THE CONVOLUTION OF FOURIER TRANSFORMS
AND ITS APPLICATION TO THE DECOMPOSITION OF
THE MOMENTUM OPERATOR ON THE HOMOGENEOUS LORENTZ GROUP

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A B S T R A C T

We study the analytic structure of that convolution kernel which permits us to find the Fourier transforms of a product of two bicovariant functions on $SL(2, \mathbb{C})$ from the Fourier transforms of its factors. We show that the poles of this kernel are connected with tensor operators, in particular that four pairs of poles give rise to the four types of generalized Dirac matrices. As an application we decompose the momentum operator of the inhomogeneous Lorentz group into a direct integral of generalized Dirac matrices for the homogeneous Lorentz group.

INTRODUCTION

In this article we resume our investigations on the problem of determining the Fourier transform of a product

$$Z(a) = x(a) y(a)$$

of two functions on the group $SL(2, \mathbb{C})$ if the Fourier transforms of both factors are known. This issue has already been studied in a previous article ¹⁾ [later referred to as (C)] where we saw that it is connected with the problem of reducing the tensor product of two representations of the principal series into irreducible components. It resulted that the two Fourier transforms have to be folded by means of a kernel which can be expressed as the product of two of those kernels that were used by Naimark ²⁾ to reduce such tensor product representations. In the special case where all functions considered are bi-invariant on double cosets of $SU(2)$ we were able to replace the general convolution kernel by a simple elementary function, which we denoted $G(\varphi, \varphi', \varphi'')$.

In Chapter 1 of this article we investigate the case where all functions to be Fourier transformed are bicovariant of an arbitrary type (instead of bi-invariant) on double cosets of $SU(2)$. The notion of bicovariance was introduced in our lectures ³⁾, Section 4.7. This work will be referred to as (L). The problem to determine the Fourier transforms of products of bicovariant functions leads us to a function $G(\chi, \chi', \chi'')$ which reduces to the function $G(\varphi, \varphi', \varphi'')$ in the simplest possible case and can in all cases be expressed as a finite sum of logarithmic derivatives of the Γ function. In addition we define Clebsch-Gordan coefficients for $SL(2, \mathbb{C})$ and derive a relation between the function G and these Clebsch-Gordan coefficients.

In Chapter 2 we reduce a unitary irreducible representation of the group $SL(2, \mathbb{C}) \times T_4$ for positive mass squared and arbitrary spin into a direct integral of unitary representations of the principal series of $SL(2, \mathbb{C})$ ⁴⁾. This is easily done by submitting the covariant wave functions which span the representation space of $SL(2, \mathbb{C}) \times T_4$ to a Fourier transformation on $SL(2, \mathbb{C})$. The problem which is of interest for us is the simultaneous decomposition of a given unbounded operator on the representation space of $SL(2, \mathbb{C}) \times T_4$ like the momentum operator (infinitesimal translation operator). This problem is related with the decomposition of the S matrix in the t channel in the theory of scattering of elementary particles. The latter problem leads to Regge and Toller poles. In the case of the momentum operator it is the poles of the function $G(\chi, \chi', \chi'')$ which play the corresponding role.

At first sight this decomposition of the momentum operator might seem to cause a paradoxical situation. If the spin of the representation of $SL(2, \mathbb{C}) \times T_4$ to be reduced is zero, it is easy to see that only representations with $m=0$, φ real (for the notations see Section 1.1) arise in the reduction. The momentum operator is a four-vector and should correspondingly go into a direct integral of four-vector operators on $SL(2, \mathbb{C})$. From the work of Naimark⁵⁾ it is known, however, that only four four-vector operators exist, each of which connects a pair of representations of $SL(2, \mathbb{C})$ with $m'-m=\pm 2$, $\varphi'-\varphi=0$, respectively $m'-m=0$, $\varphi'-\varphi=\pm 2i$. Such pairs of representations are not involved in the direct integral. The solution of this paradox is typical for the decomposition of unbounded operators. We note that the four four-vector operators are caused by four pairs of poles of the function G which pinch the integration contour.

In order to understand the poles and zeros of the function G we add a Chapter 3 on trilinear invariant functionals on $SL(2, \mathbb{C})$. It allows us to identify the residues of the poles computed in Chapter 2 with matrix elements of the four-vector operators of $SL(2, \mathbb{C})$. In addition we trace the zeros of G back to the existence of invariant subspaces of the representation spaces of $SL(2, \mathbb{C})$ which annihilate the trilinear functional.

1. THE CONVOLUTION KERNEL FOR BICOVARIANT FUNCTIONS

1.1 Definitions and notations

The group $SL(2, \mathbb{C})$ consists of complex unimodular 2×2 matrices. Matrices of the triangular shape

$$k = \begin{pmatrix} \bar{\lambda}^{-1} & \mu \\ 0 & \lambda \end{pmatrix}$$

form a subgroup K of $SL(2, \mathbb{C})$. The one-dimensional unitary representations of K can be labelled by the symbol $\chi = (m, \rho)$ where m is an integer and ρ is real. These representations are obtained by the mapping

$$k \rightarrow \left(\frac{\lambda}{|\lambda|} \right)^{-m} |\lambda|^{i\rho}$$

The right cosets of K in $SL(2, \mathbb{C})$ can be parameterized by a complex number z , $z \neq \infty$ included. An element $a \in SL(2, \mathbb{C})$ is in the coset z if

$$a = k \xi, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \bar{\lambda}^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

and in the coset $z = \infty$ if $a_{22} = 0$.

We define a measure on $SL(2, \mathbb{C})/K$ as

$$z = x + iy, \quad Dz = dx dy, \quad -\infty < x, y < +\infty$$

and consider the Hilbert space $L^2(Z)$ of measurable functions $f(z)$ whose scalar product norm

$$\|f\| = \left\{ \int |f(z)|^2 Dz \right\}^{\frac{1}{2}}$$

is finite. The unitary irreducible representations of the principal series of $SL(2, \mathbb{C})$ realized in the space $L^2(Z)$ are obtained by induction from the unitary one-dimensional representations of K .

We write

$$T_a^\chi f(z) = \lambda(z, a)^{-\frac{1}{2}m + \frac{1}{2}p - 1} \overline{\lambda(z, a)^{\frac{1}{2}m + \frac{1}{2}p - 1}} f(z_a) \quad (1.1)$$

where

$$\begin{aligned} \zeta a &= k \zeta a, \quad \zeta a = \begin{pmatrix} 1 & 0 \\ z_a & 1 \end{pmatrix} \\ k &= \begin{pmatrix} \lambda(z, a)^{-1} & \mu \\ 0 & \lambda(z, a) \end{pmatrix} \\ z_a &= (a_{11}z + a_{21}) \lambda(z, a)^{-1} \end{aligned} \quad (1.2)$$

In the sequel we denote the multiplier in front of $f(z_a)$, (1.1), by $\alpha^\chi(z, a)$.

The subgroup of unitary matrices is denoted $SU(2)$, its elements by u . Special elements of $SU(2)$ are the matrices

$$\gamma = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \quad (1.3)$$

Measurable functions $\varphi(u)$ on $SU(2)$ which obey the covariance constraint

$$\varphi(\gamma u) = e^{im\omega} \varphi(u) \quad (1.4)$$

and have finite scalar product norm

$$\|\varphi\| = \left\{ \int |\varphi(u)|^2 d\mu(u) \right\}^{\frac{1}{2}}, \quad \int d\mu(u) = 1$$

span a Hilbert space $L_m^2(U)$. $d\mu(u)$ is the invariant measure on $SU(2)$. The matrices γ (1.3) form a subgroup $U(1)$ of $SU(2)$.

The right cosets of $U(1)$ in $SU(2)$ can be mapped one-to-one on the right cosets $SL(2, \mathbb{C})/K$. This leads us to an alternative realization of the principal series in the space $L^2_m(U)$ by

$$T_a^x \varphi(u) = \alpha^x(u, a) \varphi(u_a) \quad (1.5)$$

where

$$\begin{aligned} u_a &= k u_a \\ k &= \begin{pmatrix} \lambda(u, a)^{-1} & \mu \\ 0 & \lambda(u, a) \end{pmatrix} \\ \alpha^x(u, a) &= \lambda(u, a)^{-\frac{1}{2}m + \frac{i}{2}\rho - 1} \frac{\lambda(u, a)^{\frac{1}{2}m + \frac{i}{2}\rho - 1}}{\lambda(u, a)} \end{aligned} \quad (1.6)$$

Explicit formulae for the connection between z and u and the point transformation mapping $L^2(Z)$ isometrically on $L^2_m(U)$ can be found in (L), (3.14) to (3.16).

In $L^2_m(U)$ an orthonormal basis $\{\varphi_q^j(u)\}$, the "canonical basis", can be defined by

$$\begin{aligned} \varphi_q^j(u) &= (2j+1)^{\frac{1}{2}} D_{\frac{1}{2}m, q}^j(u) \\ j &= \frac{1}{2}|m|, \frac{1}{2}|m|+1, \dots \\ -j &\leq q \leq j \end{aligned} \quad (1.7)$$

where we use Edmonds' notation ⁶⁾ $D_{q_1 q_2}^j(u)$ for matrix elements of a unitary representation to "spin j " of $SU(2)$. The corresponding basis $\{f_q^j(z)\}$ in $L^2(Z)$ is given explicitly in (L), (3.83), (3.84). With this basis we define the matrix elements of the operators T_a^x by

$$D_{j_1 q_1 j_2 q_2}^x(a) = \int \overline{\varphi_{q_1}^{j_1}(u)} T_a^x \varphi_{q_2}^{j_2}(u) d\mu(u) \quad (1.8)$$

They satisfy the covariance constraint

$$D_{j_1 q_1 j_2 q_2}^{\chi}(a) = \sum_{q_3 q_4} D_{q_1 q_3}^{j_1}(u_1) D_{q_4 q_2}^{j_2}(u_2) D_{j_1 q_3 j_2 q_4}^{\chi}(\bar{u}_1^{-1} a \bar{u}_2^{-1}) \quad (1.9)$$

on double cosets of $SU(2)$. Consequently they are expressible by the functions

$$d_{j_1 j_2 q}^{\chi}(\eta) = D_{j_1 q j_2 q}^{\chi}(d) \quad (1.10)$$

where d denotes the matrix

$$d = \begin{pmatrix} e^{\frac{1}{2}\eta} & 0 \\ 0 & e^{-\frac{1}{2}\eta} \end{pmatrix}, \quad \eta \geq 0$$

Two representations $\chi = (m, \rho)$ and $-\chi = (-m, -\rho)$ are equivalent. This fact reflects itself in the relation

$$d_{j_1 j_2 q}^{\chi}(\eta) = (\beta^{j_1})^{-1} \beta^{j_2} d_{j_1 j_2 q}^{-\chi}(\eta) \quad (1.11)$$

with

$$\beta^j = \beta^j(\chi) = \frac{\Gamma(j+1+\frac{1}{2}\rho)}{\Gamma(j+1-\frac{1}{2}\rho)} \quad (1.12)$$

The proportionality factor in (1.11) is a pure phase (on the principal series, other representations occur first in Section 1.4). For the derivation of (1.11) see (L), (3.94), (3.96). The functions $d_{j_1 j_2 q}^{\chi}(\eta)$ are given explicitly in Section 1.4.

