–red cutoff has to reduce to the $T = 0$ case when the temperature vanishes. The metric will be taken to be conformally flat, $g_{\mu\nu} = \Omega^2 \delta_{\mu\nu}$, with

$$\Omega = \frac{h(\beta, R^2)}{h(\beta, R^2) + (\beta^2/\pi^2)|\sinh(\pi z/\beta)|^2},$$  \hfill (3.1)

satisfying $\Omega(x_2 + \beta) = \Omega(x_2)$, and where the function $h(\beta, R^2)$ satisfies

$$\lim_{\beta \to \infty} h(\beta, R^2) = h(0, R^2) = 4R^2,$$  \hfill (3.2)

to ensure $\lim_{\beta \to \infty} \Omega = \Omega_0$. The Euclidean action $S$ at finite temperature is then given by the same expression as (2.5), with the replacements $\Omega_0 \to \Omega$ and $\int_{-\infty}^{\infty} d^2x \to \int_0^\beta dx_2 \int_{-\infty}^{\infty} dx_1$, and where bosonic (fermionic) fields are periodic (antiperiodic) under $x_2 \to x_2 + \beta$. 8
At finite temperature there still exists an exact instanton solution, which can be deduced from the charge $k = 1$ instanton by adding an infinite string of such instantons, located at $x_2 = nT^{-1} = n\beta$ with identical sizes and rotations,

$$\phi_{\text{inst}} = y \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - z_0 - in\beta} \right) + c = \frac{y\pi}{\beta} \coth \left[ \frac{\pi(z - z_0)}{\beta} \right] + c . \quad (3.3)$$

This describes a periodic $k = 1$ instanton solution with the same number of collective coordinates as at $T = 0$. Note that when $\text{Re}(z - z_0) \to \pm \infty$ the solution becomes

$$\lim_{\text{Re}(z - z_0) \to \pm \infty} \phi_{\text{inst}} = c \pm \frac{y\pi}{\beta} ,$$

implying that when $T > 0$ the collective coordinate $y$ is entangled with the isospin rotations parametrized by $c$. In other words, already at this level it is clear that the rotational degrees of freedom will be more complicated than in the $T = 0$ case. This behavior should be contrasted with the case of Yang–Mills theory in four dimensions, where there is no mixing of the collective coordinates at any temperature [14].

From the instanton solution (3.3) we find the bosonic zero modes by taking derivatives with respect to the collective coordinates, and the fermionic zero modes, satisfying antiperiodic boundary conditions, are found from the zero temperature solutions (2.9):

$$\psi_0^{(1)\alpha}(z) = \theta^{(1)\delta\alpha_1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{y}{(z - z_0 - in\beta)} \theta^{(1)\delta\alpha_1} \left( \frac{y\pi}{\beta} \right) \frac{1}{\sinh \left[ \frac{\pi(z - z_0)}{\beta} \right]} , \quad (3.4)$$

and

$$\psi_0^{(2)\alpha}(z) = \theta^{(2)\delta\alpha_1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{y}{(z - z_0 - in\beta)^2} \theta^{(2)\delta\alpha_1} \left( \frac{y\pi^2}{\beta^2} \right) \frac{\cosh \left[ \frac{\pi(z - z_0)}{\beta} \right]}{\sinh^2 \left[ \frac{\pi(z - z_0)}{\beta} \right]} . \quad (3.5)$$

At zero temperature, the Greens functions $\Pi^{(n)}$ that are not trivially vanishing but receive instanton contributions, are determined by the chiral selection rule. Since we are interested in the instanton corrections at finite temperature, it is necessary to establish a corresponding selection rule at $T > 0$. Considering the correlation function at finite temperature and denoting by $O_i = O(\vec{x}_i)$,

$$\langle T(O_1 \cdots O_n) \rangle = \int D\bar{\psi} D\psi D\phi^* D\phi (O_1 \cdots O_n) e^{-S} , \quad (3.6)$$

we now perform a global chiral rotation of $\psi$,

$$\psi(\vec{x}) \to \psi'(\vec{x}) = e^{i\alpha\gamma_5} \psi(\vec{x}) . \quad (3.7)$$
Note that any transformation of the fields has to respect the appropriate boundary conditions. For a generic chiral parameter \( \alpha(x) \), the antiperiodicity of the fermionic field requires \( \alpha(x) \) to be periodic, which is trivially satisfied in this particular case, \( \alpha(x) = \alpha = \text{constant} \). The crucial step is then to notice that the relation for the axial anomaly remains the same at any temperature; intuitively this is rather clear, since the anomaly relation can be viewed as a short distance effect and should therefore be independent of the influence from the medium [19, 20, 21]. Using the Fujikawa method [22], we can write

\[
\langle T(O_1 \cdots O_n) \rangle = \int D\overline{\psi}' D\psi' D\phi D\phi^* (O_1' \cdots O_n') e^{-S'} = \\
= \int D\overline{\psi} D\psi D\phi D\phi^* \exp \left[ 2i n \alpha - 4i \alpha \int d^2 x \tilde{k} \right] (O_1 \cdots O_n) e^{-S},
\]

(3.8)

where \( \tilde{k} \) is the topological charge density,

\[
k = \int d^2 x \tilde{k},
\]

(3.9)

and the integration is performed over the strip \( \mathcal{M} \). The first term in the exponential comes from the rotation of \( O_1 \cdots O_n \) and the second from the change in the measure. Comparing (3.8) with (3.6), we get the integrated chiral Ward identity

\[
(2n - 4k) \langle T(O_1 \cdots O_n) \rangle = 0.
\]

(3.10)

