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THE $o(3,1)$ ANALYSIS OF CURRENT MATRIX ELEMENTS
AND THE ASYMPTOTIC EXPANSION OF FORM FACTORS

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ABSTRACT

Matrix elements of current density operators between single-particle states are submitted to harmonic analysis of the homogeneous Lorentz group.

1. INTRODUCTION

The matrix elements of current density operators between single-particle states can be submitted to harmonic analysis on the homogeneous Lorentz group applying a method which is quite similar to that used for the reduction of the momentum operator on the homogeneous Lorentz group ¹⁾. The result of this harmonic analysis can be described as follows. The invariant functions involved in the matrix elements, the form factors, are Fourier transformed, eventually in the generalized sense of distributions. In this respect and in its consequences our formalism resembles the group theoretical approach to Regge poles and Toller's analysis of the forward scattering amplitude ²⁾. For the presentation of these theories in the context of Fourier transformations of distributions on $SL(2, C)$ and $SU(1, 1)$ see Ref. ³⁾, Chapter 8. The covariant nature of the current, e.g., the fact that the current is a four-vector, complicates this simple procedure considerably. The Fourier transforms of the form factors are entangled with direct integrals of covariant operators on the homogeneous Lorentz group. In the case of the vector current which we study in detail, these covariant operators are the four possible types of infinite dimensional, generalized, Dirac matrices ^{1), 3), 4), 5)}.

The Regge pole theory of scattering processes enables us to derive asymptotic expansions of the scattering amplitude. Assuming that the Fourier transforms of the form factors are meromorphic in the group invariants, we obtain asymptotic expansions of the form factors by the same arguments. The physical image attributed to these poles is, however, quite different in both theories. Whereas Regge and Toller poles are usually identified with interpolations of strongly interacting particles and resonances, which are exchanged between the particles scattered, it turns out that the poles in the Fourier transforms of the form factors are possibly connected with a tower structure of the external particles. Of course, it has to be assumed that the same pole appears in the matrix elements of any pair of particles of these towers, but such an assumption has its counterpart in Regge pole theory. The towers are described by irreducible representations of the homogeneous Lorentz group.

Nevertheless, there exists a domain where both types of $O(3, 1)$ analyses overlap. We assume, with Salam ⁶⁾, that a family of Regge

trajectories which belong to one Toller pole at $t=0$, stay almost parallel and fit into one representation of $O(3,1)$ also at $t \neq 0$. It might then happen that this family of trajectories creates a tower of particles or resonances which have almost the same mass. When such a family of Regge poles is exchanged in a process induced by a real or virtual photon, we have a situation of the asserted kind.

The main difference between the two types of $O(3,1)$ analyses can be summarized in the formula : Toller's analysis is based on the property of $O(3,1)$ to act as a little group; our analysis uses the fact that any simple Lie group applied in a theory of non-compact symmetry groups of the type investigated by Barut ⁷⁾ ("dynamical groups") or Fronsdal ⁸⁾ ("infinite component fields") has to contain $O(3,1)$ as a subgroup.

The applications of our formalism are numerous. It can first be used as a systematic scheme to classify the asymptotic behaviour of form factors without reference to any physical picture. It may further serve as a tool to overcome the trial-and-error period of the theories of non-compact symmetry groups, since our analysis starts with phenomenological form factors and allows us to study boundary conditions imposed on such models by analyticity, current algebras, etc. In this context, an attempt to classify the solutions of the Gell-Mann - Dashen model of current algebra might be of particular interest. We propose to enquire into the restrictions on the singularities of the Fourier transforms of the form factors imposed by the algebra, and to see whether these singularities fit into representations of $O(3,1)$.

Space inversion, time reflection and internal symmetries, are not yet included in our formalism, but this can be done easily by extending the homogeneous Lorentz group or by using the known transformation properties of the vertex functions in the brick-wall frame ⁹⁾ as subsidiary information. In the latter fashion we can also introduce current conservation. Instead of taking account of PCT invariance in Toller's manner ²⁾, which would imply a simultaneous treatment of the timelike and spacelike region of the form factors, it is probably also more convenient to use dispersion relations for the form factors and directly analyze the cut functions. In order to accommodate those

readers who are mainly interested in applications, we have added two appendices. In Appendix 1, we translate the elegant formalism of co-variant wave functions which is throughout used in the formal parts of this article for the description of single particle states, into the familiar formalism of wave functions in momentum space and plane wave states. As a standard for the notations we use the classical articles by Joos ¹⁰⁾. In Appendix 2, we give the formulae translating between the familiar vector basis and the somewhat less familiar canonical basis of the vector representation of $O(3,1)$. For the representations of $O(3,1)$ we use the notations introduced in (I), Section 1.1, which we do not repeat in this article. Instead of $O(3,1)$ we shall always use the group $SL(2,C)$. The relation between these groups is also standardized in Appendix 1.

