Hamiltonian structure of hamiltonian chaos

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From a kinematical point of view, the geometrical information of hamiltonian chaos is given by the (un)stable directions, while the dynamical information is given by the Lyapunov exponents. The finite time Lyapunov exponents are of particular importance in physics. The spatial variations of the finite time Lyapunov exponent and its associated (un)stable direction are related. Both of them are found to be determined by a new hamiltonian of same number of degrees of freedom as the original one. This new hamiltonian defines a flow field with characteristically chaotic trajectories. The direction and the magnitude of the phase flow field give the (un)stable direction and the finite time Lyapunov exponent of the original hamiltonian. Our analysis was based on a $1 \frac{1}{2}$ degree of freedom hamiltonian system.

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I. INTRODUCTION

Many systems that are chaotic such as the stochastic magnetic field line in toroidal plasma confinement devices have a hamiltonian representation [1,2]. The KAM theorem [3] deals with the onset of chaos in hamiltonian systems, and the description of the kinematics of developed chaos involves the ergodic theorem [4]. Hamiltonian dynamics naturally give rise to a differentiable dynamical system where the multiplicative ergodic theorem of Oseledec is applicable [5]. Oseledec’s theorem gives two aspects of chaos. First the sensitivity of the dependence on initial conditions is measured by various infinite time Lyapunov exponents, i.e., the dynamical information of chaos. Secondly the characteristic directions associated with these Lyapunov exponents give the geometrical aspect of chaos. That is: if two points are separated along different directions at the initial time, they can diverge or converge exponentially at different characteristic Lyapunov exponents.

Ergodic theory treats the time asymptotic limit, in which the infinite time Lyapunov exponents are constants and the characteristic directions are functions of position only. For finite time, there is a convergence issue for both the Lyapunov exponents and their associated characteristic directions. The convergence of the characteristic directions is exponential, so the geometrical aspect of chaos at finite time is well described by its time asymptotic limit. This is not the case for Lyapunov exponents. Finite time Lyapunov exponents suffer from a notoriously slow convergence problem [6]. Its spatial and temporal dependence was discussed by Tang and Boozer [7], who gave a direct link between the convergence function in finite time Lyapunov exponent and the geometry of the vector field defined by the corresponding characteristic direction. It should be emphasized that most applications of practical interest involve a finite duration of time. Hence the finite time properties of chaos, rather than the asymptotic properties of chaos, are of real concern.

In [7] we found that the spatial variation of the finite time Lyapunov exponent and the corresponding characteristic direction are not independent of each other. The exact relation was shown in [7] and is restated later in this paper, equations (7,8,9). Besides its practical importance such as those in transport studies [7], it also leads to a conceptual advance in the understanding of chaos. The finite time Lyapunov exponent (e.g. $\lambda$) and its characteristic direction (e.g. $\hat{s}_{\infty}$) can be described by another hamiltonian in the same phase space with the same number of degrees of freedom as the original hamiltonian. If one constructs the corresponding vector field (necessarily divergence-free) associated with the phase space trajectory of this new hamiltonian, the magnitude of this vector field gives the local finite time Lyapunov exponent while the direction of the vector field gives the corresponding characteristic direction. The new hamiltonian is also chaotic in the same region as the parent hamiltonian. One could characterize the chaos in this new hamiltonian by invoking a third generation hamiltonian. Consequently, a hierarchy of hamiltonian can be bootstrapped from the original hamiltonian of a conservative system that is chaotic.

II. GLOBALLY DIVERGENCE-FREE FIELDS AND HAMILTONIAN MECHANICS

A large class of conservative systems which exhibit chaotic behavior has a hamiltonian representation. Two of the well known examples are the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{v}$ of a divergence-free fluid [8]. An arbitrary divergence-free vector $\mathbf{G}(x,y,z)$ can be written in the so-called canonical form,
\[ G = \nabla \psi \times \nabla \theta + \nabla \phi \times \nabla \chi. \] (1)

The function \( \chi(\psi, \theta, \phi) \) is the Hamiltonian of the \( G \) field lines. The \( G \) field line is the trajectory \( x(\tau) \) given by equation

\[ \frac{dx}{d\tau} = G(x). \] (2)