As in the zero temperature case, we expand the action around the instanton solution up to quadratic fluctuations. The classical action associated with the instanton solution is not affected by the temperature, and thus \( S_0 = 4\pi/g^2 \) at any temperature. We again get a Jacobian associated with the integrations over the bosonic collective coordinates and also a Jacobian from the expansion of the fermionic field in Grassmann coefficients, belonging to the zero modes. Furthermore, the bosonic non–zero eigenvalues give a determinant for a differential operator defined on the space of periodic functions, and the fermionic non–zero modes give a determinant for a differential operator defined on antiperiodic functions. The correlation function is then calculated to this order by using the integration measure.
derived from the semiclassical expansion and replacing the fields in $O_i$ by the bosonic instanton solution and the fermionic zero modes. At $T = 0$ there is a cancellation between the bosonic and fermionic contributions to the determinants, that can be understood as a consequence of supersymmetry. At $T \neq 0$ the different boundary conditions for the differential operators belonging to the bosonic and fermionic sector seem to remove the a priori reason for such a cancellation, but as we show below and in Appendix A, it still takes place. This calculation is along the same lines as the corresponding one at zero temperature, given in [10, 16]. Following that calculation we first remove the singularities in the quadratic fluctuations by making the following rescalings:

$$\tilde{\phi} = \frac{\phi}{\chi} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2 \left[ \pi(z - z_0)/\beta \right], \quad (3.11)$$

$$\tilde{\psi} = \frac{\psi}{\chi} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2 \left[ \pi(z - z_0)/\beta \right],$$

where

$$\tilde{\chi} = \chi_0 \left( \frac{\beta^2}{\pi^2} \right) \sinh \left[ \pi(z - z_0)/\beta \right]. \quad (3.12)$$

In the bilinear approximation, we now rewrite the action (2.7) in terms of the rescaled variables (3.11),

$$S = S_0 + \int d^2x \Omega^2 \left[ \tilde{\phi}^* \left( -4\Omega^{-2} \frac{\partial}{\partial z} \tilde{\chi}^{-2} \frac{\partial}{\partial \tilde{z}} \tilde{\chi} \right) \tilde{\phi} + 2i\Omega^{-3/2} \tilde{\psi} \left( 0 \tilde{\chi}_0^{-1} \tilde{\chi} \tilde{\partial}_{\tilde{z}} \tilde{\chi}_0^{-1} 0 \right) \Omega^{1/2} \tilde{\psi} \right]$$

$$= S_0 + \int d^2x \Omega^2 \left[ \tilde{\phi}^* D_B \tilde{\phi} + \tilde{\psi} D_F \tilde{\psi} \right]. \quad (3.13)$$

As shown in Appendix A, by defining

$$f_\mu = \begin{pmatrix} (\beta/2\pi) \sinh(2\pi z/\beta) \\ (\beta^2/\pi^2) \sinh^2(\pi z/\beta) \end{pmatrix}, \quad f_i' = \begin{pmatrix} \cosh(\pi z/\beta) \\ (\beta/\pi) \sinh(\pi z/\beta) \end{pmatrix}, \quad (3.14)$$

and

$$R_{\lambda\rho} = \int d^2x \frac{\Omega^2}{\chi^2} f_\lambda f_\rho, \quad R_{kl}' = \int d^2x \frac{\Omega}{\tilde{\chi}^2} f_k f_l', \quad (3.15)$$

the total Jacobian can be written as

$$J = \left( \frac{1}{|y|^2} \right) \frac{\text{Det}R}{\text{Det}R'}, \quad (3.16)$$
where $R'$ denotes the part from the fermion Jacobian and $R$ from the boson Jacobian.

As for the calculation of the determinants $\text{Det}'D_B$ and $\text{Det}'D_F$, we find, by varying with respect to the instanton parameters (see Appendix A for details):

$$
\delta (\ln \text{Det}'D_B) = \delta (\ln \text{Det}R) + \int d^2x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} - \frac{1}{2\pi} \partial_\mu \partial_\mu \ln \Omega \right], \quad (3.17)
$$

$$
\delta (\ln \text{Det}'D_F) = \delta (\ln \text{Det}R') + \int d^2x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} \right]. \quad (3.18)
$$

The second term in the square brackets of (3.17) is calculated in the limit where the IR–regularization is removed, $R \to \infty$;

$$
\delta \left( \frac{1}{2\pi} \int d^2x \ln(\tilde{\chi})\partial_\mu \partial_\mu \ln \Omega \right) = -\delta \ln \left( (1 + |c + \tilde{y}|^2)(1 + |c - \tilde{y}|^2) \right), \quad (3.19)
$$

where $\tilde{y} = y\pi/\beta$. Hence, up to some numerical factor,

$$
J (\text{Det}'D_B)^{-1} (\text{Det}'D_F) = M^2 \left( \frac{1}{|y|^2} \right) \left( \frac{1}{1 + |c + \tilde{y}|^2}(1 + |c - \tilde{y}|^2) \right), \quad (3.20)
$$

where we have inserted the ultraviolet cut–off $M$.

The integration measure at finite temperature can now be written as,

$$
I = \Lambda^2 \int d^2x_0 \frac{d^2c}{(1 + |c + \tilde{y}|^2)} \frac{d^2y}{|y|^2} d\theta^{(1)} d\theta^{(1)} d\theta^{(2)} d\theta^{(2)}, \quad (3.21)
$$

and this measure is invariant under O(3)–transformations, as it should.