2. VERTEX FUNCTIONS

We introduce the vertex functions as matrix elements of a current density operator $j_\mu(x)$ between single-particle states of arbitrary mass M and spin S (Appendix 1)

$$\langle p_2, q_2 | j_\mu(0) | p_1, q_1 \rangle = N_1 N_2 \Gamma_\mu(p_2, p_1)_{q_2 q_1} \quad (1)$$

$N_{1,2}$ are normalization factors

$$N^2 = \frac{2M}{(2\pi)^3}$$

The current density is assumed to satisfy

$$\begin{aligned} u_a^{-1} j_\mu(0) u_a &= \Lambda_\mu^\nu(a) j_\nu(0) \\ a &\in SL(2,C), \quad \Lambda \in O(3,1) \end{aligned} \quad (2)$$

We emphasize that a vertex function is usually identified with the quantity

$$N_1^{-1} N_2^{-1} [D_F(p_2 - p_1)]^{-1} D_F^0(p_2 - p_1) \langle p_2, q_2 | j(0) | p_1, q_1 \rangle$$

where D_F (D_F^0) is the Feynman propagator for the interacting (free) particle coupled to the current. In weak and electromagnetic interactions to lowest order, in particular in the case of the electromagnetic and weak nucleon form factors, we can neglect this difference.

Those vectors Φ of the Hilbert space $L^2(M, S)$ (Appendix 1) whose wave functions $\Phi_q(p)$ have compact support and are infinitely differentiable, form a linear subspace $C_c^\infty(M, S)$ of $L^2(M, S)$. For two such vectors, we have from (A.10)

$$\begin{aligned} \langle \Phi^2 | j_\mu | \Phi^1 \rangle &= \\ &= N_1 N_2 \sum_{q_1, q_2} \int d\mu(p_1) d\mu(p_2) \overline{\Phi_{q_2}^2(p_2)} \Gamma_\mu(p_2, p_1)_{q_2 q_1} \Phi_{q_1}^1(p_1) \quad (3) \end{aligned}$$

We assume throughout that (3) defines a continuous linear functional on $C_c^\infty(M_1, S_1)$ when Φ^2 is held fixed, and an antilinear continuous functional on $C_c^\infty(M_2, S_2)$ when Φ^1 is held fixed. This assumption is called "premise zero". In addition, it may happen that

$$\sum_{q_1} \int d\mu(p_1) \Gamma_\mu(p_2, p_1)_{q_2 q_1} \Phi_{q_1}^1(p_1)$$

is a function of $L^2(M_2, S_2)$ for all $\Phi_{q_1}^1(p_1) \in C_c^\infty(M_1, S_1)$. An assumption of this kind is denoted by "premise one". Premise one is fulfilled if the form factors are continuous and have an appropriate asymptotic behaviour in $q^2 = (p_2 - p_1)^2 \rightarrow -\infty$. In the case of the electromagnetic form factors of the nucleons whose mass is assumed to be identical, it can be verified easily that

$$\begin{aligned} \overline{\lim}_{q^2 \rightarrow -\infty} |F_1(q^2) q^{2\alpha}| &< \infty \\ \overline{\lim}_{q^2 \rightarrow -\infty} |F_2(q^2) q^{2+2\alpha}| &< \infty \end{aligned}$$

for any $\alpha > \frac{3}{2}$ is sufficient for the validity of premise one.