To find the \( G \) field line in \( \psi, \theta, \phi \) coordinates, one has to invert the transformation equation \( x(\psi, \theta, \phi) \). Combining the transformation equation \( x(\psi, \theta, \phi) \) and the field line trajectory, equation (2), one arrives at the familiar Hamilton's equations,

\[ \frac{d\psi}{d\phi} = -\frac{\partial \chi}{\partial \theta}, \quad \frac{d\theta}{d\phi} = \frac{\partial \chi}{\partial \psi}, \quad \frac{d\phi}{d\phi} = 1, \] (3)

where \( \chi \) is the Hamiltonian, \( \theta \) the canonical position, \( \psi \) the canonical momentum, and \( \phi \) the canonical time 1. Since continuous transformation, such as \( x(\psi, \theta, \phi) \), preserve topological properties, questions regarding the integrability of the field line are then answered by the Hamiltonian \( \chi(\psi, \theta, \phi) \) alone.

It should be noted that only a globally divergence-free field can be represented by the form of equation (1), a requirement set by Poincare's lemma [11]. Furthermore, the divergence-free field \( G \) should not vanish in the region of interest.

A Hamiltonian representation of such a vector field is desirable since an array of well-developed techniques from Hamiltonian mechanics are available. For example, the Hamiltonian description of a magnetic field has played a major role in the theory of toroidal plasmas [8].

Before we proceed to the next section which is on the ergodic theorem, a digression on the applicability of the multiplicative ergodic theorem to Hamiltonian chaos is useful. A Hamiltonian flow, even when it is chaotic, can preserve topological properties, questions regarding the integrability of the field line are then answered by the Hamiltonian \( \chi(\psi, \theta, \phi) \) alone.

Despite the lack of a rigorous theory [9], these new numerical evidence and the simple Markov transport model put forward by Meiss should boost our confidence in the ergodicity of Hamiltonian chaos.

III. MULTIPlicative ergodic THEORem in LAGRangian COORDinates

The multiplicative ergodic theorem of Oseledec [5] complements KAM theorem [3] in understanding Hamiltonian chaos, especially the limiting case in which the regular components occupy a small portion of the phase space. This theorem can be understood in terms of a general coordinate transformation [11] between ordinary space \( x \) and the Lagrangian coordinates \( \xi \), a widely used coordinate system in fluid mechanics. A point with Lagrangian coordinates \( \xi \) is related to \( x \) in ordinary space by the integral curve of \( dx/dt = G(x) \) from initial time \( t_0 \) to later time \( t \) with the initial condition \( x(\xi, t_0) = \xi \). For a pair of fixed \((t_0, t)\), there is a one-to-one mapping between \( \xi \) and \( x(\xi, t) \). There is freedom in the choice of Lagrangian coordinates \( \xi \) due to the arbitrariness of \( t_0 \), a point that we will revisit in section IV. Of course, \( t_0 \) should be fixed for any chosen set of Lagrangian coordinates. \( t_0 \) is usually set to zero for convenience of bookkeeping.

In addition to a set of coordinates, one needs a metric to specify the physical distance between two neighboring points. Let \( x \) be ordinary Cartesian coordinates, then its metric is a unit matrix, \( dl^2 = dx \cdot dx = dx^i dx^j \). The same differential distance can also be specified in Lagrangian coordinates, \( dl^2 = g_{ij} d\xi^i d\xi^j \), where \( g_{ij} = \partial x^i / \partial \xi^j \cdot \partial x^j / \partial \xi^i \) is the metric tensor of the Lagrangian coordinates \( \xi \). Since \( x(\xi, 0) = \xi \), one could also understand \( dx^2 = g_{ij} d\xi^i d\xi^j \) as an equation which relates the initial separation \( d\xi^i d\xi^j \) between two neighboring points to their later separation \( d\xi^i d\xi^j \). Once this is understood, the interpretation of the metric tensor \( g_{ij} \) in Oseledec's multiplicative ergodic theorem becomes transparent, i.e., \( g_{ij} \) is the Oseledec matrix \( \Lambda_{ij} \) (for its definition, see [5] or [4]).

