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PION FORM FACTOR AND RHO MESON WIDTH

Matts Roos
CERN - Geneva

and

Ján Pišút
Katedra Teoretickej Fyziky
Comenius University, Bratislava

A B S T R A C T

Several different parametrizations of the pion form factor $F_\pi(s)$ are given using either the Omnès function or the N/D method. In order to obtain the ρ resonance parameters we fit all formulae to data from $e^+e^- \rightarrow \pi^+\pi^-$ colliding beam experiments. The resonance mass, M_{ρ^0} , and the width, Γ_0 , turn out to be very dependent on the parametrization and the approximations chosen. The most realistic parametrizations lead to a width of $\Gamma_0 = 122^{+7}_{-6}$ MeV and a mass of $M_{\rho^0} = (770 \pm 5)$ MeV. We also make some comparisons with the pion electroproduction data, which give indirect information on $F_\pi(s)$ in the region $s < 0$.

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1. INTRODUCTION

A reaction which offers particular advantages for the study of $\pi\pi$ interactions is

$$e^+e^- \rightarrow \pi^+\pi^- \quad (1)$$

realized with colliding-beam techniques^{1,2)}. Firstly, this reaction is "clean" in the sense that no further hadron is present to complicate the dynamics and perturb the pure $\pi\pi$ interactions in the final state. Secondly, reaction (1) proceeds dominantly through the one-photon intermediate state. The vector nature of the photon then requires the $\pi\pi$ system to be in a state of unique angular momentum, $J = 1$. It follows from this and from Bose statistics that the isospin must be $T = 1$. Thus the interaction between the final-state pions in reaction (1) is described by only one partial wave, $J = T = 1$.

In this partial wave the $\pi\pi$ interaction at low energies is dominated by the well-known ρ resonance at 765 MeV. The colliding-beam experiments^{1,2)} have to date concentrated on this energy region, thereby adding important information to the understanding of the ρ meson. One purpose of the present paper is to extract information on the ρ resonance parameters by carrying out a detailed analysis of the energy dependence of the pion electromagnetic form factor, $F_\pi(s)$, which describes the coupling at the $\gamma\pi^+\pi^-$ vertex. Reaction (1) furnishes direct information on $F_\pi(s)$ for $s > 4\mu^2$, where μ is the mass of the charged pion.

Most of our present knowledge of the ρ meson comes from the reactions $\pi N \rightarrow \pi\pi N$ and $\gamma N \rightarrow \pi\pi N$, where the final ρ meson is produced in the presence of a strongly interacting nucleon. The analysis of these reactions depends strongly on the dynamical assumptions and, as we have shown in the $\pi N \rightarrow \pi\pi N$ case³⁾, the mass and width of the ρ meson become very model-dependent. A better understanding of the pion form factor will therefore shed light on the dynamics of ρN production.

Another reaction where the pion form factor comes in is pion electroproduction

$$e^-p \rightarrow e^-\pi^+n. \quad (2)$$

With the kinematical conditions suitably chosen, this reaction may be dominated by the diagram in Fig. 1, which contains the $\gamma \pi^+ \pi^-$ vertex and thus $F_\pi(s)$ (for $s < 0$). The extraction of information on $F_\pi(s)$ from experimental data on reaction (2) is, however, less direct than for reaction (1). We shall return to this point in Section 3.

Information about the resonance mass ω_r , and width Γ_0 , is usually sought in a narrow energy region around ω_r . A difficulty then may be the lack of sufficient statistics, in particular for the determination of Γ_0 , if it is large. If the statistics are sufficient, or if one considers a wider energy region, one soon learns that the shape of a broad resonance can no longer be described with only two parameters. Thus the wide-spread use of narrow-resonance approximations to analyse broad resonances cannot be justified, and can lead to apparent success only when the statistics are insufficient.

The present statistics on reaction (1) are too small to permit restriction to a narrow region. Independently of this fact, we prefer to use formulae which may be valid over a larger energy range, since the ρ meson is broad. Thus we must allow for more parameters. The freedom in the choice of further parameters should then be used judiciously, so that the same theory may describe features far away from the resonance (the principle of economic parametrization), and so that it does not produce unwanted effects (even if far away).

In the previous analysis³⁾ we have shown, and we shall show it again in Section 3 below, that different parametrizations and approximations to the shape of a resonance may lead to quite different values of ω_r and Γ_0 . In particular, narrow-resonance formulae notoriously give too small values for ω_r and Γ_0 when applied to broad resonances. Since many formulae may be equally good from a theoretical point of view, the variation caused by different (good) parametrizations may be called the "theoretical error" of the parameter. Clearly, to give the result of a fit of just one formula implies deliberate neglect of the theoretical error.