4. The correlation function

In this section we perform the explicit calculation of the spatial correlation function

$$
\Pi^{(2)}(x_1) = \left\langle \left( \frac{\overline{\psi}(x_1,0) (1 + \gamma_5)\psi(x_1,0)}{\chi^2(x_1,0)} \overline{\psi}(0,0) (1 + \gamma_5)\psi(0,0) \right) \right\rangle. \quad (4.1)
$$

In the semiclassical approximation, when we replace the fields $\chi$ and $\psi$ by their classical values, we have

$$
(1 + \gamma_5)\psi = \frac{2y\pi}{\beta} \left( \theta^{(1)} \frac{1}{\sinh [\pi(z - z_0)/\beta]} + \theta^{(2)} \frac{\pi}{\beta} \frac{\cosh [\pi(z - z_0)/\beta]}{\sinh^2 [\pi(z - z_0)/\beta]} \right),
$$

$$
\chi = \chi_0 + \phi^{\text{inst}}_0 \theta^{\text{inst}}, \quad (4.2)
$$
and inserting also the integration measure (3.21) from the previous section, the correlator becomes

\[
\Pi^{(2)}(z) = N\Lambda^2 \left(\frac{\pi}{\beta}\right)^6 \int d^2x_0 \frac{d^2c}{(1 + |c + \tilde{y}|^2)(1 + |c - \tilde{y}|^2)} \frac{d^2y}{|y|^2} \frac{d\theta^{(1)} d\theta^{(2)} d\theta^{(3)}}{d^2y |y|^2} \times
\]

\[
\frac{\theta^{(1)} \theta^{(2)} \theta^{(3)}}{|\sinh[\pi z/\beta]|^2} \frac{|y|^4 |\sinh[\pi z/\beta]|^2}{|\sinh[\pi (z - z_0)/\beta]|^4 |\sinh[\pi z_0/\beta]|^4} \left[\frac{1}{(1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(z))} \left\{1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(z)\right\} \left\{1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(0)\right\}\right]^2,
\]

where \(N\) is a numerical factor. Although not written out explicitly, the instanton field \(\phi_{\text{inst}}\) also depends on \(c, z_0\) and \(y\). In Appendix B we show how to perform the \(c\)-integration by using the Fadeev–Popov method, after which the correlation function becomes

\[
\Pi^{(2)}(z) = N\pi \Lambda^2 \left(\frac{\pi}{\beta}\right)^6 \int d^2x_0 \frac{d^2y}{|y|^2} \left(\frac{1 - |\tilde{y}|^2}{1 + |\tilde{y}|^2}\right) \times
\]

\[
\frac{|y|^4 |\sinh[\pi z/\beta]|^2}{|\sinh[\pi (z - z_0)/\beta]|^4 |\sinh[\pi z_0/\beta]|^4} \left[\frac{1}{(1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(z))} \left\{1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(z)\right\} \left\{1 + \phi^*_{\text{inst}} \phi_{\text{inst}}(0)\right\}\right]^2
\]

\[
= K\Lambda^2 \pi^2 \beta^2 |\sinh(\pi z/\beta)|^2 \int_0^\infty dx_0 \int_0^\beta dy_0 \int_0^\infty d\rho \rho^3 \left(\frac{1 - (\rho^2 \pi^2/\beta^2)}{1 + (\rho^2 \pi^2/\beta^2)}\right) \times
\]

\[
\left\{\beta^2 |\sinh[\pi (z - z_0)/\beta]|^2 + \rho^2 \pi^2 |\cosh[\pi (z - z_0)/\beta]|^2\right\} \times
\]

\[
\left\{\beta^2 |\sinh[\pi z_0/\beta]|^2 + \rho^2 \pi^2 |\cosh[\pi z_0/\beta]|^2\right\}^{-2},
\]

(4.4)

where \(K\) is some new constant and \(\rho = |y|\). Now, writing \(z_0 = x_0 + iy_0\), putting Re\((z) = x_1,\)

\[\text{Im}(z) = 0\]

and thus neglecting any time dependence, and defining

\[
\tilde{x} = 2\pi x_1/\beta, \quad \tilde{x}_0 = 2\pi x_0/\beta, \quad \tilde{y}_0 = 2\pi y_0/\beta, \quad t = (\rho \pi/\beta)^2 - 1 \quad \text{and} \quad u = \tilde{x}_0 - \tilde{x}/2,
\]

we can write (4.4) as (dropping the tildes on the integration variables)

\[
\Pi^{(2)}(x_1) = \frac{K\pi \Lambda^2}{4} \sinh^2 \left(\frac{\tilde{x}}{2}\right) \int_{-\infty}^\infty du \int_0^{2\pi} dy_0 \int_{-1}^1 dt \left(1 - t^2\right)|t| \times
\]

\[
\left[\left(\cosh[u - (\tilde{x}/2)] + t \cos y_0\right) \left(\cosh[u + (\tilde{x}/2)] + t \cos y_0\right)\right]^2.
\]

(4.5)
The remaining integrals are in principle straightforward, although rather nontrivial. Referring the details of the calculation to Appendix C, we find

$$
\Pi^{(2)}(x_1) = \frac{K \pi^2 \Lambda^2}{3} \left[ 1 + 2 \pi x_1 T \coth(\pi x_1 T) \left( 1 - 2 \sinh^2(\pi x_1 T) \right) \right] + 2 \sinh^2(\pi x_1 T) \ln \left( 4 \sinh^2(\pi x_1 T) \right) \right] \tag{4.6}
$$

where we have assumed that $x_1 > 0$ for simplicity. This result is exact in the one-instanton background and the bilinear approximation.

The correlation function is thus seen to depend on the dimensionless combination $x_1 T$ in a rather complicated way, although it is always a decreasing function of $x_1 T$, as shown in Fig. 1.

Figure 1: The correlator $\Pi$ in units of $K \Lambda^2$ as a function of $\pi x_1 T$.

However, for special limits it simplifies considerably. For vanishing temperature or distance,

$$
\Pi^{(2)}(x_1) \xrightarrow{x_1 T \to 0} K \pi^2 \Lambda^2 \propto \Lambda^2. \tag{4.7}
$$
Since the limit $x_1 \ll T^{-1}$ corresponds to distances much shorter than the average separation between the constituents of the medium, $\Delta x_{\text{mean}} \sim T^{-1}$, there should be no temperature effects. Unsurprisingly, (4.7) is in agreement with the direct $T = 0$ calculation. On the other hand, for asymptotically large values, $x_1T \to \infty$, the correlator falls of exponentially:

$$\Pi^{(2)}(x_1) \xrightarrow{x_1T \to \infty} 2K\pi^2\Lambda^2 (\pi x_1T) e^{-2\pi x_1T}.$$  (4.8)

Note that the inverse correlation length is given by twice the lowest Matsubara frequency, $\pi T$.