Instead of taking the limit of \( \lim_{t \to \infty} \lambda_{1t}^{1/2t} = \Lambda \) and diagonalizing the matrix \( \Lambda \), we diagonalize the matrix \( g_{ij} \) first and then take the limit of \( t \to \infty \). Since \( g_{ij} \) is a positive definite and symmetric matrix, it can be diagonalized with positive eigenvalues and real eigenvectors,

\[ g_{ij} = \Lambda_1 \hat{e}_i \hat{e}_j + \Lambda_2 \hat{m}_i \hat{m}_j + \Lambda_3 \hat{s}_i \hat{s}_j, \]

with \( \Lambda_1 \geq \Lambda_2 \geq \Lambda_3 > 0 \). There are three Lyapunov characteristic exponents associated with vector field \( G \),

\[ \lambda_1^\infty = \lim_{t \to \infty} \frac{\ln \Lambda_1}{2t}, \quad \lambda_2^\infty = \lim_{t \to \infty} \frac{\ln \Lambda_2}{2t}, \quad \lambda_3^\infty = \lim_{t \to \infty} \frac{\ln \Lambda_3}{2t}. \]

For a divergence-free field with a Hamiltonian representation, \( \lambda_1^\infty = -\lambda_2^\infty > 0 \) and \( \lambda_3^\infty = 0 \). These are usually called infinite time Lyapunov exponents and they are

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1In this extended ‘phase’ space, there are three independent coordinates and hence three Lyapunov exponents.
constants in one ergodic region. The eigenvectors \( \hat{e}, \hat{m}, \hat{s} \) have well-defined time-asymptotic limits as well, but they are position dependent, \( \hat{e}_\infty(\xi), \hat{m}_\infty(\xi), \hat{s}_\infty(\xi) \). There is a stable manifold theorem in the vicinity of an arbitrary point \( \xi \) [4]. The \( \hat{s}_\infty(\xi) \) is the tangent vector of the local stable manifold associated with the Lyapunov characteristic exponent \( \lambda^\infty \), the so-called stable direction. There is also an unstable direction which is tangent to the local unstable manifold. The unstable direction is defined using the time-reversed dynamics,

\[
\frac{dx}{dt} = -G(x),
\]

in which \( d^2 = (dx)^2 = g_{ij}^{(-)}d\xi^id\xi^j \) and \( g_{ij}^{(-)} = \Lambda_1^{(-)} \hat{e}^{(-)} + \Lambda_\infty^{(-)} \hat{m}^{(-)} + \Lambda_\infty^{(-)} \hat{s}^{(-)} \). The local stable manifold for the backward time dynamics defines the local unstable manifold for the forward time dynamics. Two linearly independent vectors define a plane. The intersection of two planes gives rise to another vector. One can use the intersection of plane \( \hat{s}_\infty, \hat{m}_\infty \) and plane \( \hat{s}^{(-)}, \hat{m}^{(-)} \) to construct another vector \( \hat{m} \). For the case we are interested in this paper, \( \lambda^\infty \) could be nonzero, \( \hat{m} \) is then the tangent vector of another local (un)stable manifold, corresponding to a (positive) negative \( \lambda^\infty \). In either case, these three vectors \( \hat{s}_\infty, \hat{m}_\infty, \hat{s}^{(-)} \), and \( \hat{m} \) span the three dimensional space.

The negative Lyapunov exponent means neighboring points converge exponentially in time, while a positive Lyapunov exponent means neighboring points diverge exponentially in time, while a positive Lyapunov exponent means neighboring points converge exponentially in time, while a positive Lyapunov exponent means neighboring points diverge exponentially in time. The ones of most importance are \( \lambda \) and \( \lambda^\infty \). The negative Lyapunov exponent means neighboring points diverge exponentially in time. The ones of most importance are \( \lambda \) and \( \lambda^\infty \). The second one is on geometry. Along different directions, neighboring points do behave differently, \( \lambda^\infty \) is the infinite time Lyapunov exponent. We note that the finite time Lyapunov exponent \( \lambda(\xi, t) \) can always be asymptotically decomposed into three main parts with the addition of an exponentially small fourth (correction) term

\[
\lambda(\xi, t) = \hat{\lambda}(\xi) / t + f(\xi, t) / \sqrt{t} + \lambda^\infty + O(\exp[-2\lambda(\xi, t)t]),
\]  