1.2 The convolution kernel

Let χ_1, χ_2 denote two representations of the principal series. The Hilbert space of measurable functions $f(z_1, z_2)$ of two complex variables z_1, z_2 with finite scalar product norm

$$\|f\| = \left\{ \int |f(z_1, z_2)|^2 D z_1 D z_2 \right\}^{\frac{1}{2}}$$

is denoted $L^2(Z_1 \times Z_2)$. This space carries the tensor product representation $\chi_1 \times \chi_2$ which is defined by

$$T_a^{\chi_1 \times \chi_2} f(z_1, z_2) = \chi^{\chi_1}(z_1, a) \chi^{\chi_2}(z_2, a) f((z_1)_a, (z_2)_a) \quad (1.13)$$

with the notations (1.2). This tensor product can be decomposed into irreducible components in a fashion described first by Naimark²⁾.

We introduce the direct integral Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \int^{\oplus} L^2(Z, \chi) d\chi$$

where $d\chi$ is the Plancherel measure of $SL(2, \mathbb{C})$, see (I), Section 4.2. Each Hilbert space $L^2(Z, \chi)$ consists of functions $f(z, \chi)$ which are measurable in z and have finite scalar product norm

$$\|f\|_{\chi} = \left\{ \int |f(z, \chi)|^2 D z \right\}^{\frac{1}{2}}$$

and carries the representation χ of the principal series. Since the representations χ and $-\chi$ are equivalent, strictly speaking each representation χ occurs twice in $\mathcal{H}$.

The spaces $L^2(Z_1 \times Z_2)$ and $\mathcal{H}$ can be mapped isometrically on each other by the integral transformations

$$f(z, \chi) = \int N(z, z_1, z_2 | -\chi, \chi_1, \chi_2) f(z_1, z_2) D z_1 D z_2 \quad (1.14)$$

$$f(z_1, z_2) = \int N(z, z_1, z_2 | \chi, -\chi_1, -\chi_2) f(z, \chi) D z d\chi \quad (1.15)$$

which converge in the sense of the norm of the respective image spaces. Naimark's kernel N is defined as

$$N(z_1, z_2, z_3 | \chi_1, \chi_2, \chi_3) = \begin{cases} \frac{1}{2}(2\pi)^{-2} (z_1 - z_2)^{A_3} (\bar{z}_1 - \bar{z}_2)^{B_3} \\ \times (z_2 - z_3)^{A_1} (\bar{z}_2 - \bar{z}_3)^{B_1} (z_3 - z_1)^{A_2} (\bar{z}_3 - \bar{z}_1)^{B_2} \\ \text{if } m_1 + m_2 + m_3 \text{ is even} \\ 0 \text{ if } m_1 + m_2 + m_3 \text{ is odd} \end{cases} \quad (1.16)$$

With the shorthand notation

$$n_i = -\frac{1}{2}m_i + \frac{i}{2}\rho_i, \quad n'_i = +\frac{1}{2}m_i + \frac{i}{2}\rho_i \quad (1.17)$$

we can give the powers A_i in the form

$$\begin{aligned} A_1 &= -\frac{1}{2}(-n_1 + n_2 + n_3 + 1) \\ A_2 &= -\frac{1}{2}(n_1 - n_2 + n_3 + 1) \\ A_3 &= -\frac{1}{2}(n_1 + n_2 - n_3 + 1) \end{aligned} \quad (1.18)$$

and the B_i by replacing the n_i by n'_i . The kernels in (1.14), (1.15) are complex conjugates of each other. The correct transformation behaviour of the function $f(z, \chi)$ defined by (1.14) is guaranteed by the isotropy relation

$$\begin{aligned} N(z_1, z_2, z_3 | \chi_1, \chi_2, \chi_3) &= \\ &= \left[\prod_{i=1,2,3} \alpha^{-\chi_i} (z_i, \alpha) \right] N((z_1)_\alpha, (z_2)_\alpha, (z_3)_\alpha | \chi_1, \chi_2, \chi_3) \end{aligned} \quad (1.19)$$

which is valid for all $\alpha \in \text{SL}(2, \mathbb{C})$. Summarizing (1.14), (1.15) we get the orthogonality relation

$$\int d\lambda D\mathbf{z} N(\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2 | -\lambda, \lambda_1, \lambda_2) N(\mathbf{z}, \mathbf{z}_1', \mathbf{z}_2' | \lambda, -\lambda_1, -\lambda_2) \\ = \delta(\mathbf{z}_1 - \mathbf{z}_1') \delta(\mathbf{z}_2 - \mathbf{z}_2') \quad (1.20)$$

We define Clebsch-Gordan coefficients for $SL(2, \mathbb{C})$ by

$$\sum_{jQ} (\lambda j Q | \lambda_1 j_1 q_1, \lambda_2 j_2 q_2) f_Q^j(\mathbf{z}) = \\ = \int N(\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2 | -\lambda, \lambda_1, \lambda_2) f_{q_1}^{j_1}(\mathbf{z}_1) f_{q_2}^{j_2}(\mathbf{z}_2) D\mathbf{z}_1 D\mathbf{z}_2 \quad (1.21)$$

and

$$\overline{(\lambda_1 j_1 q_1, \lambda_2 j_2 q_2 | \lambda j Q)} = (\lambda j Q | \lambda_1 j_1 q_1, \lambda_2 j_2 q_2) \quad (1.22)$$

where the functions $f_Q^j(\mathbf{z})$ are elements of the canonical basis (Section 1.1). From the explicit form of these functions, (I), (3.83), (3.84), we deduce

$$\overline{f_Q^j(\mathbf{z})}^\chi = (-1)^{\frac{1}{2}m-Q} f_{-Q}^j(\mathbf{z})^{-\chi} \quad (1.23)$$

and have therefore instead of (1.22)

$$(\lambda_1 j_1 q_1, \lambda_2 j_2 q_2 | \lambda j Q) = \\ = (-1)^{\frac{1}{2}(m-m_1-m_2)-(Q-q_1-q_2)} (-\lambda j -Q | -\lambda_1 j_1 -q_1, -\lambda_2 j_2 -q_2) \quad (1.24)$$

We recall that the Clebsch-Gordan coefficients of $SU(2)$ obey [Edmonds ⁶⁾, Eq. (4.3.1)]

$$D_{q_1 q_3}^{j_1}(u) D_{q_2 q_4}^{j_2}(u) =$$

$$= \sum_{jQ, Q_2} (j_1 q_1, j_2 q_2 | jQ)(jQ | j_1 q_3, j_2 q_4) D_{Q_1 Q_2}^j(u) \quad (1.25)$$

which implies

$$(j_1 q_1, j_2 q_2 | jQ)(jQ | j_1 q_3, j_2 q_4)$$

$$= (2j+1) \int d\mu(u) \overline{D_{Q_1 Q_2}^j(u)} D_{q_1 q_3}^{j_1}(u) D_{q_2 q_4}^{j_2}(u)$$

$$(1.26)$$

and the orthogonality relation

$$\sum_{jQ} (j_1 q_1, j_2 q_2 | jQ)(jQ | j_1 q_3, j_2 q_4) =$$

$$= \delta_{q_1 q_3} \delta_{q_2 q_4} \quad (1.27)$$

if we set $u=e$. Analogous relations hold for the Clebsch-Gordan coefficients of $SL(2, \mathbb{C})$.

Operating with T_a^{χ} on the left-hand side of (1.21), multiplying with the conjugate of (1.21), integrating over z and χ , and taking into account (1.8) and (1.20), we get

$$D_{j_1 q_1 j_1' q_1'}^{\chi_1}(a) D_{j_2 q_2 j_2' q_2'}^{\chi_2}(a) =$$

$$= \int d\chi \sum_{jQ j'Q'} (\chi, j_1 q_1, \chi_2 j_2 q_2 | \chi jQ)(\chi j'Q' | \chi, j_1' q_1', \chi_2 j_2' q_2') \times D_{jQ j'Q'}^{\chi}(a) \quad (1.28)$$

With the help of (1.9), (1.10) and (1.26), we simplify (1.28) to

$$\int d\chi (\chi, j_1 q_1, \chi_2 j_2 q_2 | \chi jQ)(\chi j'Q' | \chi, j_1' q_1', \chi_2 j_2' q_2') d_{jQ j'Q'}^{\chi}(\eta)$$

$$= \sum_{\substack{q_1'' q_2'' \\ (Q'' \text{ fixed})}} (j_1 q_1, j_2 q_2 | jQ)(jQ'' | j_1 q_1'', j_2 q_2'') (j_1' q_1'', j_2' q_2'' | j'Q'') (j'Q' | j_1' q_1', j_2' q_2') \times d_{j_1 j_1' q_1''}^{\chi_1}(\eta) d_{j_2 j_2' q_2''}^{\chi_2}(\eta) \quad (1.29)$$

In connection with (1.11) we define the symmetric part of a function $F_{jj'}(\chi)$ by

$$S_{\chi} F_{jj'}(\chi) = \frac{1}{2} \{ F_{jj'}(\chi) + \beta^j(\chi) (\beta^{j'}(\chi))^{-1} F_{jj'}(-\chi) \} \quad (1.30)$$

such that

$$\int d\chi d_{jj'q}^{\chi}(\eta) F_{jj'}(\chi) = \int d\chi d_{jj'q}^{\chi}(\eta) S_{\chi} F_{jj'}(\chi)$$

The Plancherel theorem for a function $F_{jj'}(\chi)$ and its symmetric part (1.30) looks like (I), (4.82), (4.84), (4.86)

$$S_{\chi} F_{jj'}(\chi) = [8\pi(2j+1)(2j'+1)]^{-1} \times \int_0^{\infty} d\eta \, \eta^2 \, d_{jj'q}^{\bar{\chi}}(\eta) \int d\chi F_{jj'}(\chi) d_{jj'q}^{\chi}(\eta) \quad (1.31)$$

where we denote $\bar{\chi} = (m, -p)$ if $\chi = (m, p)$. With the notation

$$G(\chi_1, \chi_2, \chi_3 | j_1 j'_1 q_1, j_2 j'_2 q_2, j_3 j'_3 q_3) = \int_0^{\infty} d\eta \, \eta^2 \, d_{j_1 j'_1 q_1}^{\chi_1}(\eta) d_{j_2 j'_2 q_2}^{\chi_2}(\eta) d_{j_3 j'_3 q_3}^{\chi_3}(\eta) \quad (1.32)$$

we finally obtain from (1.29)

$$\begin{aligned} & S_{\chi} (\chi_1 j_1 q_1, \chi_2 j_2 q_2 | \chi \gamma Q) (\chi \gamma' Q' | \chi_1 j'_1 q'_1, \chi_2 j'_2 q'_2) \\ &= [8\pi(2\gamma+1)(2\gamma'+1)]^{-1} \\ & \times \sum_{\substack{q''_1 q''_2 \\ (Q'' \text{ fixed})}} (\gamma_1 q_1, \gamma_2 q_2 | \gamma Q) (\gamma Q' | \gamma_1 q'_1, \gamma_2 q'_2) (\gamma_1 q''_1, \gamma_2 q''_2 | \gamma' Q'') (\gamma' Q' | \gamma_1 q'_1, \gamma_2 q'_2) \\ & \times G(\bar{\chi}, \chi_1, \chi_2 | \gamma \gamma' Q'', j_1 j'_1 q''_1, j_2 j'_2 q''_2) \end{aligned} \quad (1.33)$$

The function G defined by (1.32) and analytically continued in $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$ is the quantity of main interest in the following Sections. This function solves also the problem to give the Fourier transform of a product of two bicovariant functions if the Fourier transforms of the factors are known.