In the present case, we require from a "good" formula for the pion form factor $F_\pi(s)$ that

- i) it describes the data from reaction (1) in the region $s \sim \omega_r^2$,

- ii) it agrees with theoretical knowledge on the threshold behaviour,
 $s \gtrsim 4\mu^2$,
- iii) it is correctly normalized^{*)} at $s = 0$, $F_\pi(0) = 1$,
- iv) it can describe the data from reaction (2) in the region $s < 0$.

We do not try to relate the knowledge of $F_\pi(s)$ to nucleon form factors. The present status of this problem may be traced back starting from the work by Höhler et al.⁴⁾. It has previously been treated in detail in the excellent book on the subject by Drell and Zachariasen⁵⁾.

The organization of the present paper is as follows. Section 2 gives a brief survey of well-known properties of the pion form factor. In general, this is a multichannel problem, requiring knowledge of processes like $e^+e^- \rightarrow K\bar{K}$, $e^+e^- \rightarrow 4\pi$, etc. Lacking experimental data on these processes, we can at most account for them by some phenomenological parameters. The formulae we use will, therefore, rest on the assumption that the $\pi\pi$ intermediate state dominates the contribution to $\text{Im } F_\pi(s)$ well above the ρ -meson mass (up to about 1.5 GeV). In this case the pion form factor $F_\pi(s)$ can be expressed in terms of the D-function for $\pi\pi$ scattering in the $T = J = 1$ partial wave.

Section 2 proceeds to discuss two ways of parametrizing $F_\pi(s)$. Either the phase shift $\delta_1^1(s)$ is parametrized in an intuitively reasonable way, and then inserted into the Omnès formula for the pion form factor. Or the D-function is calculated starting from simple assumptions on the left-hand cut singularities in the $T = J = 1$ partial wave. In both cases we discuss the assumptions under which $F_\pi(s)$ simplifies to the expressions used frequently for $F_\pi(s)$. Discussions of details can be found in the Appendix.

The analysis of the data from reaction (1), using formulae for $F_\pi(s)$ given in Section 2, is presented in Section 3. We also compare predictions of the most realistic formulae in the region $s < 0$ with data from reaction (2). The parameters of the ρ meson are given and compared with those obtained previously³⁾.

Let us finally draw your attention to a different matter. In Ref. 3 we defined a radius of interaction R to describe the skewness of the

*) We still use "normalize" in the honest sense of the word.

ρ meson. We have since discovered that all the numerical values for R^2 in Ref. 3 are erroneously too small by a factor of 4. The definition of R^2 , Eq. (30) in Ref. 3, and the dimension $(\text{GeV}/c)^{-2}$ are unchanged.

2. PARAMETRIZATIONS OF THE PION FORM FACTOR

Let the total energy of the pion pair in reaction (1) be denoted $\omega = \sqrt{s}$. The virtual photon four-momentum squared is then s . For the elastic $\pi\pi$ scattering amplitude in the $T = J = 1$ partial wave, we write

$$a(s) \equiv A_{\pi\pi, \pi\pi}(s) = \frac{\omega}{2q^3} e^{i\delta_1^1} \sin \delta_1^1, \quad (3)$$

and for the inelastic amplitudes $A_{\pi\pi, K\bar{K}}(s)$, etc. Denoting the pion c.m.s. momentum $q = |\vec{q}|$ and defining $\nu \equiv q^2/\mu^2$, we have $s = 4\mu^2(\nu + 1)$.

The analytic properties of the pion form factor $F_\pi(s)$ are simple^{5,6,7}). $F_\pi(s)$ is a real analytic function $F_\pi(s) = F_\pi^*(s^*)$, with only one cut extending from $s_0 = 4\mu^2$ to $s = \infty$. As usual, we take $F_\pi(s)$ to be normalized at $s = 0$ to 1.

The behaviour of $F_\pi(s)$ is controlled by $\text{Im } F_\pi(s + i\epsilon)$ for $s > s_0$, and $\text{Im } F_\pi$ in turn by the unitarity condition [see, for example, Bjorken⁶]):

$$\text{Im } F_\pi(s) = \sum_n \rho_n(s) \Theta(s - s_n) F_n^*(s) A_{n, \pi\pi}(s). \quad (4)$$

Here the $\rho_n(s)$ are the real kinematical factors for the process $\pi\pi \rightarrow n$ with threshold s_n , and $F_n(s)$ is the amplitude for the process $\gamma \rightarrow n$.