5. Conclusions.

We have derived, within the semiclassical expansion, an analytical expression for the instanton induced, finite temperature correlation function in the SUSY 2d nonlinear $\sigma$-model. For large values of the product $x_1T$ we find that the correlator is exponentially decaying, and when $x_1T \to 0$ it reduces to the well known $T = 0$ calculation. We would now like to comment on the validity of these expressions and any implications for the condensate, $\langle O \rangle$.

First of all, we have neglected all extra instanton–antiinstanton pairs and also the higher order effects beyond the semiclassical approximation. Hence, the result for the correlator can only be accurate to the extent that the coupling is small enough for these other effects to be neglectable. Therefore, the relevant question is what sets the scale of the running coupling.

At zero temperature, the only scale available is the distance, so in that case the semiclassical approximation is a priori trustworthy only at small distances. But supersymmetry guarantees that the correlation function has to be independent of the distance and thus makes the short distance calculation valid at all distances. However, at finite temperature the supersymmetric arguments are not applicable. This is rather obvious, since the correlator now depends on $x_1$ even in the lowest approximation. Nonetheless, we
believe the semiclassical approximation to be accurate in the high temperature regime. The reason is that when $x_1 T \gg 1$, the correlation function is saturated by an instanton size of the order $\rho \leq T^{-1}$. Since the relevant scale in the correlation function should be set by the instanton size, it seems reasonable to expect the semiclassical approximation to be valid when both $x_1 T \gg 1$ and $T \gg \Lambda$. Similarly, when $x_1 T \ll 1$ the correlator should be well described by the semiclassical approximation in the limit of vanishing distance, $x_1 \to 0$.

If the above scenario is correct, which seems plausible, the exponential decay of the correlator should be reliable at high temperatures. By using the cluster decomposition we then arrive at

$$
\langle O \rangle_{T \gg \Lambda} = \pm \sqrt{\lim_{x_1 \to \infty} \left[ \Pi^{(2)}(x_1) \right]_{T \gg \Lambda} = 0},
$$

and the discrete symmetry is restored.

Since the theory is defined in one spatial dimension, general arguments based on the free energy would imply that the symmetry is restored at any positive temperature. This argument is not in conflict with the above semiclassical calculation, although it is not easy to justify why the approximations should be qualitatively correct even at low temperatures.

Finally, we would like to connect this model to the full, 4d SUSY Yang–Mills theory, although we must emphasize that this is highly speculative at the present stage. However, if the results of this toy model at finite temperature has any generalizations to the supersymmetric Yang–Mills theory, it would indicate that there is a high temperature phase in the 4d theory, where the discrete symmetry is restored. Such a restoration would clearly be important in connection with the formation of domain walls [5], and a phase transition at the temperature of symmetry restoration can in that case not be excluded.

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Appendix A. Calculation of the Jacobian and the determinants.

In this appendix we calculate the contribution from the Jacobian and the boson and fermion determinant at finite temperature. We will follow the $T = 0$ approach \[10, 16\] and generalize it to the finite temperature case.

Starting with the boson Jacobian $J_B$, let $\alpha_\mu$ denote the collective coordinates in the following way, $\alpha_1 = z_0$, $\alpha_2 = y$, $\alpha_3 = c$. The Jacobian is then given by

$$J_B = \text{Det} M^{(B)} , \quad M^{(B)}_{\mu \nu} = \int d^2x \frac{\Omega^2}{\lambda_0^2} \left( \frac{\partial \phi^*_{\text{inst}}}{\partial \alpha_\mu} \right) \left( \frac{\partial \phi_{\text{inst}}}{\partial \alpha_\nu} \right),$$

(A.1)

where the integration is taken over the IR–regulated Euclidean strip. By using the explicit form of the instanton solution (3.3) we can write,

i) $\frac{\partial \phi_{\text{inst}}}{\partial \alpha_1} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2(\pi(z - z_0)/\beta) = y$

ii) $\frac{\partial \phi_{\text{inst}}}{\partial \alpha_2} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2(\pi(z - z_0)/\beta) = \frac{\beta}{2\pi} \left[ -\sinh(2\pi z_0/\beta) + \cosh(2\pi z_0/\beta) \sinh(2\pi z/\beta) - 2 \sinh(2\pi z_0/\beta) \sinh^2(\pi z/\beta) \right]$

iii) $\frac{\partial \phi_{\text{inst}}}{\partial \alpha_3} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2(\pi(z - z_0)/\beta) = \frac{\beta^2}{\pi^2} \left[ \sinh^2(\pi z_0/\beta) - \frac{1}{2} \sinh(2\pi z_0/\beta) \sinh(2\pi z/\beta) + \cosh(2\pi z_0/\beta) \sinh^2(\pi z/\beta) \right].$

(A.2)

If we now define the transposed vector

$$f^{T}_\mu = \left[ 1 , \quad (\beta/2\pi) \sinh(2\pi z/\beta) , \quad (\beta^2/\pi^2) \sinh^2(\pi z/\beta) \right],$$

(A.3)

and the following matrix

$$U_{\mu \nu} = \begin{pmatrix} y & 0 & 0 \\ -(\beta/2\pi) \sinh(2\pi z_0/\beta) & \cosh(2\pi z_0/\beta) & -(\pi/\beta) \sinh(2\pi z_0/\beta) \\ (\beta^2/\pi^2) \sinh^2(\pi z_0/\beta) & -(\beta/\pi) \sinh(2\pi z_0/\beta) & \cosh(2\pi z_0/\beta) \end{pmatrix},$$

(A.4)

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we can write the set of equations (A.2) in a compact form as
\[
\frac{\partial \phi_{\text{inst}}}{\partial \alpha_{\mu}} \left( \frac{\beta^2}{\pi^2} \right) \sinh^2(\pi(z - z_0)/\beta) = U_{\mu\nu} f_\nu .
\] (A.5)