In fact, two arrays of functions

$$x_{j_1 q_1 j_2 q_2}(a), \quad -j_1 \leq q_1 \leq j_1, \quad -j_2 \leq q_2 \leq j_2$$

$$y_{j'_1 q'_1 j'_2 q'_2}(a), \quad -j'_1 \leq q'_1 \leq j'_1, \quad -j'_2 \leq q'_2 \leq j'_2$$

of C_c^∞ are given. They are bicovariant, i.e., they are to exhibit the property (1.9). With the Clebsch-Gordan coefficients of $SU(2)$ we define new bicovariant functions by

$$\begin{aligned} z_{j_1 Q_1 j_2 Q_2}(a) &= \\ &= \sum_{q_1 q_2 q'_1 q'_2} (j_1 Q_1 | j_1 q_1 j'_1 q'_1) (j_2 Q_2 | j_2 q_2 j'_2 q'_2) \\ &\quad \times x_{j_1 q_1 j_2 q_2}(a) y_{j'_1 q'_1 j'_2 q'_2}(a) \end{aligned} \quad (1.34)$$

We define the Fourier transforms by (I), (4.84)

$$\begin{aligned} K_{j_1 j_2 Q}(\bar{x})_x &= [4\pi (2j_1+1)(2j_2+1)]^{-1} \\ &\quad \times \int_0^\infty d\eta \sinh^2 \eta \, d_{j_1 j_2 Q}^{\bar{x}}(\eta) x_{j_1 q_1 j_2 q_2}(d(\eta)) \end{aligned} \quad (1.35)$$

with $d(\eta)$ as in (1.10). The solution of our issue can be found by direct application of (I), (4.82), (4.84), (4.86) in the form

$$\begin{aligned} K_{j_1 j_2 Q}(\bar{x})_z &= [16\pi (2j_1+1)(2j_2+1)]^{-1} \sum_{\substack{q q' \\ (Q \text{ fixed})}} (j_1 Q | j_1 q j'_1 q') \\ &\quad \times (j_2 q_2 j'_2 q' | j_2 Q) \int d\chi_1 d\chi_2 G(\bar{x}, \chi_1, \chi_2 | j_1 j_2 Q, j_1 j_2 q, j'_1 j'_2 q') \\ &\quad \times K_{j_1 j_2 Q}(\bar{x}_1)_x K_{j'_1 j'_2 Q}(\bar{x}_2)_y \end{aligned} \quad (1.36)$$

We see that the two Fourier transforms are folded with the kernel G . The result (1.36) can also be obtained if we use the general basis independent convolution kernel derived in (C), Section 4, insert the Fourier transforms of the bicovariant functions in Gelfand's integral operator form, (L), (4.85), and take into account the orthogonality relation, (L), (4.87).

1.3 The computation of the function G

We set out now to evaluate the integral (1.32) in terms of elementary functions. The integral (1.8) can be seen to lead to [L], (4.59) and its derivation]

$$\begin{aligned}
 d_{j_1 j_2 q}^{\chi}(\eta) = & \left\{ (2j_1+1)(2j_2+1) (j_1+\frac{1}{2}m)! (j_1-\frac{1}{2}m)! (j_2+\frac{1}{2}m)! (j_2-\frac{1}{2}m)! \right. \\
 & \times (j_1+q)! (j_1-q)! (j_2+q)! (j_2-q)! \left. \right\}^{\frac{1}{2}} \\
 & \times (2\sinh \eta)^{-j_1-j_2-1} \sum_{\nu, \mu} C_{\nu \mu} \frac{2\sinh(\frac{i}{2}\eta - j_2 + \nu)\eta}{\frac{i}{2}\eta - j_2 + \nu} e^{(j_1+\mu)\eta}
 \end{aligned}
 \tag{1.37}$$

where the $C_{\nu \mu}$ are certain rational numbers. They are defined by

$$\begin{aligned}
 C_{\nu \mu}(j_1 j_2 q \frac{1}{2}m) = & (-1)^{\mu} \sum_{n_1, n_2} [n_1! (j_1+q-n_1)! (n_1-q+\frac{1}{2}m)! \\
 & \times (j_1-n_1-\frac{1}{2}m)! n_2! (j_2+q-n_2)! (n_2-q+\frac{1}{2}m)! \\
 & \times (j_2-n_2-\frac{1}{2}m)!]^{-1} \begin{pmatrix} j_1+j_2-n_1-n_2+q-\frac{1}{2}m \\ -n_1+\frac{1}{2}(\nu+\mu+q-\frac{1}{2}m) \end{pmatrix} \\
 & \times \begin{pmatrix} n_1+n_2-q+\frac{1}{2}m \\ n_2-\frac{1}{2}(\nu-\mu+q-\frac{1}{2}m) \end{pmatrix}
 \end{aligned}
 \tag{1.38}$$

This definition (1.38) is supplemented by the requirements that the support of the coefficients $C_{\nu\mu}$ (i.e., those numbers ν, μ for which $C_{\nu\mu} \neq 0$) is restricted by

$$\nu + \mu = \max(\nu + \mu) - 2s_1 = \min(\nu + \mu) + 2t_1$$

$$\nu - \mu = \max(\nu - \mu) - 2s_2 = \min(\nu - \mu) + 2t_2$$

where $s_{1,2}$ and $t_{1,2}$ are integers and a simple computation gives for the extrema

$$\max(\mu - \varepsilon \nu) = (1 - \varepsilon)j_2 + 2j_1 - |q + \frac{1}{2}\varepsilon m|$$

$$\min(\mu - \varepsilon \nu) = -(1 + \varepsilon)j_2 + |q + \frac{1}{2}\varepsilon m|$$

$$\varepsilon = \pm 1$$

The computation of the coefficients $C_{\nu\mu}$ simplifies considerably if we take certain symmetries into account which can be easily read from the definition (1.38). We find the following three identities

$$C_{\nu\mu}(j_1 j_2 q \frac{1}{2}m) = C_{\nu\mu}(j_1 j_2 \frac{1}{2}mq)$$

$$C_{\nu\mu}(j_1 j_2 q \frac{1}{2}m) = (-1)^{2j_1} C_{\nu, 2j_1 - \mu}(j_1 j_2 q - \frac{1}{2}m)$$

$$C_{\nu\mu}(j_1 j_2 q \frac{1}{2}m) = (-1)^{j_1 + j_2} C_{j_1 + j_2 - \nu, j_1 + j_2 - \mu}(j_1 j_2 q \frac{1}{2}m)$$

(1.39)

In addition, we know, (L), Section 4.5, that

$$\lim_{\eta \rightarrow 0} \eta^{-|j_1 - j_2|} d_{j_1 j_2 q}^X(\eta) < \infty$$

This fact can be exploited to derive "sum rules" for the numbers $C_{\nu\mu}$. Indeed, with the definition

$$\text{sh} \eta e^{\beta \eta} = \alpha \eta \sum_{n=0}^{\infty} \gamma_n(\alpha, \beta) \eta^n$$

we have from (1.37)

$$\sum_{\nu\mu} C_{\nu\mu} \gamma_n(\frac{1}{2}q - j_2 + \nu, j_1 - \mu) = 0 \quad \text{for } 0 \leq n \leq 2\max(j_1, j_2) - 1$$

Besides the functions (1.37) we need the functions of "the second kind" defined by

$$e_{j_1 j_2 q}^x(\eta) = \left\{ \dots \right\}^{\frac{1}{2}} (2 \sinh \eta)^{j_1 - j_2 - 1} \times \sum_{\nu, \mu} C_{\nu \mu} \left(\frac{1}{2} \eta - j_2 + \nu \right)^{-1} \exp \eta \left(\frac{1}{2} \eta + j_1 - j_2 + \nu - \mu \right) \quad (1.40)$$

where the curly bracket is the same as in (1.37). If we make use of the combination

$$C_{\nu \mu}(j_2 j_1, q - \frac{1}{2} m) = (-1)^{j_1 - j_2} C_{j_1 + j_2 - \nu, j_1 - j_2 + \mu}(j_1 j_2, q \frac{1}{2} m) \quad (1.41)$$

of the second and third identity (1.39), we recognize the following relation between functions of the first and second kind

$$d_{j_1 j_2 q}^x(\eta) = e_{j_1 j_2 q}^x(\eta) + (-1)^{j_1 - j_2} e_{j_2 j_1 q}^{-x}(\eta) \quad (1.42)$$

The main problem in the analytic evaluation of the integral (1.32) is due to the following complication. The e functions (1.40) have a pole at $\eta = 0$ which cancels in (1.42). This pole makes it impossible to use (1.42) and express the function G as a sum of integrals over e functions. In order to make the regular behaviour of the d functions at $\eta = 0$ explicit, we are, on the other hand, forced to use infinite hypergeometric series which can be integrated termwise but leave us the problem to sum up infinite series after the integration. We circumvent these difficulties by choosing the first alternative and handling the divergent integrals with an analytic regularization technique.

We define the function (χ_i on the principal series)

$$F(\alpha | \chi_1 \chi_2 \chi_3 | j_1 j_1' q_1, j_2 j_2' q_2, j_3 j_3' q_3) = \int_0^\infty d\eta \sinh \eta (1 - e^{-2\eta})^\alpha e_{j_1 j_1' q_1}^{\chi_1}(\eta) e_{j_2 j_2' q_2}^{\chi_2}(\eta) e_{j_3 j_3' q_3}^{\chi_3}(\eta) \quad (1.43)$$

which is holomorphic in α for $\text{Re } \alpha > J+J'$ where

$$\begin{aligned} J &= \sum_{i=1,2,3} j_i \\ J' &= \sum_{i=1,2,3} j'_i \end{aligned} \quad (1.44)$$

Using the identity (1.42), we have

$$\begin{aligned} G(x_1, x_2, x_3 | j_1 j'_1 q_1, j_2 j'_2 q_2, j_3 j'_3 q_3) = \\ = \lim_{\alpha \rightarrow 0} \left\{ F(\alpha | x_1 x_2 x_3 | j_1 j'_1 q_1, \dots) \right. \\ \quad + (-1)^{j_1 - j'_1} F(\alpha | -x_1 x_2 x_3 | j'_1 j_1 q_1, \dots) \\ \quad \left. + \dots \right\} \end{aligned} \quad (1.45)$$

This trick works in practice only because the function $F(\alpha)$ possesses poles in α of first order only at the positions

$$\alpha = J + J' - n, \quad n = 0, 1, 2, \dots$$

in particular one such pole at $\alpha = 0$. A simple cancellation between the residues of these poles gives us the desired integral.