The state with the lowest threshold s_n contributing to the r.h.s. of Eq. (4) is the $\pi\pi$ state. Next follow the 4π state, $\pi\eta$, $K\bar{K}$, etc. None of these channels, however, seem to be coupled to the $T = J = 1 \pi\pi$ channel. The ρ meson has never been seen decaying into either 4π or $\pi\eta$, the upper limit of the branching ratio into $4\pi + \pi\eta$ being 1%⁷), and the first possibility for a resonance decaying into both $\pi\pi$ and 4π arises only around 1.7 GeV⁸).

Let us then take $s_0 < s < s_H$ as the region where the $\pi\pi$ intermediate state saturates the unitarity condition (4). From the reality of

$\text{Im } F_\pi(s)$ and $\rho_{\pi\pi}(s)$, it follows that

$$\varphi(s) \equiv \arg F_\pi(s) = \delta_1^1(s) \quad \text{in the region} \quad s_0 < s < s_H. \quad (5)$$

Dividing $F_\pi(s)$ by a polynomial $P(s)$ which has the same zeros as $F_\pi(s)$, and writing the dispersion relation for the function $\ln [F_\pi(s)/P(s)]$ we arrive at the Omnès formula^{*} for $F_\pi(s)$:

$$F_\pi(s) = P(s) D^{-1}(s) = P(s) \exp \left\{ \frac{s}{\pi} \int_{s_0}^{\infty} \frac{\varphi(s') ds'}{s'(s'-s)} \right\} \quad (6)$$

where $D^{-1}(s)$ is called the Omnès function. The Omnès formula respects explicitly the analyticity properties of $F_\pi(s)$, the normalization condition $F_\pi(0) = 1$ and it exhibits clearly possible zeros in $F_\pi(s)$. Through Eq. (5), it expresses $F_\pi(s)$ directly in terms of $\delta_1^1(s)$ and it is therefore particularly convenient and flexible for extracting $\delta_1^1(s)$ from the data on $F_\pi(s)$.

The polynomial $P(s)$ always remains undetermined [up to the requirement $P(0) = 1$]. The choice $P(s) = 1$ for all s is equivalent to the requirement of no zeros in $F_\pi(s)$. In most cases it is assumed that $P(s) = 1$. The lack of convincing arguments for this assumption has been stressed already in one of the basic papers on nucleon and pion form factors⁹⁾. It is, however, the simplest assumption, and we shall use it in several of our formulae.

Phenomenology enters into the Omnès formula (6) through the phase $\varphi(s)$ or the phase shift $\delta_1^1(s)$, which may be parametrized in different ways to suit the experimental needs. However, it is not easy to see from Eq. (6) to what left-hand singularities in the partial wave amplitude, $a(s)$, a particular parametrization of $\delta_1^1(s)$ corresponds. Instead, one may, therefore, choose to let phenomenology enter in a different way, so as to leave full control over the left-hand singularities. Although formally different, such an approach is completely equivalent to the Omnès function approach. The phase shift $\delta_1^1(s)$ is then given implicitly by the parametrization chosen for the left-hand singularities.

^{*}) Details about the Omnès function approach to $F_\pi(s)$ and properties of the Omnès function may be found in Refs. 5, 9, 10, 11 and 12.

Let

$$a(s) = \frac{N(s)}{D(s)}, \quad (7)$$

where $D(s)$ is real analytic, with a cut $s_0 < s < \infty$, and $N(s)$ is real analytic, with cuts $-\infty < s < 4\mu^2$ and $s_H < s < \infty$. Since $N(s)$ is real for $s_0 < s < s_H$, we easily obtain from Eqs. (3) and (7)

$$D(s) = |D(s)| e^{-i\delta_1^+(s)}, \quad \text{for } s_0 < s < s_H.$$

Due to the properties of $F_\pi(s)$ and $D(s)$ the function $F_\pi(s) D(s)$ is real analytic with only a cut extending from s_H to ∞ . We shall, in the following, ignore any detailed structure of this cut which (with more data) can be at least phenomenologically accounted for. With this omission, we can write

$$F_\pi(s) = \frac{D(0)}{D(s)} P(s), \quad (8)$$

where $P(s)$ is again a polynomial. It must have at least the same zeros as $F_\pi(s)$. Thus the polynomial ambiguity again appears. In the following, we shall use Eq. (8) with $P(s) = 1$, risking in certain cases unwanted poles in $F_\pi(s)$.