The notion of the vertex function can be extended in several directions. We need not only consider vector currents, but may admit any covariant operators transforming like a completely irreducible representation of the group $SL(2, \mathbb{C})$. In particular, it is crucial for our approach to permit unitary representations of the principal series. Operators transforming like a finite-dimensional representation of $SL(2, \mathbb{C})$ and their decomposition on $SL(2, \mathbb{C})$ are obtained by analytic continuation in the group invariants of corresponding operators transforming like representations of the principal series. As usual, we give the infinite-dimensional representations in the canonical basis, $[(L), (3-83)-(3-84)]$. We write the vertex functions correspondingly

$$\Gamma_Q^J(p_2, p_1 | \chi)_{q_2 q_1}$$

Another generalization obtains if we replace the wave functions in (3) by covariant wave functions (A.13)-(A.14). With the definitions

$$\Gamma_Q^J(a(p_2), a(p_1) | \chi)_{q_2 q_1} = \Gamma_Q^J(p_2, p_1 | \chi)_{q_2 q_1} \quad (4)$$

and

$$\begin{aligned} & \Gamma_Q^J(u_2 a_2, u_1 a_1 | \chi)_{q_2 q_1} = \\ & = \sum_{q'_1 q'_2} D_{q_2 q'_2}^{S_2}(u_2) D_{q'_1 q_1}^{S_1}(u_1^{-1}) \Gamma_Q^J(a_2, a_1 | \chi)_{q'_2 q'_1} \end{aligned} \quad (5)$$

where $a(p)$ is a boost and $u_{1,2}$ are elements of $SU(2)$, and with the unique decomposition

$$\begin{aligned} a &= u a(p) \\ d\mu(u) d\mu(p) &= c d\mu(a), \quad c = 8\pi^2 M^2 \end{aligned} \quad (6)$$

we have from (3)

$$\begin{aligned}
 & \langle \Phi^2 | j_Q^j(x) | \Phi^1 \rangle \\
 &= N_1 N_2 c_1 c_2 \int d\mu(a_1) d\mu(a_2) \sum_{q_1 q_2} \overline{\Phi_{q_2}^2(a_2)} \Gamma_Q^j(a_2, a_1 | \chi) \Phi_{q_1}^1(a_1) \\
 &= (2S_1 + 1) N_1 c_1 (2S_2 + 1) N_2 c_2 \\
 &\quad \times \int d\mu(a_1) d\mu(a_2) \overline{\Phi_{q_2}^2(a_2)} \Gamma_Q^j(a_2, a_1 | \chi)_{q_2 q_1} \Phi_{q_1}^1(a_1)
 \end{aligned} \tag{7}$$

In the second form of (7), q_1 and q_2 are arbitrary but fixed. The covariance relation (2) implies

$$\begin{aligned}
 & \Gamma_Q^j(a_2 a^{-1}, a, a^{-1} | \chi)_{q_2 q_1} = \\
 &= \sum_{j' q'} D_{j q j' q'}^{\chi}(a) \Gamma_{Q'}^{j'}(a_2, a_1 | \chi)_{q_2 q_1}
 \end{aligned} \tag{8}$$

3. REDUCED MATRIX ELEMENTS OF THE VECTOR OPERATORS OF $SL(2, C)$

Our investigations in (I) suggest that any vector operators like the momentum or vector current operators if decomposed on the group $SL(2, C)$, lead to a direct integral of the four types of generalized Dirac matrices. These generalized Dirac matrices were given in the form of differential operators in $z, \bar{z}$ and first order polynomials in $x, \bar{x}$ in [(I), (3-14)-(3-17)] or in [(L), (3-100)-(3-103)]. With the help of the canonical basis vectors (A.22) they can be decomposed as

$$\begin{aligned}
 T_{\nu\mu}(z|x) &= \sum_{jq} f_q^j(x) T_q^j(z|\nu, \mu) \\
 \nu, \mu &= 0, 1
 \end{aligned} \tag{9}$$

where the operators T_q^j are differential operators in $z, \bar{z}$ satisfying the covariance condition

$$T_{a^{-1}}^{\chi_2} T_q^j T_a^{\chi_1} = \sum_{j'q'} D_{jqj'q'}^{\chi}(\alpha) T_{q'}^{j'} \quad (10)$$

In (10) χ denotes the vector representation. χ_1 and χ_2 are certain completely irreducible representations of $SL(2, \mathbb{C})$ which were shown in [(I), Chapter 3], to be necessarily related by

$$\begin{aligned} \chi_2 &= (m_2, p_2), \quad \chi_1 = (m_1, p_1), \quad \chi_1 = (\chi_2)_{\nu\mu} \\ p_2 &= p_1 + 2i(\nu + \mu - 1) \\ m_2 &= m_1 + 2(\nu - \mu) \end{aligned}$$