With (1.40) and the curly brackets as in (1.37) we get

$$\begin{aligned} F(\alpha) = \frac{1}{4} \{1\}^{\frac{1}{2}} \{2\}^{\frac{1}{2}} \{3\}^{\frac{1}{2}} \sum_{\nu_i \mu_i} C_{\nu_i \mu_i}^{(1)} C_{\nu_i \mu_i}^{(2)} C_{\nu_i \mu_i}^{(3)} \\ \times \left[\left(\frac{i}{2} p_1 + \nu_1 - j'_1 \right) \left(\frac{i}{2} p_2 + \nu_2 - j'_2 \right) \left(\frac{i}{2} p_3 + \nu_3 - j'_3 \right) \right]^{-1} \\ \times \int_0^\infty d\eta (2\cosh \eta)^{-J-J'-1} (1-e^{-2\eta})^\alpha \exp \eta \left(\frac{i}{2} R + J - J' + N - M \right) \end{aligned} \quad (1.46)$$

where the notations (1.44) and

$$R = \sum_i \rho_i, \quad N = \sum_i \nu_i, \quad M = \sum_i \mu_i \quad (1.47)$$

are used. The integral in (1.46) is elementary and gives a beta function

$$\frac{1}{2} \frac{\Gamma(\alpha - \gamma - \gamma') \Gamma(\frac{1}{2} - \frac{\epsilon}{4} R + \gamma' + \frac{1}{2}(M - N))}{\Gamma(\alpha - \gamma + \frac{1}{2} - \frac{\epsilon}{4} R + \frac{1}{2}(M - N))}$$

Inserting this result in (1.45) and using the identity (1.41) yields

$$G(x_1, x_2, x_3) = \lim_{\alpha \rightarrow 0} \Gamma(\alpha - \gamma - \gamma') \frac{1}{8} \{1\}^{\frac{1}{2}} \{2\}^{\frac{1}{2}} \{3\}^{\frac{1}{2}} \\ \times \sum_{\nu_i \mu_i} C_{\nu_1 \mu_1}^{(1)} C_{\nu_2 \mu_2}^{(2)} C_{\nu_3 \mu_3}^{(3)} \left[\left(\frac{\epsilon}{2} \rho_1 + \nu_1 - \gamma'_1 \right) \left(\frac{\epsilon}{2} \rho_2 + \nu_2 - \gamma'_2 \right) \left(\frac{\epsilon}{2} \rho_3 + \nu_3 - \gamma'_3 \right) \right]^{-1} \\ \times \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 \frac{\Gamma(A(\epsilon))}{\Gamma(\alpha - \gamma - \gamma' + A(\epsilon))} \quad (1.48)$$

where the ϵ_i are sign factors taking both values ± 1 , and

$$A(\epsilon) = \frac{1}{2} - \frac{\epsilon}{4} R(\epsilon) + \frac{1}{2}(\gamma' + \gamma'(\epsilon)) + \frac{1}{2}(M - N(\epsilon)) \\ R(\epsilon) = \sum_i \epsilon_i \rho_i, \quad \gamma'(\epsilon) = \sum_i \epsilon_i \gamma'_i, \quad N(\epsilon) = \sum_i \epsilon_i \nu_i \quad (1.49)$$

With the expansion around $z = -n$

$$\frac{\Gamma(x)}{\Gamma(x+z)} = \left[\prod_{k=1}^n (x-k) \right] \left[1 - (z+n) \frac{\Gamma'(x-n)}{\Gamma(x-n)} + \dots \right]$$

and the familiar notation

$$\Psi(z) = \frac{d}{dz} \ln \Gamma(z)$$

we get finally

$$\begin{aligned}
 G(x_1, x_2, x_3) &= (-1)^{J+J'+1} [(J+J')!]^{-1} \frac{1}{8} \{1\}^{\frac{1}{2}} \{2\}^{\frac{1}{2}} \{3\}^{\frac{1}{2}} \\
 &\times \sum_{\nu_i \mu_i} C_{\nu_1 \mu_1}^{(1)} C_{\nu_2 \mu_2}^{(2)} C_{\nu_3 \mu_3}^{(3)} \\
 &\times \left[\left(\frac{\epsilon}{2} p_1 + \nu_1 - j_1' \right) \left(\frac{\epsilon}{2} p_2 + \nu_2 - j_2' \right) \left(\frac{\epsilon}{2} p_3 + \nu_3 - j_3' \right) \right]^{-1} \\
 &\times \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 \left[\prod_{k=1}^{J+J'} (A(\epsilon) - k) \right] \Psi(A(\epsilon) - J - J')
 \end{aligned} \tag{1.50}$$

We still have to show that the sum

$$\begin{aligned}
 &\sum_{\nu_i \mu_i} C_{\nu_1 \mu_1}^{(1)} C_{\nu_2 \mu_2}^{(2)} C_{\nu_3 \mu_3}^{(3)} \left[\left(\frac{\epsilon}{2} p_1 + \nu_1 - j_1' \right) \left(\frac{\epsilon}{2} p_2 + \nu_2 - j_2' \right) \left(\frac{\epsilon}{2} p_3 + \nu_3 - j_3' \right) \right]^{-1} \\
 &\times \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 \prod_{k=1}^{J+J'} [A(\epsilon) - k]
 \end{aligned}$$

is identically zero. This follows, however, from the sum rules derivable from the regularity of the product

$$d_{j_1 j_1' q_1}^{x_1}(\eta) d_{j_2 j_2' q_2}^{x_2}(\eta) d_{j_3 j_3' q_3}^{x_3}(\eta)$$

at $\eta = 0$. In fact, the identity

$$\prod_{i=1,2,3} (2\gamma h \eta \alpha_i \eta) e^{\beta_i \eta} = \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 \exp[\eta \sum_i (\alpha_i \epsilon_i + \beta_i)]$$

and (1.37) give

$$\begin{aligned}
 &\sum_{\nu_i \mu_i} C_{\nu_1 \mu_1}^{(1)} C_{\nu_2 \mu_2}^{(2)} C_{\nu_3 \mu_3}^{(3)} \left[\left(\frac{\epsilon}{2} p_1 + \nu_1 - j_1' \right) \left(\frac{\epsilon}{2} p_2 + \nu_2 - j_2' \right) \left(\frac{\epsilon}{2} p_3 + \nu_3 - j_3' \right) \right]^{-1} \\
 &\times \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 [-2A(\epsilon) + J + J' + 1]^n = 0
 \end{aligned}$$

for $n=0, 1, 2, \dots, J+J'+2$ (and eventually for higher n). Moreover, we obtain this way that $\Psi(A(\epsilon) - J - J')$ in (1.50) can be replaced by any function $\Psi(A(\epsilon) - n)$, $n=0, 1, 2, \dots, J+J'$.

In the simplest possible case

$$m_i = j_i = j'_i = q_i = 0, \quad i = 1, 2, 3$$

(1.50) reduces to

$$G(p_1, p_2, p_3) = -\frac{1}{8} \left[\frac{i}{2} p_1 \frac{i}{2} p_2 \frac{i}{2} p_3 \right]^{-1} \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 \Psi\left(\frac{1}{2} - \frac{i}{4} R(\epsilon)\right)$$

Since

$$\Psi\left(\frac{1}{2} - z\right) - \Psi\left(\frac{1}{2} + z\right) = -\pi \operatorname{tg} \pi z$$

we get the result

$$G(p_1, p_2, p_3) = -\frac{\pi}{2 p_1 p_2 p_3} \sum_{\epsilon_i} \epsilon_1 \epsilon_2 \epsilon_3 + \frac{\pi}{4} R(\epsilon) \quad (1.51)$$

known from (C), Section 6.

1.4 The poles of the G function

So far (1.50) was considered as a function of χ_1, χ_2, χ_3 with p_1, p_2, p_3 varying over the real axis. If we continue the function G as given by (1.50) analytically in the variables p_i , the issue arises to give the positions and the residues of the poles of G . One might first suspect that the denominator

$$\left[\prod_{i=1,2,3} \left(\frac{i}{2} p_i + v_i - j'_i \right) \right]^{-1}$$

causes singularities. This is, however, not true. Poles of this type do occur in the e functions (1.40) but cancel in (1.42) leaving entire analytic d functions (1.37). The poles of G are therefore due to the Ψ function alone. The Ψ function possesses a Mittag-Leffler expansion

$$\Psi(z) = -C - \sum_{k=0}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k+1} \right) \quad (1.52)$$

where C is Euler's constant. We regard G as a sum of eight terms, each one belonging to a fixed combination $(\epsilon_1, \epsilon_2, \epsilon_3)$. The poles of these terms may superimpose later and give rise to second order poles.

From (1.50) and (1.52) we inspect that a fixed combination (ϵ_i) exhibits poles at

$$A(\epsilon) = -n, \quad n = 0, 1, 2, \dots$$

or at

$$\frac{i}{2} R(\epsilon) = j' + j'(\epsilon) + M - N(\epsilon) + 2n + 1$$

Since, (Section 1.3)

$$\min(\mu - \epsilon\nu) = -(1 + \epsilon)j' + |q + \frac{1}{2}\epsilon m|$$

and with integer s_i

$$\mu_i - \epsilon_i \nu_i = \min(\mu_i - \epsilon_i \nu_i) + 2s_i$$

we finally get

$$\frac{i}{2} R(\epsilon) = Q(\epsilon) + 2(n + s_1 + s_2 + s_3) + 1 \quad (1.53)$$

where we introduced the notation

$$Q(\epsilon) = \sum_i |q_i + \frac{1}{2}\epsilon_i m_i|$$

The residues of the Ψ function, (1.52), are minus one. For the product in front of the Ψ function, (1.50), we get, at the pole

$$\prod_{k=1}^{j+j'} (-n-k) = (-1)^{j+j'} \frac{(j+j'+n)!}{n!}$$

We use the notation

$$R_{s,\epsilon}(jj'qx) = \frac{1}{2} \{ \dots \}^{\frac{1}{2}} \epsilon \sum C_{\nu\mu} \left(\frac{i}{2} p + \nu - j' \right)^{-1} \quad (1.54)$$

where the curly bracket is the same as in (1.37) and the sum extends over all ν, μ satisfying

$$\mu - \epsilon \nu = \min(\mu - \epsilon \nu) + 2s, \quad s \text{ fixed}$$

The residue of one combination (ϵ_i) in G in the variable $-\frac{i}{4}R(\epsilon)$ can then be written as

$$\begin{aligned} \text{Res } G^{(\epsilon_i)}(x_1, x_2, x_3) &= \\ &= \sum_{n + \sum_i s_i = N} \binom{j + j' + n}{n} R_{s_1 \epsilon_1}^{(1)} R_{s_2 \epsilon_2}^{(2)} R_{s_3 \epsilon_3}^{(3)} \end{aligned} \quad (1.55)$$

In some simple cases we can find a closed form for the coefficients $R_{s\epsilon}$. In fact, from (1.40), (1.54) we have

$$\begin{aligned} e_{j_1 j_2 q}^x(\eta) &= 2(2 \sinh \eta)^{-j_1 - j_2 - 1} \exp \eta \left(\frac{i}{2} p + j_1 + j_2 - (q + \frac{1}{2} m) \right) \\ &\quad \times \sum_{s=0}^{j_1 + j_2 - (q + \frac{1}{2} m)} R_{s+} e^{-2s\eta} \end{aligned} \quad (1.56)$$

On the other hand, we obtain from (1.41), (1.54)

$$R_{s-}(j_1 j_2 q x) = (-1)^{j_1 - j_2} R_{s+}(j_2 j_1 q - x) \quad (1.57)$$

Using then the result of (L), (4.64), we find

$$\begin{aligned} R_{0+}(j_1 j_2 q x) &= \frac{1}{2} \{ (2j_1 + 1)(2j_2 + 1) \frac{(j_1 + \frac{1}{2} m)! (j_1 - \frac{1}{2} m)! (j_2 + \frac{1}{2} m)! (j_2 - \frac{1}{2} m)!}{(j_1 + q)! (j_1 - q)! (j_2 + q)! (j_2 - q)!} \}^{\frac{1}{2}} \\ &\quad \times (-1)^{j_2 - q} (q + \frac{1}{2} m)! \binom{j_1 + q}{q + \frac{1}{2} m} \binom{j_2 + q}{q + \frac{1}{2} m} \\ &\quad \times \frac{\Gamma(j_1 - \frac{i}{2} p + 1) \Gamma(-\frac{1}{2} m + \frac{i}{2} p)}{\Gamma(j_1 + \frac{i}{2} p + 1) \Gamma(q - \frac{i}{2} p + 1)} \end{aligned} \quad (1.58)$$

if $q + \frac{1}{2}m \geq 0$, and the same result with q, m replaced by $-q, -m$ respectively if $q + \frac{1}{2}m \leq 0$. Another simple case is $j = q = 0$, j' and s arbitrary. However, we do not give the result for this case, because we shall not need it in the sequel.