The actual calculation of $D(s)$ is then based on the N/D equations¹⁰⁾

$$N(\nu) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{\alpha(\nu') D(\nu') d\nu'}{\nu' - \nu} \quad (9a)$$

$$D(\nu) = 1 - \frac{\nu + 1}{\pi} \int_0^\infty \left(\frac{\nu'^3}{\nu' + 1} \right)^{1/2} \frac{N(\nu')}{(\nu' + 1)(\nu' - \nu)} d\nu' \quad (9b)$$

where $\alpha(\nu')$ is the left-hand discontinuity of the amplitude $a(\nu)$ [see Eq. (3)], e.g. $\alpha(\nu') \equiv [a(\nu' + i\epsilon) - a(\nu' - i\epsilon)]/(2i)$ for $\nu' < -1$. We shall in fact use only Eq. (9b), making simple assumptions about $N(\nu)$.

The parametrizations of Eqs. (6) and (9b) to be used in Section 3 will be discussed in subsections 2.1 and 2.2 below and, in some more detail, in the Appendix.

2.1 The Omnès function approach

We shall parametrize the phase shift $\delta_1^1(s)$ in terms of the energy-dependent width $\Gamma(\omega)$, as

$$\cot \delta_1^1(\omega) = \frac{\omega_r^2 - \omega^2}{\omega_r \Gamma(\omega)} \quad (10)$$

where ω_r is the resonance energy and $q_r = q(\omega_r)$. The "local" width $\Gamma_0 = \Gamma(\omega_r)$ is then related to the derivative of $\delta_1^1(\omega)$ at ω_r by

$$\left. \frac{d\delta_1^1(\omega)}{d\omega} \right|_{\omega=\omega_r} = \frac{2}{\Gamma_0} . \quad (11)$$

The simplest parametrization of $\Gamma(\omega)$ leading to the correct threshold behaviour of $\delta_1^1(\omega)$ is

$$\Gamma(\omega) = \Gamma_0 (q/q_r)^3 \quad (12a)$$

A more realistic parametrization of $\delta_1^1(\omega)$ uses Eq. (10) with

$$\Gamma(\omega) = \Gamma_0 \frac{\omega_r}{\omega} \left(\frac{q}{q_r} \right)^3 \frac{1 + R^2 q_r^2}{1 + R^2 q^2} . \quad (12b)$$

A few comments on Eq. (12b) may be found in Ref. 3.

A somewhat more accurate description of the threshold region may be expected from the parametrization

$$\frac{q^3}{\omega} \cot \delta_1^1(s) = \frac{2}{\pi} \frac{q^3}{\omega} \ln \left(\frac{\omega + 2q}{2\mu} \right) + a + b\omega^2 + c\omega^4 , \quad (12c)$$

which has the same number of parameters as Eq. (12b).

To each of these parametrizations one might wish to add a pole term, thus forcing the phase shift to go through π at some energy ω_c . This is necessary if one wants to have a second resonance (ρ') somewhere

beyond ω_r . A $\Gamma(\omega)$ with this property is given by

$$\Gamma(\omega) = \Gamma_0 \frac{q^3}{\omega} \left(\frac{\omega_r}{q_r^3} \right) \frac{\omega_e^2 - \omega^2}{\omega_e^2 - \omega_r^2} \cdot \frac{B - \omega_r^2}{B - \omega^2} \quad (12d)$$

This form corresponds to Eq. (36) in Ref. 3. An expression for $F_\pi(s)$ based on Eqs. (6), (10), and (12a) can be calculated in a closed analytic form (see the Appendix, Section 1). In other cases we insert $\Gamma(\omega)$ into Eq. (10) and calculate numerically $F_\pi(s)$ by Eq. (6). According to Eq. (5), we put $\varphi(s) = \delta_1^1$ below s_H and

$$\varphi(s) = \text{const.} = \delta_1^1(s_H) \text{ for } s > s_H, \quad (13)$$

and

$$P(s) = 1.$$

The condition (13) is difficult to replace by anything better owing to the lack of theoretical understanding, and since the experimental information is still meagre. In Section 3 we shall give some figures for acceptable values of "const.". Note that if one moves s_H down to the vicinity of s_r or below s_r , no information about the ρ meson parameters can be obtained anymore from data on the pion form factor. Claims^{13,14}) that the ρ parameters are strongly dependent on "const." are therefore always a result of choosing s_H nearer to s_r than we have done.