(7)

where

\[
\hat{s}_\infty \cdot \nabla f(\xi, t) = 0, \quad \hat{s}_\infty \cdot \nabla \hat{\lambda}(\xi) + \nabla_0 \cdot \hat{s}_\infty = 0,
\]

(8)

(9)

and \( \lambda^\infty \) is the infinite time Lyapunov exponent. We note that \( \hat{\lambda}(\xi) \) is a smooth function of position due to the smoothness of vector field \( \hat{s}_\infty \).

The correction to the asymptotic decomposition, i.e., the fourth term in equation (7), becomes exponentially small as \( t \) becomes large. The rate of the exponential decay is given by the magnitude of the local Lyapunov exponent. Hence the correction term becomes negligible on a time scale of a few local Lyapunov time. It should be emphasized that what we presented here is an asymptotic form of a local expression (finite time Lyapunov exponent is an explicit function of position and time). In different regions of a chaotic component, the local Lyapunov exponent can vary significantly. For example, the stochastic layer would have a much smaller finite time Lyapunov exponent and hence a longer period during

IV. THE HAMILTONIAN NATURE OF THE THEORY OF FINITE TIME LYAPUNOV EXPONENTS

Before we proceed, it is useful to summarize the main results of ergodic theorem. Chaos means sensitive dependence on initial conditions. There are two aspects of chaos which are captured by the multiplicative ergodic theorem of Oseledec. The first one is the dynamical aspect, i.e., how sensitive is the dependence on initial conditions? This is answered by the Lyapunov characteristic exponents. The second one is on geometry. Along different directions, neighboring points do behave differently, i.e., different Lyapunov exponent corresponds to different characteristic directions. As signified by its name, ergodic theorem treats the time-asymptotic limit or long time average. Hence the Lyapunov exponents in ergodic theorem are also called infinite time Lyapunov exponents which are constants in a chaotic sea. The geometrical information that is given by the ergodic theorem is always local, i.e., \( \hat{s}_\infty(\xi) \) is a function of position. The smoothness of \( G(x) \) implies the smoothness of \( \hat{s}_\infty(\xi) \) as a vector field. The field line of \( \hat{s}_\infty(\xi) \) is of great importance in transport of advection-diffusion type. For example, the rapid diffusive relaxation of an externally imposed scalar or vector field occurs only along the \( \hat{s} \) lines [7].

In applications where finite time is of concern, one needs to understand the properties of the finite time Lyapunov exponents. The finite time Lyapunov exponent has both a time and space dependence. It is also called local Lyapunov exponent for that reason. Since, for example, both \( \lambda(\xi, t) \) and \( \hat{s}_\infty(\xi) \) are local, one might feel that there could be some relationship which relates the spatial variation of these two. This was addressed by our work on finite time Lyapunov exponent [7]. We found that the finite time Lyapunov exponent \( \lambda(\xi, t) \) can always be asymptotically decomposed into three main parts with the addition of an exponentially small fourth (correction) term

\[
\lambda(\xi, t) = \hat{\lambda}(\xi) / t + f(\xi, t) / \sqrt{t} + \lambda^\infty + O(\exp[-2\lambda(\xi, t)t]),
\]  

(7)
which the exponentially small correction is still important. However that time length is fixed on the order of a few local Lyapunov time, which is the characteristic time scale over which the stochasticity of a chaotic trajectory starting from a particular location is of practical importance. This should be contrasted with a statistical description of the finite time Lyapunov exponent, whose practical applicability would be affected by the extremely long transients for hamiltonian systems to reach to an invariant distribution by following one chaotic trajectory [9].