2. THE DECOMPOSITION OF THE MOMENTUM OPERATOR ON THE
HOMOGENEOUS LORENTZ GROUP

2.1 The reduction of unitary representations of $SL(2, \mathbb{C}) \times T_4$
on the subgroup $SL(2, \mathbb{C})$

We are interested only in unitary irreducible representations of $SL(2, \mathbb{C}) \times T_4$ on positive timelike orbits with mass M and spin S . We call the Hilbert space carrying such representations $L^2(M, S)$. That realization of the space $L^2(M, S)$ which is best suited to our problem consists of measurable functions $\Phi_q(a)$, $a \in SL(2, \mathbb{C})$, $-S \leq q \leq S$, which satisfy the covariance constraint

$$\Phi_q(ua) = \sum_{q'} D_{qq'}^S(u) \Phi_{q'}(a) \quad (2.1)$$

for any $u \in SU(2)$, [L], (2.17), (2.18), we use throughout the notation of [L], Chapter 2]. If $a(p)$ is a boost we define

$$\Phi_q(p) = \Phi_q(a(p)) \quad (2.2)$$

and require that the scalar product norm

$$\begin{aligned} \|\Phi\|^2 &= \int d\mu(p) \sum_q |\Phi_q(p)|^2 \\ &= \int d\mu(p) \sum_q |\Phi_q(a)|^2, \quad a = ua(p) \\ d\mu(p) &= \Theta(p_0) \delta(p^2 - M^2) d^4p \end{aligned} \quad (2.3)$$

be finite. The unitary representation (M, S) of $SL(2, \mathbb{C}) \times T_4$ is defined by [L], (2.17)],

$$U_{(a, y)} \Phi_q(a_1) = \exp[iM \Lambda_\mu^0(a_1) y^\mu] \Phi_q(a_1 a) \quad (2.4)$$

On the subgroup $SL(2, \mathbb{C})$ (2.4) reduces to

$$U_a \Phi_q(a_1) = \Phi_q(a_1 a) \quad (2.5)$$

We call $\Phi_q(a)$ a covariant wave function.

Next we submit the covariant wave functions to Fourier transformation on $SL(2, \mathbb{C})$. For χ in the principal series we define Fourier transforms by (1.8)

$$\mathcal{F}_{jq}(\chi) = \int \Phi_{q'}(a) \overline{D_{sq'jq}^\chi(a)} d\mu(a) \quad (2.6)$$

where $d\mu(a)$ is the invariant measure on $SL(2, \mathbb{C})$ in an appropriate normalization, (L), (1.20). If we decompose $a \in SL(2, \mathbb{C})$ as

$$a = u_1 d u_2$$

with d as in (1.10), we have, [L], Appendix A.1d]

$$d\mu(a) = \frac{1}{4\pi} d\mu(u_1) d\mu(u_2) sh^2 \eta d\eta$$

and therefore from (1.9), (1.10), (2.1), (2.6)

$$\begin{aligned} \mathcal{F}_{jq}(\chi) &= [4\pi(2s+1)]^{-1} \sum_{q'} \int \Phi_{q'}(du_2) \\ &\quad \times d_{sq'jq}^\chi(\eta) \overline{D_{q'q}^\chi(u_2)} d\mu(u_2) sh^2 \eta d\eta \end{aligned} \quad (2.7)$$

which proves that $\mathcal{F}_{jq}(\chi)$ (2.6) does not depend on q' indeed.

The inverse Fourier transformation, (L), (4.32), looks like

$$\Phi_q(a) = \frac{1}{2} \sum_{m=-2s}^{2s} \int_{-\infty}^{\infty} d\varrho (\varrho^2 + m^2) \sum_{jq'} \mathcal{F}_{jq'}(\chi) D_{sqjq'}^\chi(a) \quad (2.8)$$

With [L), Appendix A.3b],

$$a = ua(p), \quad d\mu(a) = C^{-1} d\mu(u) d\mu(p), \quad C = 8\pi^2 M^2$$

Parseval's formula follows from (2.3) and (L), (4.31) in the form

$$\begin{aligned} \|\Phi\|^2 &= (2S+1)C \int |\Phi_q(a)|^2 d\mu(a) \\ &= (2S+1)C \frac{1}{2} \sum_{m=-2S}^{2S} \int_{-\infty}^{+\infty} d\varrho (\varrho^2 + m^2) \sum_{j'q} |\mathcal{F}_{j'q}(x)|^2 \end{aligned} \quad (2.9)$$

We construct Hilbert spaces $\mathcal{H}(X)$ which consist of the ℓ^2 sequences $\{\mathcal{F}_{jq}(X)\}$, the scalar product norm is defined as

$$\|\mathcal{F}(X)\|^2 = \sum_{jq} |\mathcal{F}_{jq}(X)|^2 \quad (2.10)$$

On the space $\mathcal{H}(X)$ the operator U_a (2.5) reduces to

$$U_a \mathcal{F}_{jq}(X) = \sum_{j'q'} D_{jqj'q'}^X(a) \mathcal{F}_{j'q'}(X) \quad (2.11)$$

Equations (2.9) to (2.11) give the complete solution of our reduction problem ^{4), 7)}.

For the sake of completeness we note the connection between the ℓ^2 space $\mathcal{H}(X)$ and the corresponding space $L_m^2(U)$. We define the functions of $L_m^2(U)$ by

$$\varphi(u, X) = \sum_{jq} (2j+1)^{\frac{1}{2}} \mathcal{F}_{jq}(X) D_{\frac{1}{2}m, q}^j(u) \quad (2.12)$$

These functions (2.12) have the correct transformation behaviour (1.5), (1.6)

$$U_a \varphi(u, X) = \alpha^X(u, a) \varphi(u_a, X)$$

Inserting (2.6), (1.7) and (1.8), we can perform the summation in (2.12) and obtain

$$\varphi(u, \chi) = (2S+1)^{-\frac{1}{2}} \pi \int \Phi_{\frac{1}{2}m}(ku) \tilde{\alpha}^{-\chi}(e, k) d\mu_e(k) \quad (2.13)$$

with k as in Section 1.1 and the left invariant measure on the group K defined as

$$d\mu_e(k) = (2\pi)^{-4} D\lambda D\mu$$

In terms of Gelfand's integral operator, (C), Section 2, we have the alternative form of (2.13)

$$\varphi(u, \chi) = (2S+1)^{-\frac{1}{2}} K \Phi_{\frac{1}{2}m}(e, u|-\chi) \quad (2.14)$$

2.2 The momentum operator

We define the unbounded self-adjoint operator P_0 on $L^2(M, S)$ by (2.2)

$$P_0 \Phi_q(p) = p_0 \Phi_q(p)$$

In terms of covariant wave functions we have instead

$$P_0 \Phi_q(a) = \frac{1}{2} |a|^2 M \Phi_q(a) \quad (2.15)$$

where

$$|a|^2 = \text{Tr}(aa^\dagger)$$

The other three components P_k , $k=1, 2, 3$ which together with P_0 form the momentum four-vector operator, can be obtained by applying Lorentz transformations U_a (2.5), to P_0 . The domain D of P_0 consists of those functions $\Phi_q(a) \in L^2(M, S)$ for which

$$\int |a|^4 |\Phi_q(a)|^2 d\mu(a) < \infty$$

Our first task is to characterize the image of D in $\mathcal{H}(\chi)$ or $L_m^2(U)$.

We denote the analytic continuation of $\overline{D_{j_1 q_1 j_2 q_2}^{\chi}(a)}$ in ϱ by $D_{j_1 q_1 j_2 q_2}^{\chi}(a)^*$. The analytically continued integral (2.6)

$$\mathcal{F}_{jq}(\chi) = \int |a|^2 \Phi_{q'}(a) \{ |a|^{-2} D_{sq'jq}^{\chi}(a)^* \} d\mu(a), \Phi \in D$$

converges absolutely as long as the expression in curly brackets is square integrable on $SL(2, \mathbb{C})$. This is true if and only if

$$\sum_q \int_0^\infty d\eta \, \eta^2 (ch\eta)^{-2} d_{sjq}^{(m, -\varrho)}(\eta) d_{sjq}^{(m, -\bar{\varrho})}(\eta) < \infty \quad (2.16)$$

The convergence of the integral (2.16) is determined by the asymptotic behaviour of the functions $d_{j_1 j_2 q}^{\chi}(\eta)$ at $\eta \rightarrow \infty$ which can be read off (1.42) and (1.56). It obtains that the condition (2.16) is satisfied in the strip $|\operatorname{Im} \varrho| < 2$. In this open strip each $\mathcal{F}_{jq}(\chi)$ is holomorphic in ϱ .

A refinement of this result is possible if we use the functions $\Psi(u, \chi)$ (2.12), (2.13). By simple arguments based on the representation (2.13) we can show that the inequality

$$\int d\mu(u) |\Psi(u, \chi)|^2 < \infty \quad (2.17)$$

holds for almost all ϱ on any straight line parallel to the real axis in the strip $|\operatorname{Im} \varrho| \leq 2$. Moreover, we have

$$\frac{1}{2} \sum_{m=-2s}^{2s} \int_{-\infty+i\sigma}^{+\infty+i\sigma} d\rho (\rho^2 + m^2) \int d\mu(u) |\Psi(u, \chi)|^2 < \infty \quad (2.18)$$

for any number σ in the closed interval

$$-2 \leq \sigma \leq 2$$

On the boundary $\operatorname{Im} \varrho = \pm 2$ the function $\Psi(u, \chi)$ is obtained as the L^2 limit, (2.18), of functions $\Psi(u, \chi)$ from inside the strip $|\sigma| < 2$. From (2.17) we obtain in particular

$$\sum_{jq} |\mathcal{F}_{jq}(\chi)|^2 < \infty$$

for almost all q on any straight line parallel to the real axis in the closed strip. In order to specify an additional property of the sequences $\{\mathcal{F}_{jq}(\chi)\}$ or functions $\Psi(u, \chi)$ we have to make use of later results of this and the subsequent Chapter. In fact, the sums and integrals (2.32), (3.27) converge absolutely, which means that the domain of the covariant vector operators (3.19) to (3.22) can be extended onto these functions and sequences.