It is sometimes argued that the $\Gamma(\omega)$ corresponding to Eq. (12a) is not realistic because of its q^3 dependence, and that one should choose a $\Gamma(\omega)$ linear in q instead. This corresponds to $\varphi(s)$ having an s -wave threshold dependence, and the Omnès function is then given by:

$$D^{-1}(\omega^2) = \frac{\omega_r^2 + \omega_r \Gamma_0 (\mu/q_r)}{\omega_r^2 - \omega^2 - i\omega_r \Gamma_0 (q/q_r)}. \quad (14a)$$

(See the Appendix, Section 1, for a similar calculation.)

If the resonance is very narrow, every formula is well approximated by

$$F_{\pi}(s) = \frac{C'}{\omega_r^2 - \omega^2 - i\omega_r \Gamma_0} . \quad (14b)$$

In the vicinity of ω_r one may make the further approximation $\omega + \omega_r \approx 2\omega_r$, in which case Eq. (14b) further simplifies to

$$F_{\pi}(s) = \frac{C}{\omega_r - \omega - i\Gamma_0/2} . \quad (14c)$$

Equation (14c) was used by the Novosibirsk group¹⁾ (in their first paper, which used preliminary data, and which aroused considerable attention among other things because of $\Gamma_0 = 93$ MeV), with C an adjustable parameter.

Before the publication of the Novosibirsk results¹⁾, several people applied the Omnès formula to the pion form factor, for instance Bowcock, Cottingham and Lurié¹⁵⁾, Vick¹⁶⁾, Antoniou and Bowcock¹³⁾, and Selleri¹⁷⁾.

Actual fits to the Novosibirsk 1967^{*)} data¹⁾ were done by Bergia and Mignaco¹⁸⁾, by Geffen and Walsh¹⁹⁾, and by Moffat¹⁴⁾. A fit to a different set of data points (containing some errors) was done by us³⁾.

The theoretical questions concerning the determination of $\varphi(s)$ from a knowledge of $|F_{\pi}(s)|^2$ were discussed by Bowcock and Kanellopoulos²⁰⁾ and by Truong et al.²¹⁾. The former authors pointed out that the peak in $|F_{\pi}(s)|^2$ in their model occurred slightly below the $\delta_1^+(s) = \pi/2$ point. Although this interesting conclusion was reached in a particular model, it must hold for a wider class of $\varphi(s)$, since the same behaviour had been observed previously, at least by Antoniou and Bowcock¹³⁾ and by Ball and Wong²²⁾.

*) It is important to note that the 1967 data were changed quite significantly in 1968. Thus the Novosibirsk group¹⁾ using Eq. (14c) fitted their 1967 data with $\Gamma_0 = 93 \pm 15$ MeV, and their 1968 data with $\Gamma_0 = 105 \pm 20$ MeV.

2.2 The N/D approach

A frequently used approximation is to have two poles on the left-hand cut

$$N(\nu) = \frac{C_1}{\nu + \nu_1} + \frac{C_2}{\nu + \nu_2} \quad \text{for } \nu > 0. \quad (15)$$

The function $D(s)$ is then calculated by Eq. (9b). The explicit expression for $D(s)$ following from Eq. (15a) is given in the Appendix, Section 2^{*}). With an expression for $D(s)$ given, the pion form factor squared is $|F_\pi(s)|^2 = |1/D(s)|^2$. A minor disadvantage in this approach is that the parameters c_i , ν_i , etc., are not simply related to the resonance parameters ω_r , Γ_0 . The relations between ω_r and Γ_0 and the N/D parameters can be given¹⁰⁾ in the form

$$\begin{aligned} \text{Re } D(\nu_r) &= 0 \\ \Gamma_0 &= \left| \frac{4}{\omega_r} \left(\frac{\nu_r^3}{\nu_r + 1} \right)^{1/2} \frac{N(\nu_r)}{\text{Re } D'(\nu_r)} \mu^2 \right| \end{aligned} \quad (16)$$

The narrow resonance approximation leads also in the $1/D$ formulation to an expression very similar to Eq. 14b. In fact, when $F_\pi(s) = 1/D(s)$, we have near the resonance

$$F_\pi(s) \sim \left[\nu - \nu_r - i \left(\frac{\nu^3}{\nu+1} \right)^{1/2} \frac{N(\nu)}{\text{Re } D'(\nu_r)} \right]^{-1}$$

using $\text{Re } D(\nu_r) = 0$ and $\text{Im } D = -N(\nu) [\nu^3/(\nu+1)]^{1/2}$. If we now approximate $N(\nu)/\text{Re } D'(\nu_r)$ by a constant, we are led to the usual Breit-Wigner formula for $F_\pi(s)$ with an energy-dependent width,

$$\begin{aligned} F_\pi(s) &= \frac{C}{\omega_r^2 - \omega^2 - i\omega_r \Gamma(\omega)} \\ \Gamma(\omega) &= \Gamma_0 \frac{\omega_r}{\omega} \left(\frac{q}{q_r} \right)^3. \end{aligned} \quad (17)$$

^{*}) More details about $1/D$ expressions for $F_\pi(s)$ may be found in papers by Ball and Wong²²⁾ Frazer and Fulco²³⁾ and in the book by Chew¹⁰⁾.