Furthermore, by rescaling $\chi_0 = 1 + \phi_{\text{inst}}^* \phi_{\text{inst}}$ as
\[
\tilde{\chi} = \chi_0 \frac{\beta^2}{\pi^2} |\sinh[\pi(z - z_0)/\beta]|^2 ,
\] (A.6)
we find that $M^{(B)}_{\mu\nu} = U^\dagger R U$, where
\[
R_{\lambda\rho} = \int d^2x \frac{\Omega^2}{\tilde{\chi}^2} f_{\lambda}^* f_{\rho} .
\] (A.7)

The boson Jacobian is thus given by
\[
J_B = (\text{Det}U) \left( \text{Det}U^\dagger \right) (\text{Det}R) .
\] (A.8)

Note that the definitions of $f_\mu$ and $U_{\mu\nu}$ reduce to the well known $T = 0$ formulas in the limit of vanishing temperature [16].

In the fermionic case we have to evaluate
\[
J_F = \left( \text{Det}M^{(F)} \right)^{-1} ,
\] (A.9)
where
\[
M^{(F)}_{ij} = \int d^2x \frac{\Omega}{\chi_0} \psi_i^\dagger \psi_j ,
\] (A.10)
and $\psi_i$ are the fermionic zero mode solutions,
\[
\psi_1 = \frac{y \pi^2}{\beta^2} \frac{\cosh(\pi(z - z_0)/\beta)}{\sinh^2(\pi(z - z_0)/\beta)} , \quad \psi_2 = \frac{y \pi}{\beta} \frac{1}{\sinh(\pi(z - z_0)/\beta)} .
\] (A.11)

Performing the same calculations as in the bosonic case give
\[
\psi_i \left( \frac{\beta^2}{\pi^2} \right) \sinh^2(\pi(z - z_0)/\beta) = U'_{ij} f'_j ,
\] (A.12)
where
\[
f'_i = \left( \frac{\cosh(\pi z/\beta)}{(\beta/\pi) \sinh(\pi z/\beta)} \right) , \quad U'_{ij} = \left( \begin{array}{cc}
\frac{y \cosh(\pi z_0/\beta)}{-(y\beta/\pi) \sinh(\pi z_0/\beta)} & \frac{-(y \pi/\beta) \sinh(\pi z_0/\beta)}{y \cosh(\pi z_0/\beta)} \\
\end{array} \right) .
\] (A.13)
Now, by putting
\[ R'_{kl} = \int d^2x \frac{\Omega}{\lambda^2} f_k^* f_l', \tag{A.14} \]
we get \( M^{(F)} = U'^+ R' U' \), and so the fermion Jacobian becomes
\[ J_F = \left( \text{Det} U'^+ \right)^{-1} \left( \text{Det} R' \right)^{-1} \left( \text{Det} U' \right)^{-1}. \tag{A.15} \]

Putting the two contributions \( J_B \) and \( J_F \) from (A.8) and (A.15) together, we obtain the total Jacobian \( J = J_B J_F \). By explicitly taking the determinants of \( U \) and \( U' \) we finally get the result
\[ J = \left( \frac{1}{|y|} \right) \frac{\text{Det} R}{\text{Det} R'}. \tag{A.16} \]

Now consider the calculation of the determinants appearing after the gaussian integration over the quadratic fluctuations, taken over the non–zero eigenvalues only. Beginning with the boson determinant, we have to evaluate
\[ (\text{Det}' D_B)^{-1} = \exp \left[ -\ln \text{Det}' D_B \right], \tag{A.17} \]
where the operator \( D_B \) after the rescaling is given by
\[ D_B = -4\Omega^{-2} \tilde{\chi} \partial_z \tilde{\chi}^{-2} \partial_{\bar{z}} \tilde{\chi}, \tag{A.18} \]
and \( \tilde{\chi} \) is as defined in (A.6). This expression for the determinant is of course formal, since it is divergent and needs to be regularized. In order to do this we will use the proper time method, and define the regularized part of the determinant as [16]
\[ \ln \text{Det}' D_B = \lim_{\epsilon \to 0} \left[ -\int_t^\infty \frac{dt}{t} \left( \text{Tr} e^{-tD_B} - p \right) + \alpha_1 \epsilon^{-1} - \alpha_0 \ln(\epsilon) \right], \tag{A.19} \]
where \( p \) is the number of zero modes, being six real–valued ones in our case. The coefficients \( \alpha_1 \) and \( \alpha_0 \) are independent of the instanton parameters, as will be verified in the \( t \to 0 \) limit, so by making a variation with respect to the parameters we get,
\[ \delta (\ln \text{Det}' D_B) = \int_0^\infty dt \left( \text{Tr} (\delta D_B) e^{-tD_B} \right). \tag{A.20} \]
Defining \( \tilde{D}_B = -4\tilde{\chi}^{-1}\partial_z\tilde{\chi}^2\Omega^{-2}\partial_z\tilde{\chi}^{-1} \), which satisfies \( D_B\Omega^{-2}\tilde{\chi}\partial_z\tilde{\chi}^{-1} = \Omega^{-2}\tilde{\chi}\partial_z\tilde{\chi}^{-1}\tilde{D}_B \), allows us to write

\[
\text{Tr} (\delta D_B) e^{-t\tilde{D}_B} = 2\text{Tr} \left[ \delta (\ln \tilde{\chi}) \left( D_B e^{-tD_B} - \tilde{D}_B e^{-t\tilde{D}_B} \right) \right],
\tag{A.21}
\]

and hence

\[
\delta (\ln \text{Det}'D_B) = -2\text{Tr} \left[ \delta (\ln \tilde{\chi}) \left( e^{-tD_B} - e^{-t\tilde{D}_B} \right) \right] \bigg|_{t=0}.
\tag{A.22}
\]