Next we compute the matrix element

$$\langle \Phi^1 | P_0 | \Phi^2 \rangle = \int d\mu(p) \sum_q \overline{\Phi_q^1(p)} P_0 \Phi_q^2(p) \quad (2.19)$$

with $\Phi^1 \in L^2(M, S)$ and $\Phi^2 \in D$. Inserting (2.15) we obtain in a similar fashion as in (2.9)

$$\begin{aligned} \langle \Phi^1 | P_0 | \Phi^2 \rangle &= (2S+1) c M \int d\mu(a) \overline{\Phi_q^1(a)} \frac{1}{2} |a|^2 \Phi_q^2(a) \\ &= (2S+1) c M \int d\mu(a) \frac{1}{2} |a|^2 \\ &\quad \times \frac{1}{2} \sum_{m_1=-2S}^{2S} \int_{-\infty}^{+\infty} d\rho_1 (\rho_1^2 + m_1^2) \sum_{j, q_1} \overline{\mathcal{F}_{jq_1}^1(\chi_1)} \overline{D_{Sq_1 j_1 q_1}^{\chi_1}(a)} \\ &\quad \times \frac{1}{2} \sum_{m_2=-2S}^{2S} \int_{-\infty}^{+\infty} d\rho_2 (\rho_2^2 + m_2^2) \sum_{j_2, q_2} \mathcal{F}_{j_2 q_2}^2(\chi_2) D_{Sq_2 j_2 q_2}^{\chi_2}(a) \end{aligned} \quad (2.20)$$

In order to evaluate the integral (2.20) we make some formal manipulations with the divergent integral

$$\int d\mu(a) \frac{1}{2} |a|^2 \overline{D_{Sq_1 j_1 q_1}^{\chi_1}(a)} D_{Sq_2 j_2 q_2}^{\chi_2}(a) \quad (2.21)$$

Since in the parameters

$$a = u_1 du_2,$$

as in (1.10), we have, (L), (3.36)

$$\frac{1}{2} |a|^2 = \text{ch } \eta = \frac{2 \sin \frac{1}{2} \vartheta \eta}{\vartheta \sinh \eta} \Big|_{\vartheta=4i} = d_{000}^{(0,\vartheta)}(\eta) \Big|_{\vartheta=4i}$$

we may write (2.21) as

$$\delta_{j_1 j_2} \delta_{q_1 q_2} [4\pi (2S+1)(2j_1+1)]^{-1} \lim_{\vartheta \rightarrow 4i} \sum_q G(\bar{\chi}_1, \chi_2, (0, \vartheta) | S_{j_1 q}, S_{j_1 q}, 000)$$

Inserting this expression into (2.20) and thus reaching solid ground again we get

$$\begin{aligned} \langle \Phi^1 | P_0 | \Phi^2 \rangle &= \pi M^3 \lim_{\vartheta \rightarrow 4i} \sum_{m_1=-2S}^{2S} \frac{1}{2} \int_{-\infty}^{+\infty} d\varrho_1 (\varrho_1^2 + m_1^2) \\ &\times \sum_{m_2=-2S}^{2S} \int_{C_\vartheta} d\varrho_2 (\varrho_2^2 + m_2^2) \sum_{jq} (2j+1)^{-1} \overline{F_{jq}^1(\chi_1)} F_{jq}^2(\chi_2) \\ &\times \sum_{q'} G(\bar{\chi}_1, \chi_2, (0, \vartheta) | S_{jq'}, S_{jq'}, 000) \end{aligned} \quad (2.22)$$

The contour C_ϑ has to be chosen such that it avoids the poles of the G function moving against the real axis. It turns out that the domain of holomorphy of $F_{jq}^2(\chi_2)$ is just big enough to allow for such deformations of the contour of integration.

We assume that the continuation in ϑ starts at $\vartheta = 0$, proceeds first along the imaginary axis, goes around the point $\vartheta = 2i$ in a small circle and continues along the imaginary axis till $\vartheta = 4i$. During this continuation two sets of poles of G penetrate into the strip $|\text{Im } \vartheta_2| < 2$. The first set belongs to $Q(\varepsilon) = 0$ (1.53), and $N=0$, (1.55). These poles lie at

$$\begin{aligned} \vartheta_2 &= \tau \vartheta_1 - \varepsilon_2 (2i - \vartheta) \\ \tau &= \varepsilon_1 \varepsilon_2, \quad \varepsilon = -1 \end{aligned}$$

For each value τ and for fixed ϑ_1 on the real axis the pole belonging to $\varepsilon_2 = +1$ moves from $\text{Im } \vartheta_2 = -2$ up to $\text{Im } \vartheta_2 = +2$, whereas the mirror pole $\varepsilon_2 = -1$ moves down from $\text{Im } \vartheta_2 = +2$ till

$\text{Im } \varrho_2 = -2$. They do not cross each other on the real axis, because of the small circle at $\varrho = 2i$. However, at the end points $\varrho = 4i$ there are pinches with poles belonging to $Q(\varepsilon) = 0$, $N = 1$ and $Q(\varepsilon) = 2$, $N = 0$. $Q(\varepsilon) = 0$ is possible only if

$$q' + \frac{1}{2}\varepsilon_1 m_1 = q' + \frac{1}{2}\varepsilon_2 m_2 = 0$$

We have therefore such poles only if

$$\begin{aligned} m_2 &= \tau m_1 \\ \varrho_2 &= \tau \varrho_1 - \varepsilon_2(2i - \varrho) \end{aligned}$$

In the integral (2.22) we split off the contributions of the poles as soon as these cross the real axis. These contributions are obtained by the replacement

$$\begin{aligned} \sum_{q'} G &\rightarrow \frac{8\pi}{-i\varrho} \sum_{\varepsilon_1, \varepsilon_2} R_{0\varepsilon_1}^{(1)}(\bar{\chi}_1) R_{0\varepsilon_2}^{(2)}(\chi_2) \\ &\quad \times \delta_{m_2, \tau m_1} \delta(\varrho_2 - \tau \varrho_1 + \varepsilon_2(2i - \varrho)) \end{aligned} \quad (2.23)$$

where $\delta(\varrho_2 - \dots)$ denotes an "analytic delta function" and use was made of (1.55) and (1.58). The limit $\varrho \rightarrow 4i$ can be performed because we know that $\mathcal{F}_{jq}^2(\chi_2)$ possesses an L^2 limit on the boundary of the strip of holomorphy. As can be inspected from (1.58) R_{s-} and R_{s+} are also connected by the relation

$$R_{s-}(j_1 j_2 q, \chi) = (\beta^{j_1})^{-1} \beta^{j_2} R_{s+}(j_1 j_2 q, -\chi) \quad (2.24)$$

where β^j is as in (1.12). From (1.11) and (2.7) we have moreover

$$\begin{aligned} \beta^s (\beta^j)^{-1} \overline{\mathcal{F}_{jq}^1(\chi)} &= \overline{\mathcal{F}_{jq}^1(-\chi)} \\ (\beta^s)^{-1} \beta^j \mathcal{F}_{jq}^2(\chi) &= \mathcal{F}_{jq}^2(-\chi) \end{aligned} \quad \text{on the principal series} \quad (2.25)$$

The symmetry relations (2.24) and (2.25) imply that all four combinations ϵ_1, ϵ_2 in (2.23) give the same result after inserting (2.23) into (2.22). In the case $\epsilon_1 = \epsilon_2 = +1$ we have from (1.58)

$$R_{0+}^{(1)}(\bar{\chi}_1) R_{0+}^{(2)}(\chi_2) =$$

$$= 4(2S+1)(2j+1)(S+\frac{i}{2}\varphi)(S-\frac{i}{2}\varphi+1) [(\varphi_1^2+m_1^2)((\varphi_1+2i)^2+m_1^2)]^{-1}$$

and therefore the contribution to the matrix element of P_0 (2.22)

$$32\pi^2 M^3 (2S+1) \sum_{m=-2S}^{2S} \frac{1}{2} \int_{-\infty}^{\infty} d\varphi (S+\frac{i}{2}\varphi)(S-\frac{i}{2}\varphi+1)$$

$$\times \sum_{j\varphi} \overline{\mathcal{F}_{j\varphi}^1(m, \varphi)} \mathcal{F}_{j\varphi}^2(m, \varphi+2i)$$
(2.26)

An alternative form of (2.26) is obtained if we use (2.25)

$$32\pi^2 M^3 (2S+1) \sum_{m=-2S}^{2S} \frac{1}{2} \int_{-\infty}^{+\infty} d\varphi \sum_{j\varphi} (j-\frac{i}{2}\varphi)(j+\frac{i}{2}\varphi+1)$$

$$\times \overline{\mathcal{F}_{j\varphi}^1(m, \varphi)} \mathcal{F}_{j\varphi}^2(m, \varphi-2i)$$
(2.27)

The second set of poles which penetrate into the strip $|\text{Im } \varphi_2| < 2$ belongs to $Q(\epsilon) = 1$, $N=0$, (1.53), (1.55) with positions at

$$\varphi_2 = \tau \varphi_1 - \epsilon_2 (4i - \varphi)$$

$$\tau = \epsilon_1 \epsilon_2, \quad \epsilon = -1$$

For each value τ and a fixed value φ_1 on the real axis, the pole corresponding to $\epsilon_2 = +1$ moves from $\text{Im } \varphi_2 = -4$ up towards the real axis whereas the pole corresponding to $\epsilon_2 = -1$ moves down against the real axis. To avoid this pinch we shift the integration contour down over the pole $\epsilon_2 = +1$ and compensate this by an integral over a negatively oriented circle around this pole. We compute this compensation term.

The condition

$$Q(\varepsilon) = |q' + \frac{1}{2}\varepsilon_1 m_1| + |q' + \frac{1}{2}m_2| = 1, \quad \varepsilon_2 = +1$$

can be fulfilled only if either

$$(1)_{\pm} : \quad q' = -\frac{1}{2}\varepsilon_1 m_1, \quad q' = -\frac{1}{2}m_2 \pm 1$$

or

$$(2)_{\pm} : \quad q' = -\frac{1}{2}\varepsilon_1 m_1 \pm 1, \quad q' = -\frac{1}{2}m_2$$

It follows

$$m_2 = \varepsilon_1 m_1 + 2 \quad \text{for } (1)_+, (2)_-$$

$$m_2 = \varepsilon_1 m_1 - 2 \quad \text{for } (1)_-, (2)_+$$

The compensation term is obtained from (2.22) by the replacement

$$\begin{aligned} \sum_{q'} G \rightarrow \sum_{\varepsilon_1} \left[\sum_{(1)_+, (2)_-} R_{0\varepsilon_1}^{(1)}(\bar{\chi}_1) R_{0+}^{(2)}(\chi_2) \delta_{m_2, \varepsilon_1 m_1 + 2} \right. \\ \left. + \sum_{(1)_-, (2)_+} R_{0\varepsilon_1}^{(1)}(\bar{\chi}_1) R_{0+}^{(2)}(\chi_2) \delta_{m_2, \varepsilon_1 m_1 - 2} \right] \\ \times \delta(\varphi_2 - \varepsilon \varphi_1 - \varphi + 4i) \end{aligned}$$

(2.28)

We evaluate the sum

$$\begin{aligned} \sum_{(1)_+, (2)_-} R_{0+}^{(1)}(\bar{\chi}_1) R_{0+}^{(2)}(\chi_2) = \\ = R_{0+}^{(1)}(Sj, -\frac{1}{2}m_1, (m_1, \varphi_1)) R_{0+}^{(2)}(Sj, -\frac{1}{2}m_1, (m_1+2, \varphi_1)) \\ + R_{0+}^{(1)}(Sj, -\frac{1}{2}m_1-1, (m_1, \varphi_1)) R_{0+}^{(2)}(Sj, -\frac{1}{2}m_1-1, (m_1+2, \varphi_1)) \\ = -4(2S+1)(2j+1) [(\varphi_1^2 + m_1^2)(\varphi_1^2 + (m_1+2)^2)]^{-1} \\ \times [(S - \frac{1}{2}m_1)(S + \frac{1}{2}m_1+1)(j - \frac{1}{2}m_1)(j + \frac{1}{2}m_1+1)]^{\frac{1}{2}} \end{aligned}$$

which follows directly from (1.58). The other sign of ε_1 gives the same result after the insertion into (2.22) as follows from the symmetry relations (2.24), (2.25). With the help of the identity

$$R_{S+}(j_1 j_2, q(m, \varphi)) = R_{S+}(j_1 j_2, -q(-m, \varphi)) \quad (2.29)$$

which is a trivial consequence of (1.58) we can show that the sum over $(1)_-, (2)_+$ gives the same result as before, after inserting it into (2.22). Therefore we have altogether as the compensation term

$$\begin{aligned} & -32\pi^2 M^3 (2S+1) \sum_{m=-2S}^{2S-2} \frac{1}{2} \int_{-\infty}^{+\infty} d\varphi [(s-\frac{1}{2}m)(s+\frac{1}{2}m+1)]^{\frac{1}{2}} \\ & \times \sum_{j_1 j_2} [(j-\frac{1}{2}m)(j+\frac{1}{2}m+1)]^{\frac{1}{2}} \overline{F_{j_1 j_2}^1(m, \varphi)} F_{j_1 j_2}^2(m+2, \varphi) \end{aligned} \quad (2.30)$$

or in the alternative form

$$\begin{aligned} & -32\pi^2 M^3 (2S+1) \sum_{m=-2S+2}^{2S} \frac{1}{2} \int_{-\infty}^{+\infty} d\varphi [(s+\frac{1}{2}m)(s-\frac{1}{2}m+1)]^{\frac{1}{2}} \\ & \times \sum_{j_1 j_2} [(j+\frac{1}{2}m)(j-\frac{1}{2}m+1)]^{\frac{1}{2}} \overline{F_{j_1 j_2}^1(m, \varphi)} F_{j_1 j_2}^2(m-2, \varphi) \end{aligned} \quad (2.31)$$