A multichannel version of the one pole formula [Eq. (15a) with $C_r = 0$] was derived by Ball and Parkinson²⁴⁾ and applied to reaction (1) by Parkinson²⁵⁾.

A formula similar to those based on the N/D method was recently used by Gounaris and Sakurai²⁶⁾. This is most simply obtained if one starts from the dispersion relation for the inverse amplitude, $a^{-1}(\nu)$. Neglecting left-hand singularities, CDD poles, and inelastic effects, one has

$$\frac{1}{a(\nu)} = \alpha + \beta\nu + \nu h(\nu) - i\left(\frac{\nu^3}{\nu+1}\right)^{1/2}, \quad \text{for } \nu > 0, \quad (18a)$$

where $h(\nu)$ is as defined in the Appendix, Section 2. The amplitude $a^{-1}(\nu)$ has no left-hand singularities (and hopefully no zero). Thus an acceptable solution to $F_\pi(s)$ can be constructed as

$$F_\pi(s) = \frac{a(0)}{a(s)}, \quad a(0) = \alpha - \beta - \frac{2}{\pi}. \quad (18b)$$

The resonance parameters ω_r and Γ_0 must again be solved from relations (16), setting $N(\nu) \equiv 1$ and $\text{Re } D(\nu) \equiv \alpha + \beta\nu + \nu h(\nu)$.

3. RESULTS AND CONCLUSIONS

3.1 Fits to reaction (1)

The experimental data on $|F_\pi(\omega)|^2$ from reaction (1) consist for the time being of 14 points: nine from the Novosibirsk experiment¹⁾ and five from Orsay²⁾. There may well be some systematic disagreement between the two experiments, but as the statistical errors are still large we shall, uncritically, consider all the 14 points as good measurements of the same quantity $|F_\pi(\omega)|^2$. Clearly we cannot expect to get information about too many parameters from only 14 points. Also, the errors are large enough not to allow us to distinguish between different phase-shift parametrizations, left-hand singularities or approximations. In fact, all formulae fit the data points about equally well. The present analysis, therefore, only serves to demonstrate how strongly the values of the

resonance parameters may depend on the differences in the formulae, and thus ultimately on the (theoretically equally acceptable) dynamical assumptions and parametrizations. In Table 1 we list nine solutions corresponding to nine of the formulae we have derived in the previous section. All the solutions have χ^2 values corresponding to about a 10% confidence level. In Fig. 2 we have drawn two curves and the experimental points. These curves correspond to the upper and lower bounds of the nine solutions.

Solutions 1 and 2 correspond to the crudest approximations, Eqs. (14b) and (14c), known to give notoriously too small values for both ω_r and Γ_0 . Solution 3, corresponding to Eq. (17), represents a slight improvement, with the energy-dependent width $\Gamma(\omega)$ introduced. In consequence, the mass moves up to its known value³⁾, around 767 MeV. For the remaining solutions, which all represent improvements on solution 3, the mass stays close to 770 MeV.

The tendency of the width to increase with improved formulae, indicated when going from solution 1 to solution 3, is further demonstrated by solutions 4 and 5. Note that the upper 1 standard deviation limit of all the approximate solutions 1-5 is compatible with 122 MeV.

Solutions 5-9^{*)} represent formulae which meet the requirements (i) - (iv) in the Introduction. Of these, solution 5 represents a rather strong simplification. Solution 6 is chosen to behave well close to threshold; in consequence, it fits the data less well above ω_r , and it tries to compensate for this by shifting the mass upwards. We therefore use only the remaining good solutions 7-9 to determine the ρ^0 resonance parameters. For the ρ^0 mass we immediately find

$$M_{\rho^0} \equiv \omega_r = (770 \pm 5) \text{ MeV } ,$$

which includes a 1 MeV theoretical uncertainty. This value, together with the ρ^- mass in our previous analysis³⁾, determines a new mass difference:

$$M_{\rho^0} - M_{\rho^-} = 6 \pm 5 \text{ MeV } .$$

*) Note, however, that solutions 6 and 7 may contain illegal poles in $F_\pi(s)$.