Now, when \( t \to \infty \) it is evident that only the zero modes contribute; there are six real valued zero modes associated with \( D_B \) and none with \( \tilde{D}_B \). Thus

\[
\delta (\ln \text{Det}'D_B) = -2 \int d^2 x \delta (\ln \tilde{\chi}) P_0^B(x) + \lim_{t \to 0} 2\text{Tr} \left[ \delta (\ln \tilde{\chi}) \left( e^{-tD_B} - e^{-t\tilde{D}_B} \right) \right],
\tag{A.23}
\]

where \( P_0^B \) is the projection operator on the space of bosonic zero modes, \( P_0^B = \sum_{\mu} \hat{\phi}_\mu^* \hat{\phi}_\mu \) with \( \hat{\phi}_\mu \) an orthonormal basis. The remaining trace can be evaluated by a heat kernel expansion of \( G_B(z,z) = \langle z | e^{-tD_B} | z \rangle \) and \( \tilde{G}_B(z,z) = \langle z | e^{-t\tilde{D}_B} | z \rangle \). To this end, consider

\[
\lim_{z_1 \to z} G_B(z,z_1) \text{ and let it be represented as}
\]

\[
G_B(z,z_1) = G_B^{(0)}(z,z_1) \left[ a_0(z,z_1) + a_1(z,z_1)t + \ldots \right],
\tag{A.24}
\]

where \( G_B^{(0)} \) is the free field solution. Since \( G_B^{(0)} \) has to satisfy periodic boundary conditions, it is obtained by summing the \( T = 0 \) solution:

\[
G_B^{(0)}(z,z_1) = \frac{\Omega^2}{4\pi t} \sum_{n=-\infty}^{\infty} \exp \left( -\frac{|z-z_1-in\beta|^2}{4t} \right).
\tag{A.25}
\]

The calculation of the expansion coefficients \( a_0 \) and \( a_1 \) in (A.24) is now straightforward [17]. The value of \( a_0(z,z) \) is fixed by the free field case, \( a_0(z,z) = 1 \), and since we are ultimately interested in the limit \( z_1 \to z \), the result for \( a_1(z,z) \) follows. We get

\[
G_B(z,z) = \lim_{t \to 0} \sum_{n=-\infty}^{\infty} e^{-n^2\beta^2\Omega^2/4t} \left[ \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} \right]
\]

\[
= \lim_{t \to 0} \left[ \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} \right] \left[ 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\beta^2\Omega^2/4t} \right]
\]

\[
= \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \partial_\mu \partial_\mu \ln \tilde{\chi}.
\tag{A.26}
\]
From this equation we see that the expansion is actually independent of the boundary conditions in the limit $t \to 0$, since the limit $t \to 0$ is equivalent to $\beta \to \infty$. This result can be obtained by a calculation in momentum space as well [18]. A similar calculation for $\tilde{D}_B$ gives

$$\tilde{G}_B(z, z) = \frac{\Omega^2}{4\pi t} - \frac{1}{4\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} + \frac{1}{4\pi} \partial_\mu \partial_\mu \ln \Omega,$$

(A.27)

and (A.23) is then given by

$$\delta (\ln \text{Det}'D_B) = -2 \int d^2 x \delta (\ln \tilde{\chi}) P_0^B(x) + \int d^2 x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} - \frac{1}{2\pi} \partial_\mu \partial_\mu \ln \Omega \right].$$

(A.28)

Since the variation of the determinant is unaffected by the boundary conditions, the formal dependence on $\tilde{\chi}$ and $\Omega$ is the same as at $T = 0$, but both these functions themselves are of course very different at finite temperature.

The remaining part in the boson determinant is to calculate the projection on the space of the zero modes. For this purpose, we take as a non–orthonormal basis the periodic functions $g_\mu = \tilde{\chi}^{-1}f_\mu$. The projection operator is then given by $P_0^B = \sum_{\mu, \nu} g^*_\nu g_\mu R^{-1}_{\mu\nu} \Omega^2$, with $R_{\mu\nu}$ defined in (A.7). Substituting this into the first term in (A.28),

$$-2 \int d^2 x \delta (\ln \tilde{\chi}) P_0^B(x) = -2 \int d^2 x \delta (\ln \tilde{\chi}) g^*_\nu g_\mu R^{-1}_{\mu\nu} \Omega^2
= -2 \int d^2 x \delta (\tilde{\chi}) \tilde{\chi}^{-3} \Omega^2 f^*_\nu f_\mu R^{-1}_{\mu\nu} = \delta (\ln \text{Det} R).$$

(A.29)

The boson determinant thus becomes

$$\delta (\ln \text{Det}'D_B) = \delta (\ln \text{Det} R) + \int d^2 x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu \partial_\mu \ln \tilde{\chi} - \frac{1}{2\pi} \partial_\mu \partial_\mu \ln \Omega \right].$$

(A.30)

The fermionic part is calculated much in the same way; after taking the determinant over the spinor indices the fermion operator with positive definite eigenvalues becomes

$$D_F = -4\Omega^{-3/2} \partial_z \tilde{\chi}^{-2} \Omega^{-1} \partial_z \tilde{\chi} \Omega^{1/2},$$

and varying the regularized expression with respect to the instanton parameters we are left with

$$\delta (\ln \text{Det}'D_F) = -2 \text{Tr} \left[ \delta (\ln \tilde{\chi}) \left( e^{tD_F} - e^{-tD_F} \right) \right]_{t=0}.$$
where we have defined $\tilde{D}_F = -4\Omega^{-3/2}\chi^{-1}\partial_\mu\chi^{-1}\Omega^{-1}\partial_\mu\chi^{-1}\Omega^{1/2}$, satisfying
\begin{equation}
D_F\Omega^{-3/2}\chi^{-1}\partial_\mu\chi^{-1}\Omega^{1/2} = \Omega^{-3/2}\chi^{-1}\partial_\mu\chi^{-1}\Omega^{1/2} \tilde{D}_F .
\end{equation}