If we restrict the integration over φ in (2.26), (2.27) and (2.30), (2.31) to positive values eliminating in this fashion superfluous equivalent representations we get the unique result

$$\begin{aligned} & + 16\pi^2 M^3 (2S+1) \sum_{m=-2S}^{2S} \int_0^{\infty} d\varphi \sum_{j_1 j_2} \overline{F_{j_1 j_2}^1(m, \varphi)} \\ & \times \{ (s+\frac{1}{2}\varphi)(s-\frac{1}{2}\varphi+1) \overline{F_{j_1 j_2}^2(m, \varphi+2i)} \\ & + (j-\frac{1}{2}\varphi)(j+\frac{1}{2}\varphi+1) \overline{F_{j_1 j_2}^2(m, \varphi-2i)} \\ & - [(s-\frac{1}{2}m)(s+\frac{1}{2}m+1)(j-\frac{1}{2}m)(j+\frac{1}{2}m+1)]^{\frac{1}{2}} \overline{F_{j_1 j_2}^2(m+2, \varphi)} \\ & - [(s+\frac{1}{2}m)(s-\frac{1}{2}m+1)(j+\frac{1}{2}m)(j-\frac{1}{2}m+1)]^{\frac{1}{2}} \overline{F_{j_1 j_2}^2(m-2, \varphi)} \} \end{aligned} \quad (2.32)$$

This result (2.32) is in fact the complete matrix element (2.22) since one can prove that

$$\sum_q G(\bar{\chi}_1, \chi_2, (0, 2ni) | s_j q; s_j q, 000) = 0 \quad (2.33)$$

for all $n = \pm 1, \pm 2, \dots$ in the sense of analytic functions, whereas the distribution theoretic limit $q \rightarrow 2ni$ yields a finite sum of analytic delta functions as we have just shown. The proof of (2.33) is postponed till Section 3.2 where we shall resume the discussion of the result (2.32).

3. TRILINEAR INVARIANT FUNCTIONALS

3.1 The geometric interpretation of poles and zeros

For three representations χ_1, χ_2, χ_3 of the principal series we can construct a linear functional

$$F(f_1, f_2, f_3) = \int N(z_1, z_2, z_3 | \chi_1, \chi_2, \chi_3) f_1(z_1) f_2(z_2) f_3(z_3) D z_1 D z_2 D z_3 \quad (3.1)$$

by means of Naimark's kernel (1.16). This functional F exists for any triplet of functions $f_i \in L^2(Z_i)$, is linear in each of its three arguments, and continuous in each argument separately with respect to the scalar product norm in $L^2(Z_i)$. It is moreover invariant under transformations of the group $SL(2, \mathbb{C})$. In the sequel we should always have in mind that this functional F can be translated into a functional on the spaces $L_{m_i}^2(U_i)$ with a similar kernel N^2). In the Hilbert space $L_m^2(U)$ we can select the linear subspace $D_\chi(U)$ of infinitely differentiable functions. This linear space can be made a closed countably normed space in a standard fashion. By means of the one-to-one point transformation we can single out a similar closed countably normed subspace $D_\chi(Z)$ of $L^2(Z)$. It consists of infinitely differentiable functions in z possessing a definite asymptotic expansion at $z = \infty$ which depends on χ . The norms in $D_\chi(Z)$ are understood to be the norms carried over from $D_\chi(U)$. On the subspaces $D_{\chi_i}(Z_i)$ the functional F is also continuous. If we continue the functional F analytically in ϱ_i [for the definition of analytic continuation of F we consider first F on $D_{\chi_i}(U_i)$, these spaces are independent of ϱ_i] off the real axis, the spaces $D_{\chi_i}(Z_i)$ remain countably normed closed spaces and the functional F remains invariant and continuous on them. F is still generated by the kernel N (1.16) which has now to be understood as a distribution. For a deeper investigation of the spaces $D_\chi(Z)$ we refer to Gelfand's textbook⁸⁾ or to (I).

In order to prepare the discussion of the poles of the functional F we study the distribution

$$\begin{aligned} p_{\sigma_1 \sigma_2}(z) &= z^{\sigma_1} \bar{z}^{\sigma_2} \\ z &= x+iy, \quad \sigma_1 - \sigma_2 = m \text{ integer} \end{aligned} \quad (3.2)$$

operating on a space of test functions $C_c^\infty(R_2)$. This distribution is analytic in σ_1, σ_2 , in particular it is holomorphic if

$$\operatorname{Re}(\sigma_1 + \sigma_2) > -2$$

If both $\operatorname{Re} \sigma_1$ and $\operatorname{Re} \sigma_2$ are in the open interval $(-1, 0)$ the Fourier transform of $p_{\sigma_1 \sigma_2}(z)$ is a convergent integral which can be computed by standard techniques

$$\begin{aligned} & \int p_{\sigma_1 \sigma_2}(z) e^{-i(x\xi + y\eta)} dx dy \\ &= 2\pi 2^{\sigma_1 + \sigma_2 + 1} e^{-\frac{i}{2}\pi |m|} \frac{\Gamma(\frac{1}{2}|m| + \frac{1}{2}(\sigma_1 + \sigma_2) + 1)}{\Gamma(\frac{1}{2}|m| - \frac{1}{2}(\sigma_1 + \sigma_2))} \omega^{-\sigma_2 - 1} \bar{\omega}^{-\sigma_1 - 1} \\ & \quad (\omega = \xi + i\eta) \end{aligned} \quad (3.3)$$

Note that a distribution and its Fourier transform have the same analytic behaviour in a parameter. The distribution $\omega^{-\sigma_2 - 1} \bar{\omega}^{-\sigma_1 - 1}$ is holomorphic in the domain $\operatorname{Re}(\sigma_1 + \sigma_2) < 0$. In this domain the only singularities can be caused by the Γ function in the numerator. This Γ function has first order poles at

$$\frac{1}{2}|m| + \frac{1}{2}(\sigma_1 + \sigma_2) + 1 = -n, \quad n = 0, 1, 2, \dots$$

or at

$$\begin{aligned} \sigma_1 &= -n - 1 - \frac{1}{2}(|m| - m) \\ \sigma_2 &= -n - 1 - \frac{1}{2}(|m| + m) \end{aligned} \quad (3.4)$$

The Γ function in the denominator is finite non-zero at these points. We emphasize that the parameters σ_1, σ_2 have both to be negative integers at the poles.

Our next aim is to compute the residue of $p_{\sigma_1, \sigma_2}(z)$ at these poles. We make use of the fact that both powers in $w^{-\sigma_2-1} \bar{w}^{-\sigma_1-1}$ are non-negative integers. From

$$\begin{aligned} \frac{\partial}{\partial z} e^{-i \operatorname{Re}(z \bar{w})} &= + \frac{1}{2i} \bar{w} e^{-i \operatorname{Re}(z \bar{w})} \\ \frac{\partial}{\partial \bar{z}} e^{-i \operatorname{Re}(z \bar{w})} &= + \frac{1}{2i} w e^{-i \operatorname{Re}(z \bar{w})} \end{aligned}$$

we obtain immediately the residue in the variable $\frac{1}{2}(\sigma_1 + \sigma_2)$

$$\begin{aligned} \operatorname{Res} p_{\sigma_1, \sigma_2}(z) &= \\ &= \frac{\pi(-1)^m}{n!(n+m)!} \left(\frac{\partial}{\partial z} \right)^{-\sigma_1-1} \left(\frac{\partial}{\partial \bar{z}} \right)^{-\sigma_2-1} \delta(z) \end{aligned} \quad (3.5)$$

It is obvious from the representation (1.16) of the kernel N that the functional F may have poles if any pair (A_i, B_i) consists of negative integers, and that this is the only possible source of singularities of F . We are interested only in the more restricted problem to find the singularities of F if one representation is a fixed integer representation, say the two numbers n_3, n'_3 (1.17) are either both negative integers or both positive integers. In the former case, the space D_{χ_3} possesses an invariant subspace F_{χ_3} consisting of all functions of D_{χ_3} whose moments

$$\int z_3^k \bar{z}_3^l f(z_3) D z_3 \quad (3.6)$$

vanish for all $0 \leq k \leq -n_3-1, 0 \leq l \leq -n'_3-1$. In the same case the space $D_{-\chi_3}$ possesses an invariant subspace $E_{-\chi_3}$ of dimension $n_3 n'_3$. In fact, the asymptotic expansion of the functions $D_{-\chi_3}$ at $z_3 = \infty$ is, (L), (3.5),

$$f(z_3) \cong \sum_{k, l} d_{kl} z_3^{-n_3-k-1} \bar{z}_3^{-n'_3-l-1} \quad (3.7)$$

so that the polynomials

$$P(z_3, \bar{z}_3) = \sum_{k=0}^{-n_3-1} \sum_{\ell=0}^{-n'_3-1} d_{k\ell} z_3^{-n_3-k-1} \bar{z}_3^{-n'_3-\ell-1} \quad (3.8)$$

are in $D_{-\chi_3}$. It is easy to check that they span an invariant subspace. If we write the functional F (3.1) in the form

$$F(f_1, f_2, f_3) = \int D z_3 f_3(z_3) \int N(z_1, z_2, z_3 | \chi_1, \chi_2, \chi_3) f_1(z_1) f_2(z_2) D z_1 D z_2 \quad (3.9)$$

the inner integral defines an element of $D_{-\chi_3}$. We may then ask under which condition this function is moreover in $E_{-\chi_3}$. The answer is simple: this happens only if A_3 and B_3 are simultaneously negative integers

$$A_3 = -\nu - 1, \quad B_3 = -\mu - 1, \quad \nu, \mu = 0, 1, 2, \dots$$

If $f_3(z_3)$ is not in F_{χ_3} , the functional F has a pole with the residue in the variable $\frac{1}{2}(A_3 + B_3)$

$$\begin{aligned} \text{Res}_{\nu, \mu} F(f_1, f_2, f_3) &= \frac{1}{8\pi} \left[\left(\frac{1}{2}(\nu + \mu) - \frac{1}{2}(\nu - \mu) \right)! \left(\frac{1}{2}(\nu + \mu) + \frac{1}{2}(\nu - \mu) \right)! \right]^{-1} \\ &\times \int D z_3 f_3(z_3) \int D z f_2(z, z_3) \left(\frac{\partial}{\partial z} \right)^\nu \left(\frac{\partial}{\partial \bar{z}} \right)^\mu f_1(z, z_3) \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} f_1(z, z_3) &= (z_3 - z)^{A_2} (\bar{z}_3 - \bar{z})^{B_2} f_1(z) \\ f_2(z, z_3) &= (z - z_3)^{A_1} (\bar{z} - \bar{z}_3)^{B_1} f_2(z) \end{aligned}$$

Since (1.18)

$$\begin{aligned} A_1 + A_2 &= -n_3 - 1 \\ B_1 + B_2 &= -n'_3 - 1 \end{aligned}$$

the inner integral in (3.10) defines a polynomial of $E_{-\chi_3}$ indeed if moreover

$$\begin{aligned} 0 \leq \nu &\leq -n_3 - 1 \\ 0 \leq \mu &\leq -n'_3 - 1 \end{aligned} \quad (3.11)$$

If the function $f_3(z_3)$ is in F_{χ_3} , the residue of F is zero.