Our previous value $(2.4 \pm 2.1) \text{ MeV}^3$ is still more reliable, since most of the theoretical errors are correlated in the $\pi^- p \rightarrow \pi^- \pi^+ n$ and $\rightarrow \pi^- \pi^0 p$ cases, and do therefore cancel. The width Γ_0 is directly determined only in the good solution 8, whereas in the formulae used for solutions 6, 7, and 9 the width is not an explicit parameter. Note that the errors quoted are not 1 standard deviation values: they merely correspond to taking the free parameters at the limits of their 1 standard deviation confidence intervals.

Taking solution 8 to be the most reliable determination of Γ_0 , we find, in agreement with solutions 6, 7, and 9

$$\Gamma_0 = 122_{-6}^{+7} \text{ MeV} .$$

No information on the radius of interaction R can be obtained from the present data on reaction (1). We have set it equal to zero where it enters (solutions 5 and 8).

We have taken the asymptotic constant for the phase shift above $(1.5 \text{ GeV})^2$, entering through Eq. (28) into solutions 8 and 9 to be

$$\text{const.} = \delta(t_H) \approx \pi .$$

In order to see how sensitive solution 8 was to this value, we have also left it freely adjustable, with the result

$$\text{const.} = 3.1_{-1.0}^{+0.5} .$$

We have also tried to include a CDD zero and a CDD pole into the Omnès formula, corresponding to Eq. (12d). Since the zero then moves out to infinite energy (and so does, a fortiori, the CDD pole) this solution becomes identical to solution 9.

3.2 Discussion of reaction (2)

The situation concerning pion electroproduction, reaction (2), is less clear for both theoretical and experimental reasons. The models which relate $F_\pi(s)$ to the experimentally observed quantities give quite different results^{27, 28)}. In principle, reaction (2) should be as important a source of information on $\pi\pi$ interactions as is the reaction $\pi N \rightarrow \pi\pi N$. On the other hand, the experiments^{29, 30)} are quite difficult, and the statistics are therefore still too small to permit any definite conclusion. We can therefore not yet use pion electroproduction data to distinguish different formulae.

Instead, we shall use the best formula above, Eqs. (12b)-(13), to predict $F_\pi(s)$ for $s < 0$. In Fig. 3 we plot the experimental points^{29, 30)}, and a curve which corresponds to solution 8 in Table 1. The curve is found to be very stable against variations in Γ_0 of the order of ± 6 MeV, and against variations in R between 0 and 4 GeV^{-1} . The slope of the curve at $s = 0$ determines the charge radius of the pion to be

$$\langle r_\pi \rangle = \left[6 \left. \frac{d F_\pi(s)}{ds} \right|_{s=0} \right]^{1/2} = 0.7 \text{ F} .$$

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Table 1
Best fit solutions to reaction (1)

Solution	χ^2	$\langle \chi^2 \rangle$	ω_r (MeV)	Γ_0 (MeV)	Equation	Other parameters
1	17.0	11	759 ± 4	$112 \begin{smallmatrix} + 13 \\ - 11 \end{smallmatrix}$	14b	$(C')^2 = 0.59 \pm 0.04$
2	16.3	11	756 ± 4	$113 \begin{smallmatrix} + 13 \\ - 11 \end{smallmatrix}$	14c	$C^2 = 0.39 \pm 0.03$
3	17.0	11	769 ± 4	$114 \begin{smallmatrix} + 13 \\ - 11 \end{smallmatrix}$	17	$C^2 = 0.60 \begin{smallmatrix} + 0.05 \\ - 0.04 \end{smallmatrix}$
4	17.4	12	763 ± 4	$118 \begin{smallmatrix} + 6 \\ - 5 \end{smallmatrix}$	14a	
5	17.0	12	772 ± 4	$116 \begin{smallmatrix} + 6 \\ - 5 \end{smallmatrix}$	12a, A.2	$R = 0$
6	22.7	12	786 ± 4	$121 \pm 1^*)$	18	$\beta = 222 \pm 9$
7	17.0	10	771 ± 4	$121^*)$	15, A.5 ^{**)}	
8	17.4	12	770 ± 4	$122 \begin{smallmatrix} + 7 \\ - 6 \end{smallmatrix}$	12b, 13	$R = 0, \text{const.} = \pi$
9	17.4	12	771 ± 4	$123 \pm 7^*)$	12c ^{**)}	

*) Derived parameter, error not 1 standard deviation.