Because \( f(\xi, t) \) does not vary along the \( \hat{s} \) direction, thus to exponential accuracy the variation of the finite time Lyapunov exponent along the stable manifold is determined by the geometry of the stable manifold alone, equation (9). Notice that everything is local, i.e. functions of position. In particular, \( \tilde{\lambda}(\xi) \) and \( \hat{s}_\infty(\xi) \) do not have a time dependence. That is: they represent time asymptotic structures. They are of great practical importance because the equation (9) accurately describes the spatial variation of the finite time Lyapunov exponent along the stable manifold on a time scale of a few local Lyapunov time. The effect of geometry on a dynamical quantity like the finite time Lyapunov exponent is captured by the function \( \tilde{\lambda}(\xi) \) alone, a function that is completely determined by the \( \mathbf{G}(\mathbf{x}) \) field.

The main points of Tang and Boozer’s work on finite time Lyapunov exponents [7] are the function \( \tilde{\lambda} \) and its relationship with \( \hat{s}_\infty \). These results actually find their roots in hamiltonian dynamics. To see that, one can construct a new vector field

\[
\mathbf{S}(\xi) \equiv \tilde{\lambda}(\xi) \hat{s}_\infty(\xi).
\]

Obviously \( \mathbf{S}(\xi) \) does not vanish anywhere and is globally defined in a chaotic region, or using Meiss’s term, on an ergodic irregular component. More importantly, \( \mathbf{S} \) is divergence-free because of equation (9). All the necessary information on chaos in field \( \mathbf{G}(\mathbf{x}) \) associated with negative Lyapunov exponent for forward time, are contained in this new vector field \( \mathbf{S} \).

On a regular component in phase space, i.e. KAM surface, \( \mathbf{S} \) is also well defined and divergence-free. The property of a trajectory on a KAM surface is determined by the rotational transform \( \iota(\psi) \). In explicit form, the trajectory follows

\[
\psi = \psi_0; \quad \phi = \phi_0 + \nu_0(\psi)t; \quad \theta = \theta_0 + \iota(\psi)\nu_0(\psi)t \tag{10}
\]

with \( \nu_0(\psi) \) the Jacobian of \( (\psi, \phi, \theta) \) coordinates. The form of equation (10) is generic for integrable trajectories [13]. An explicit construction of the proper coordinate system which gives rise to equation (10) can be found in [12]. The \( \hat{s}_\infty \) introduced earlier now takes the form

\[
\hat{s}_\infty \propto (0, \nu_0', \iota' \nu_0 + \nu_0''), \tag{11}
\]

where prime denotes a derivative with respect to \( \psi \), the action. A detailed derivation of equation (11) can be found in [12]. Vanishing \( \psi \) component of \( \hat{s}_\infty \) means that \( \hat{s}_\infty \) vector is tangent to the KAM surface. By definition, two neighboring points along \( \hat{s}_\infty \) direction will converge, but quadratically in time, which should be contrasted to an exponential rate in a chaotic region. The \( \mathbf{S} \) vector can be simply given by

\[
\mathbf{S} = (0, \nu_0, (\omega_\nu(\psi)' \nu_0)'.
\]

This would correspond to an integrable trajectory with rotational transform \((\omega_\nu(\psi)' / \nu_0')\) on the constant action \( \psi_0 \) surface. In other words, if one writes the original field as

\[
\mathbf{G} = \nabla \psi \times \nabla \theta + \nabla \phi \times \nabla \chi
\]

and on the constant \( \psi_0 \) surface

\[
\iota(\psi_0) = \partial \chi / \partial \psi |_{\psi = \psi_0},
\]

then the \( \hat{\mathbf{S}} \) field will be

\[
\hat{\mathbf{S}} = \nabla \psi \times \nabla \theta + \nabla \phi \times \nabla \chi
\]

and the hamiltonian \( \tilde{\chi} \) for the field line of \( \hat{\mathbf{S}} \) satisfies

\[
\partial \tilde{\chi} / \partial \psi |_{\psi = \psi_0} = (\omega_\nu(\psi)' / \nu_0') |_{\psi = \psi_0}.
\]

Cautions should be taken regarding the derivative with respect to \( \psi \). The surviving KAM tori are parameterized on a fractal set of action \( \psi \) in a perturbed hamiltonian system. The proper definition of derivative with respect to \( \psi \) on the surviving KAM tori invokes Whitney’s notion, which was discussed in detail by Pöschel [14].