Now we consider the case that both n_3 and n'_3 are positive integers. In this case we may ask under which condition the inner integral in (3.9) is in $F_{-\chi_3}$. We shall show that this happens always when A_3 and B_3 are not simultaneously integers, provided we hold A_1, B_1 and A_2, B_2 in regular positions. For the proof we show that the integrals

$$\begin{aligned} J_{k\ell}(z_1, z_2) &= \\ &= \int D\bar{z} \, z^k \bar{z}^\ell (z - z_1)^{A_2} (\bar{z} - \bar{z}_1)^{B_2} (z_2 - z)^{A_1} (\bar{z}_2 - \bar{z})^{B_1} \end{aligned} \quad (3.12)$$

vanish in the sense of distributions for all

$$\begin{aligned} 0 \leq k &\leq n_3 - 1 \\ 0 \leq \ell &\leq n'_3 - 1 \end{aligned}$$

In order to evaluate the integrals (3.12), we use Fourier transformations, in particular formula (3.3) with the shorthand notation

$$\int z^{\sigma_1} \bar{z}^{\sigma_2} e^{-i \operatorname{Re}(z \bar{w})} = C_{\sigma_1 \sigma_2} w^{-\sigma_2 - 1} \bar{w}^{-\sigma_1 - 1}$$

We obtain in this fashion

$$\begin{aligned} J_{k\ell}(z_1, z_2) &= \sum_{\mu=0}^k \sum_{\nu=0}^{\ell} (2i)^{\nu+\mu} \binom{k}{\mu} \binom{\ell}{\nu} (A_1+1)_\mu (B_1+1)_\nu \\ &\quad \times C_{A_1 B_1} C_{A_2 B_2} [C_{A_1+A_2+\mu+1, B_1+B_2+\nu+1}]^{-1} \\ &\quad \times z_2^{\ell-\nu} \bar{z}_2^{k-\mu} (z_2 - z_1)^{A_1+A_2+\mu+1} (\bar{z}_2 - \bar{z}_1)^{B_1+B_2+\nu+1} \end{aligned} \quad (3.13)$$

So far no use was made of

$$A_1 + A_2 + 1 = -n_3, \quad B_1 + B_2 + 1 = -n'_3$$

and of the fact that n_3, n'_3 are positive integers. In this case the coefficients $C_{-n_3+\mu, -n'_3+\nu}$ are infinite, since these coefficients contain the factor (3.3)

$$\Gamma\left(\frac{1}{2}|-n_3+\mu+n'_3-\nu| + \frac{1}{2}(-n_3+\mu-n'_3+\nu)+1\right) = \infty$$

This proves our assertion. If A_3 and B_3 are simultaneously integers and moreover

$$A_3 \leq n_3 - 1, \quad B_3 \leq n'_3 - 1$$

some of the distributions

$$(z_2 - z_1)^{A_3 - n_3 + \mu} (\bar{z}_2 - \bar{z}_1)^{B_3 - n'_3 + \nu}$$

are themselves singular and the resultant expression $I_{kl}(z_1, z_2)$ may be non-zero.

If the inner integral in (3.9) is in $F_{-\chi_3}$ and $f_3(z_3) \in E_{\chi_3}$ the whole functional F becomes zero.

3.2 Covariant vector operators

We consider the case $n_3 = n'_3 = -2$ in somewhat more detail. This is the case where the space $E_{-\chi_3}$ carries the four-vector representation of $SL(2, C)$. It leads us to the covariant vector operators (generalized Dirac matrices) ⁵⁾, which have first been derived in this fashion by Wess ⁹⁾. In order that the inner integral of (3.9) defines a function in $E_{-\chi_3}$ we have necessarily

$$A_3 = -\nu - 1, \quad B_3 = -\mu - 1$$

$$\nu, \mu = 0, 1$$

There exist therefore four different vector operators (in singular cases some of them may coincide). Writing (3.10) as

$$\text{Res}_{\nu, \mu} F(f_1, f_2, f_3) = \frac{1}{8\pi} [\dots]^{-1} (-1)^{A_2 - B_2} \\ \times \int D z_3 f_3(z_3) \int D z f_2(z) T_{\nu, \mu}(z|z_3) f_1(z)$$

we get

$$T_{00}(z|z_3) = (z - z_3)(\bar{z} - \bar{z}_3) \quad (3.14)$$

$$T_{10}(z|z_3) = (z - z_3)(\bar{z} - \bar{z}_3) \frac{\partial}{\partial z} - (n_1 - 1)(\bar{z} - \bar{z}_3) \quad (3.15)$$

$$T_{01}(z|z_3) = (z - z_3)(\bar{z} - \bar{z}_3) \frac{\partial}{\partial \bar{z}} - (n'_1 - 1)(z - z_3) \quad (3.16)$$

$$T_{11}(z|z_3) = (z - z_3)(\bar{z} - \bar{z}_3) \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} - (n_1 - 1)(\bar{z} - \bar{z}_3) \frac{\partial}{\partial \bar{z}} \\ - (n'_1 - 1)(z - z_3) \frac{\partial}{\partial z} + (n_1 - 1)(n'_1 - 1) \quad (3.17)$$

The zeroth component of the corresponding four-vector may be defined by ⁹⁾

$$\Gamma(\nu, \mu) = \lim_{z_3 \rightarrow 0} \left(1 + \frac{\partial}{\partial z_3} \frac{\partial}{\partial \bar{z}_3}\right) T_{\nu, \mu}(z|z_3) \quad (3.18)$$

which implies

$$\Gamma(0,0) = (1 + z\bar{z}) \quad (3.19)$$

$$\Gamma(1,0) = (1 + z\bar{z}) \frac{\partial}{\partial z} - (n_1 - 1)\bar{z} \quad (3.20)$$

$$\Gamma(0,1) = (1 + z\bar{z}) \frac{\partial}{\partial \bar{z}} - (n'_1 - 1)z \quad (3.21)$$

$$\Gamma(1,1) = (1+z\bar{z}) \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} - (u_1-1) \bar{z} \frac{\partial}{\partial \bar{z}} - (u_1'-1) z \frac{\partial}{\partial z} + (u_1-1)(u_1'-1) \quad (3.22)$$

On the canonical basis $\{f_q^j(z)\}$ (I), (3.83), we have

$$\Gamma(v,\mu) f_q^j(z)^{\chi_1} = \gamma_{v\mu}^j(\chi_1) f_q^j(z)^{\chi_2} \quad (3.23)$$

with

$$\gamma_{00}^j(\chi_1) = 1$$

$$\gamma_{10}^j(\chi_1) = + \left[(j + \frac{1}{2}m_1 + 1) (j - \frac{1}{2}m_1) \right]^{\frac{1}{2}}$$

$$\gamma_{01}^j(\chi_1) = - \left[(j - \frac{1}{2}m_1 + 1) (j + \frac{1}{2}m_1) \right]^{\frac{1}{2}}$$

$$\gamma_n^j(\chi_1) = - (j + \frac{1}{2}g_1) (j - \frac{1}{2}g_1 + 1)$$

$$m_2 = m_1 + 2(v-\mu), \quad g_2 = g_1 + 2i(v+\mu-1)$$

(3.24)

If we introduce the function

$$f(z, \chi) = \sum_{jg} \overline{f}_{jg}(\chi) f_q^j(z)^{\chi}$$

(3.25)

in analogy to (2.12), we may write

$$\begin{aligned} & \int D\bar{z} \overline{f_1(z, \chi)} \Gamma(v, \mu) f_2(z, \chi_{v\mu}) \\ &= \sum_{jg} \gamma_{v\mu}^j(\chi_{v\mu}) \overline{f_{jg}^1(\chi)} f_{jg}^2(\chi_{v\mu}) \end{aligned}$$

(3.26)

where χ is in the principal series and $\chi_{v,\mu}$ is chosen such that it goes into χ under application of $\Gamma(v,\mu)$. This allows us to bring (2.32) in the concise form

$$\begin{aligned} \langle \Phi^1 P_0 | \Phi^2 \rangle &= 16\pi^2 M^3 (2S+1) \sum_{m=-2S}^{2S} \int_0^\infty d\rho \\ &\times \sum_{\nu, \mu} (-1)^{\nu+\mu+1} \gamma_{1-\nu, 1-\mu}^S(\chi) \int D\bar{z} \overline{f_1(z, \chi)} \\ &\times \Gamma(\nu, \mu) f_2(z, \chi_{\nu, \mu}) \end{aligned} \quad (3.27)$$

We close our investigations with a proof of (2.33). From (1.33), the Clebsch-Gordan coefficients, [Ref. 6], Eq. (3.5.13)]

$$(S q_1, S q_2 | 00) = \delta_{q_1, -q_2} (-1)^{S-q_1} (2S+1)^{-\frac{1}{2}}$$

the symmetry relations (1.23), (1.24), and the definition (1.21) we get for all χ_1, χ_2, χ on the principal series

$$\begin{aligned} \sum_q G(\bar{\chi}_1, \chi_2, \chi | S \frac{1}{2} q; S \frac{1}{2} q; 000) &= \\ &= 4\pi (2S+1)(2\frac{1}{2}+1) (-1)^{q_1 - q_1' + \frac{1}{2}(m_1 - m_2)} \\ &\times \left\{ \int N(z, z_1, z_2 | \chi, \chi_1, -\chi_2) f_0^0(z)^{\chi} f_{-q_1}^S(z_1)^{\chi_1} f_{q_1}^S(z_2)^{-\chi_2} D\bar{z} D\bar{z}_1 D\bar{z}_2 \right. \\ &\times \int N(z', z'_1, z'_2 | -\chi, -\chi_1, \chi_2) f_0^0(z')^{-\chi} f_{q_1}^{\frac{1}{2}}(z'_1)^{-\chi_1} f_{-q_1}^{\frac{1}{2}}(z'_2)^{\chi_2} D\bar{z}' D\bar{z}'_1 D\bar{z}'_2 \\ &\left. + \text{the same expression with } \chi \text{ replaced by } -\chi \right\} \end{aligned} \quad (3.28)$$

If we continue in q till $2ni$, $n=1,2,3,\dots$ but stay off the poles, $D_{-\chi}$ obtains an invariant subspace $E_{-\chi}$ which contains the function (L), (3.83), (3.84)

$$f_0^0(z)^{-\chi} = \pi^{-\frac{1}{2}} (1+z\bar{z})^{n-1}$$

Since the integral over z'_1 and z'_2 in the first term in the curly bracket (respectively z_1 and z_2 in the second term), gives a

function of F_χ (Section 3.1), in both terms one integral is zero whereas the other is finite. This completes the proof.

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