As in the bosonic case there is only a contribution from the zero modes in the $t \to \infty$ limit, and when $t \to 0$ we can make a heat kernel expansion. Expanding in powers of $t$ around the free field solution, that now satisfies anti–periodic boundary conditions, we find
\begin{equation}
\lim_{t \to 0} \langle z | e^{-t\tilde{D}_F} | z \rangle = \lim_{t \to 0} \left[ \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \left( -\partial_\mu\partial_\mu \ln \tilde{\chi} + \frac{1}{2} \ln \Omega \right) \right] \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\beta^2\Omega^2/4t} \right] = \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \left( -\partial_\mu\partial_\mu \ln \tilde{\chi} + \frac{1}{2} \ln \Omega \right) .
\end{equation}
\begin{equation}
\lim_{t \to 0} \langle x | e^{-t\tilde{D}_F} | x \rangle = \lim_{t \to 0} \left[ \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \left( -\partial_\mu\partial_\mu \ln \tilde{\chi} + \frac{1}{2} \ln \Omega \right) \right] \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\beta^2\Omega^2/4t} \right] = \frac{\Omega^2}{4\pi t} + \frac{1}{4\pi} \left( -\partial_\mu\partial_\mu \ln \tilde{\chi} + \frac{1}{2} \ln \Omega \right) .
\end{equation}

Thus we see again that in the limit of vanishing $t$, the expansion becomes independent of the boundary conditions. Including the projection over the zero modes as well,
\begin{equation}
\delta (\ln \text{Det}'D_F) = -2 \int d^2x \delta (\ln \tilde{\chi}) P_0^F (x) + \int d^2x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu\partial_\mu \ln \tilde{\chi} \right] .
\end{equation}

Taking the anti–periodic functions $g_i' = (1/\sqrt{\Omega})\chi^{-1}f_i'$ as a basis, we can write $P_0^F = \sum_{i,j} g_j'^* g_i' R_{ij}^{-1} \Omega^2$, with $R_{ij}^{-1}$ being the same as in (A.14). Hence
\begin{equation}
-2 \int d^2x \delta (\ln \tilde{\chi}) P_0^F (x) = -2 \int d^2x \delta (\ln \tilde{\chi}) g_j'^* g_i' R_{ij}^{-1} \Omega^2
= -2 \int d^2x \delta (\tilde{\chi}) \chi^{-3} f_j'^* f_i' R_{ij}^{-1} \Omega = \delta (\ln \text{Det} R) ,
\end{equation}
and the variation of the fermion determinant finally becomes
\begin{equation}
\delta (\ln \text{Det}'D_F) = \delta (\ln \text{Det} R) + \int d^2x \delta (\ln \tilde{\chi}) \left[ \frac{1}{\pi} \partial_\mu\partial_\mu \ln \tilde{\chi} \right] .
\end{equation}
Appendix B. Integrating out the coordinate $c$.

Here we derive the formula (4.4), by using the Fadeev–Popov method. The starting point is the expression for the correlation function with all integrations over the collective coordinates remaining,

$$\Pi^{(2)}(z) = \int \frac{d^2c\,d^2y\,d^2z_0}{|y|^2 \left( 1 + |c + \tilde{y}|^2 \right) \left( 1 + |c - \tilde{y}|^2 \right)} f(z, c, y, z_0) ,$$  \hspace{1cm} (B.1)

where $\tilde{y} = y\pi/\beta$ and we have defined a function $f(z, c, y, z_0)$ that contains the scale factor $\Lambda$, the contribution from the fields in the correlation function and some irrelevant numerical factor.

The basic idea is that by an O(3) transformation we can always make $c = 0$. The integration over $c$ can thus be replaced by an integral over O(3) parameters and a suitable Jacobian. To find the Jacobian we define

$$\frac{1}{F(c, y)} = \int d\Omega \, \delta(c') ,$$  \hspace{1cm} (B.2)

where $c'$ is the value of $c$ after an O(3)–rotation,

$$c' = \frac{e^{i\gamma}}{2} \left( \frac{(c + \tilde{y})e^{i\alpha} - \lambda}{1 + \lambda(c + \tilde{y})e^{i\alpha}} + \frac{(c - \tilde{y})e^{i\alpha} - \lambda}{1 + \lambda(c - \tilde{y})e^{i\alpha}} \right) ,$$  \hspace{1cm} (B.3)

and $d\Omega$ is the invariant group measure, which in terms of our O(3)–parameters reads

$$\int d\Omega = \int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\gamma \int_0^\infty d\lambda \lambda \frac{\lambda}{(1 + \lambda^2)^2} .$$  \hspace{1cm} (B.4)

This construction ensures that $F$ is rotationally invariant. Inserting the trivial factor

$$F(c, y) \int d\Omega \, \delta(c') = 1$$  \hspace{1cm} (B.5)

into the correlation function and performing a rotation to new values $c \to c'$, $\tilde{y} \to \tilde{y}'$ and $z_0 \to z'_0$ we obtain, by making use of the O(3) invariance of $f(z, c, y, z_0)$,

$$\Pi^{(2)}(z) = \int d\Omega \int \frac{d^2c'\,d^2y'\,d^2z'_0}{|y'|^2 \left( 1 + |c' + \tilde{y}'|^2 \right) \left( 1 + |c' - \tilde{y}'|^2 \right)} F(c', y')f(z', c', y', z'_0)\delta(c')$$

$$= 2\pi^2 \int \frac{d^2y\,d^2z_0}{|y|^2 \left( 1 + |\tilde{y}|^2 \right)^2} F(0, y)f(z, 0, y, z_0) ,$$  \hspace{1cm} (B.6)
\[
\begin{align*}
\frac{1}{F(0, y)} &= \lim_{c \to 0} \frac{1}{F(c, y)} \\
&= \lim_{c \to 0} \int d\alpha \, d\gamma \int_0^\infty \frac{d\lambda}{(1 + \lambda^2)^2} \left[ e^{\gamma} \left( \frac{c + y e^{i\alpha} - \lambda}{1 + \lambda(c + y e^{i\alpha})} + \frac{(c - y) e^{i\alpha} - \lambda}{1 + \lambda(c - y) e^{i\alpha}} \right) \right] \\
&= \lim_{c \to 0} 2 \pi \int \frac{d\alpha \, d\lambda}{(1 + \lambda^2)^2} \left| 1 - \lambda^2 y e^{2i\alpha} \right|^2 \delta \left( c - \lambda(e^{-i\alpha} + y^2 e^{i\alpha}) \right). \quad (B.7)
\end{align*}
\]