**) The formula was slightly rearranged so that one of the free parameters would be ω_r .

APPENDIX

1. We shall first give the explicit form of $F_{\pi}(s)$ obtained when the phase shift δ_1^1 given by Eqs. (10) and (12a) is inserted into Eq. (6). In fact, it is not necessary to integrate Eq. (6) explicitly. Let us use Eqs. (10) and (12) to construct the function

$$b(s) = \frac{1}{q^3 [\cot \delta_1^1(s) - 1]} = \frac{B}{A(s) - iq^3 B} \quad (A.1)$$

with $A = \omega_r^2 - \omega^2$, $B = q_r^3 / (\omega_r \Gamma_0)$. The function $b(s)$ has the phase δ_1^1 for $s > s_0$. It is however not the Omnès D^{-1} function, since $D^{-1}(s)$ must not have poles in the complex s -plane. To get rid of poles or inadequate branch points, we observe that the denominator in Eq. (A.1) has roots q_i in the q plane, located symmetrically with respect to the imaginary axis. It is then necessary only to multiply $a(s)$ by a polynomial $C(s)$ in s (or q^2) with real coefficients and with zeros at q_i^2 . The resulting function is identical to the corresponding Omnès $D^{-1}(s)$ function, as may easily be checked by considering its asymptotic properties. Following the outlined procedure, we obtain

$$D^{-1}(\omega^2) = C \frac{\omega^2 + \omega_1^2}{\omega_r^2 - \omega^2 - i\omega_r \Gamma_0 (q/q_r)^3}$$

$$C = [\omega_r^2 - \omega_r \Gamma_0 (\mu/q_r)^3] / \omega_1^2 \quad (A.2)$$

$$\omega_1^2 = 4(|q_1|^2 + \mu^2)$$

and q_1 is the positive imaginary root of the expression $[\omega_r^2 - \omega^2 - i\omega_r \Gamma_0 (q/q_r)^3]$. The results given in Eq. (A.2) were originally obtained in a slightly different way by Isaev and Meshcheryakov³¹⁾.

2. Here we shall give some details about formulae for $F_{\pi}(s)$ based on N/D with simple parametrizations of $N(\nu)$. Let us define first the function $h(\nu)$ for $\nu < -1$ and $\nu > 0$ as:

$$h(\nu) = \frac{2}{\pi} \left(\frac{\nu}{\nu+1} \right)^{1/2} \ln [|\nu|^{1/2} + |\nu+1|^{1/2}] . \quad (\text{A.3})$$

For $-1 < \nu < 0$ the appropriate formula is

$$h(\nu) = \frac{2}{\pi} \left(\frac{-\nu}{\nu+1} \right)^{1/2} \text{arctg} \left(\frac{\nu+1}{-\nu} \right) . \quad (\text{A.4})$$

The function $D(\nu)$ corresponding to Eq. (15) is

$$D(\nu) = 1 - \frac{\nu+1}{\pi} \int_0^{\infty} d\nu' \left(\frac{\nu'^3}{\nu'+1} \right)^{1/2} \left[\frac{C_1}{\nu'+\nu_1} + \frac{C_2}{\nu'+\nu_2} \right] \frac{1}{(\nu'+1)(\nu'-\nu)} . \quad (\text{A.5a})$$

The integration may be done explicitly, giving for $\nu > 0$

$$\text{Re } D(\nu) = 1 + C_1 H(\nu, \nu_1) + C_2 H(\nu, \nu_2) , \quad (\text{A.5b})$$

$$\text{Im } D(\nu) = - \left(\frac{\nu^3}{\nu+1} \right)^{1/2} \left[\frac{C_1}{\nu+\nu_1} + \frac{C_2}{\nu+\nu_2} \right] , \quad (\text{A.5c})$$

where

$$H(\nu, a) = \frac{2}{\pi} \frac{1}{a-1} - \frac{a(\nu+1)}{(a-1)(a+\nu)} h(-a) + \frac{\nu}{\nu+a} h(\nu) . \quad (\text{A.6})$$

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Figure captions

- Fig. 1 : Pion electroproduction diagram including the $\gamma\pi^+\pi^-$ vertex.
- Fig. 2 : Pion form factor $|F_\pi(s)|^2$ as a function of invariant energy squared, s , in the $\pi^+\pi^-$ system. The data points are from experiments on reaction (1)^{1,2)}; the curves denote the upper and lower bounds for the solutions 1-9 in Table 1.
- Fig. 3 : Prediction of $F_\pi(s)$ for $s < 0$ from the Omnès function, solution 8.

Fig. 1

Fig. 2

Fig. 3