It should be noted that so far we have established divergence-free \( \hat{\mathbf{S}} \) field separately in chaotic region and KAM region. To achieve divergence-free globally, it actually only requires the smoothness of the direction of \( \hat{\mathbf{S}} \) fields when crossing the last KAM surface. A jump in the magnitude of \( \hat{\mathbf{S}} \) is allowed since its gradient is perpendicular to \( \hat{\mathbf{S}} (\nabla \cdot \hat{\mathbf{S}} = \nabla \cdot f \cdot \hat{s}_\infty + f \cdot \nabla \cdot \hat{s}_\infty) \). Although an analytical proof has not been found, numerical results have been obtained to support the continuity of the direction of \( \hat{s}_\infty \) at the border between order and chaos.

As pointed out in section II, a globally divergence-free field admits a hamiltonian representation. Hence the field line of \( \mathbf{S} \) (and \( \hat{s} \)) is described by a hamiltonian \( \chi_1(\psi_1, \phi_1) \) with \( \chi_1 \) the canonical position, \( \psi_1 \) the canonical momentum, and \( \phi_1 \) the canonical time. In other words, the characteristics of chaos in the hamiltonian \( \chi \) are now contained in a new hamiltonian \( \chi_1 \). \( \chi_1 \) has the same degrees of freedom as \( \chi \) and the field lines of \( \mathbf{S} \) are chaotic just as those of \( \mathbf{G} \). Similarly, one could represent chaos in \( \hat{\mathbf{S}} (\chi_1) \) by another hamiltonian \( \chi_2 \) associated with another divergence-free vector field \( \mathbf{S}' \). Henceforth a hierarchy of hamiltonians is constructed for describing the chaos in \( \mathbf{G} \).

One might be concerned as to the proper counting of the degrees of freedom if \( \mathbf{G} \) is time dependent. In reality, \( \mathbf{S} \) becomes time dependent through \( \hat{s}_\infty \). This comes
from the fact that the specification of Lagrangian coordinates depends on the choice of initial time $t_0$, as discussed in last section. For a time-dependent field $G$, the vector $\hat{s}_\infty$ is a function of position and time $\hat{s}_\infty(\xi, t_0)$. If $G(x, t)$ is periodic in time $t$, $\hat{s}_\infty(\xi, t_0)$ will be periodic in time $t_0$. The hamiltonian $\chi_1$ has the exactly same degrees of freedom as $\chi$. This small subtlety holds the key to a correct understanding of a time dependent field in two dimensions $G(x, y, t)$ which has chaotic field lines. The time dependence in $\hat{s}_\infty(\xi_x, \xi_y, t_0)$ assures that the corresponding $S$ field has a time dependence and the hamiltonian $\chi_1$ for $S$ is of one and a half degrees of freedom. Hence chaos is allowed for the field line of $\hat{s}_\infty(\xi_x, \xi_y, t_0)$ and one $\hat{s}$ line fills the entire irregular component of $G$.

As stated earlier, there is a symmetry between $(\hat{s}_\infty, \lambda_s)$ and $(\hat{s}_\infty^{(-)}, \lambda_s^{(-)})$. Hence one could construct a set of hamiltonian to describe the chaos associated with the positive Lyapunov exponent and unstable direction for the forward time dynamics, in analogy to what we have done for the negative Lyapunov exponent and stable direction.

V. SUMMARY

In this paper we study the interrelations of divergence-free field, hamiltonian dynamics, multiplicative ergodic theorem in hamiltonian chaos, general coordinate transformation, and the theory of finite time Lyapunov exponents. We argue that finite time Lyapunov exponents and its associated characteristic directions are the most important information on chaos for the purpose of physical applications. Unlike the infinite time Lyapunov exponents, spatial variation of the finite time Lyapunov exponent is directly related to the geometry of its corresponding characteristic direction. Both are found to be given by one single new hamiltonian of same degrees of freedom. The magnitude of the phase flow field of the new hamiltonian determines the finite time Lyapunov exponent, while the direction of the phase flow gives the corresponding characteristic direction. The new hamiltonian is chaotic on the irregular component of the original hamiltonian. We hope the point of view presented here could stimulate new insights into hamiltonian chaos.

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