The delta function can be rewritten as,

\[
\delta \left( c - \lambda(e^{-i\alpha} + y^2 e^{i\alpha}) \right) = \frac{1}{|c| |e^{-i\alpha} + y^2 e^{i\alpha}|} \delta \left( \lambda - \frac{|c|}{|e^{-i\alpha} + y^2 e^{i\alpha}|} \right) \times 2 \left| \frac{y^2 e^{i\alpha} - e^{-i\alpha}}{y^2 e^{i\alpha} + e^{-i\alpha}} \right|^{-1} \delta (\alpha - \alpha(\text{arg}(c))) , \quad (B.8)
\]

and by inserting this back into (B.7),

\[
\frac{1}{F(0, y)} = \frac{2\pi}{|y|^2 - 1} . \quad (B.9)
\]

With the help of (B.9), we finally write (B.6) as

\[
\Pi^{(2)}(z) = \pi \int \frac{d^2 y \, d^2 z_0}{|y|^2} \left( \frac{1 - |\tilde{y}|^2}{1 + |\tilde{y}|^2} \right) f(z, 0, y, z_0) . \quad (B.10)
\]

Note that for \( \beta \to \infty \), the effect of the \( c \)-integration reduces to just an overall constant factor, as in \[10\].

**Appendix C. Calculation of the correlation function.**

In this appendix we will explicitly calculate the integrals that remain in the expression (4.5) for the correlator. We start from

\[
\Pi^{(2)}(x_1) = \frac{K \pi \Lambda^2}{4} \sinh^2 \left( \frac{x}{2} \right) \int_{-\infty}^\infty du \int_0^{2\pi} dy_0 \int_{-1}^1 dt \left( 1 - t^2 \right) |t| \times \]

\[
\times \left[ \frac{1}{(\cosh[u - (\tilde{x}/2)] + t \cos y_0) (\cosh[u + (\tilde{x}/2)] + t \cos y_0)} \right]^2 , \quad (C.1)
\]
where $\tilde{x} = 2\pi x_1/\beta$. Next we combine the denominators by using the Feynman parameter trick and also the addition theorem for the hyperbolic functions. The result is

$$\Pi^{(2)}(x_1) = 3K\pi\Lambda^2 \sinh^2 \left(\frac{\tilde{x}}{2}\right) \int_0^1 dt \left(1 - t^2\right) t \int_0^1 d\alpha \left(1 - \alpha\right) \alpha \times$$

$$\times \int_{-\infty}^{\infty} du \int_0^{2\pi} dy_0 \frac{1}{\left(\Upsilon \cosh u + t \cos y_0\right)^4}$$

$$= \frac{K\pi\Lambda^2}{2} \sinh^2 \left(\frac{\tilde{x}}{2}\right) \int_0^1 dt \left(1 - t^2\right) t \int_0^1 d\alpha \left(1 - \alpha\right) \alpha \times$$

$$\times \int_{-\infty}^{\infty} du \int_0^{2\pi} dy_0 \int_{-\infty}^{\infty} d\gamma \gamma^3 e^{-\gamma(\Upsilon \cosh u + t \cos y_0)}$$

$$= 2K\pi^2\Lambda^2 \sinh^2 \left(\frac{\tilde{x}}{2}\right) \int_0^1 dt \left(1 - t^2\right) t \int_0^1 d\alpha \left(1 - \alpha\right) \alpha \int_{-\infty}^{\infty} d\gamma \gamma^3 K_0(\gamma\Upsilon) I_0(\gamma t) ,$$

(C.2)

where

$$\Upsilon = \sqrt{\cosh^2 \left(\frac{\tilde{x}}{2}\right) - (1 - 2\alpha)^2 \sinh^2 \left(\frac{\tilde{x}}{2}\right)} ,$$

(C.3)

and $K_0$ and $I_0$ are the modified Bessel functions of zero order. The integration over $\gamma$ gives a hypergeometric function,

$$\int_{-\infty}^{\infty} d\gamma \gamma^3 K_0(\gamma\Upsilon) I_0(\gamma t) = \frac{4}{\Upsilon^4} F \left(2; 2; 1; t^2/\Upsilon^2\right) ,$$

(C.4)

and using the explicit form of the hypergeometric function

$$F \left(2; 2; 1; z^2\right) = \frac{1 + z^2}{(1 - z^2)^3} ,$$

(C.5)

the result is,

$$\Pi^{(2)}(x_1) = 8K\pi^2\Lambda^2 \sinh^2 \left(\frac{\tilde{x}}{2}\right) \int_0^1 dt \left(1 - t^2\right) t \int_0^1 d\alpha \left(1 - \alpha\right) \alpha \frac{\Upsilon^2 + t^2}{(\Upsilon^2 - t^2)^3} .$$

(C.6)

Performing the last two integrations, and assuming $x_1 > 0$ for simplicity, we finally arrive at

$$\Pi^{(2)}(x_1) = \frac{K\pi^2\Lambda^2}{3} \left[1 + 2\pi x_1 T \coth(\pi x_1 T) \left(1 - 2 \sinh^2(\pi x_1 T)\right) + \right.$$

$$\left. + 2 \sinh^2(\pi x_1 T) \ln \left(4 \sinh^2(\pi x_1 T)\right)\right]$$

(C.7)
References