For the purposes of this article, it suffices to know the operators T_0^0 and T_0^1 in all four cases $\nu, \mu = 0, 1$. We find

$$\begin{aligned} T_0^0(z|00) &= 1 + z\bar{z} \\ \sqrt{3} T_0^1(z|00) &= T_0^0(z|00) - 2 \end{aligned} \quad (11)$$

$$\begin{aligned} T_0^0(z|10) &= (1 + z\bar{z}) \frac{\partial}{\partial z} - (n_1 - 1)\bar{z} \\ \sqrt{3} T_0^1(z|10) &= T_0^0(z|10) - 2 \frac{\partial}{\partial z} \end{aligned} \quad (12)$$

$$\begin{aligned} T_0^0(z|01) &= (1 + z\bar{z}) \frac{\partial}{\partial \bar{z}} - (n_1' - 1)z \\ \sqrt{3} T_0^1(z|01) &= T_0^0(z|01) - 2 \frac{\partial}{\partial \bar{z}} \end{aligned} \quad (13)$$

$$\begin{aligned} T_0^0(z|11) &= (1 + z\bar{z}) \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} - (n_1 - 1)\bar{z} \frac{\partial}{\partial \bar{z}} \\ &\quad - (n_1' - 1)z \frac{\partial}{\partial z} + (n_1 - 1)(n_1' - 1) \end{aligned} \quad (14)$$

$$\sqrt{3} T_0^1(z|11) = T_0^0(z|11) - 2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$$

where

$$n_1 = -\frac{1}{2}m_1 + \frac{i}{2}\rho_1, \quad n_1' = +\frac{1}{2}m_1 + \frac{i}{2}\rho_1$$

The over-all normalization has been adjusted to [I], (3-18).

Covariant operators can be constructed for every representation χ of $SL(2, \mathbb{C})$. The covariance condition is

$$T_{a^{-1}}^{\chi_2} T_q^j(\chi) T_a^{\chi_1} = \sum_{j'q'} D_{jqj'q'}^{\chi}(a) T_{q'}^{j'}(\chi) \quad (15)$$

which generalizes (10). For infinite-dimensional representations χ , χ_1 and χ_2 are only restricted by the requirement that $m+m_1+m_2$ is even. If χ_2 is in the principal series, we can define the matrix elements of $T_q^j(\chi)$ by

$$\begin{aligned} \langle \chi_2, j_2 Q_2 | T_q^j(\chi) | \chi_1, j_1 Q_1 \rangle &= \\ &= \int DZ \overline{f_{Q_2}^{j_2}(Z)^{\chi_2}} T_q^j(Z|\chi) f_{Q_1}^{j_1}(Z)^{\chi_1} \end{aligned} \quad (16)$$

If χ_2 is not in the principal series, we may instead use the definition

$$\begin{aligned} \langle \chi_2, j_2 Q_2 | T_q^j(\chi) | \chi_1, j_1 Q_1 \rangle &= \\ &= (-1)^{\frac{1}{2}m_2 - Q_2} \int DZ f_{-Q_2}^{j_2}(Z)^{-\chi_2} T_q^j(Z|\chi) f_{Q_1}^{j_1}(Z)^{\chi_1} \end{aligned} \quad (17)$$

where we used [I], (1-23). Definition (17) may be regarded as the analytic continuation of (16) off the real ρ_2 axis. In both cases we have the expansion

$$\begin{aligned} T_q^j(Z|\chi) f_{Q_1}^{j_1}(Z)^{\chi_1} &= \\ &= \sum_{j_2 Q_2} \langle \chi_2, j_2 Q_2 | T_q^j(\chi) | \chi_1, j_1 Q_1 \rangle f_{Q_2}^{j_2}(Z)^{\chi_2} \end{aligned} \quad (18)$$

Due to the covariance condition (15), the matrix elements (16), (17) can be derived from reduced matrix elements. These can be defined by

$$\begin{aligned}
 \langle \chi_{2i} j_2 Q_2 | T_q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle &= \\
 &= (-1)^{2j} (2j_1 + 1)^{-\frac{1}{2}} (j_2 Q_2 i j_1 j_1 Q_1) \langle \chi_{2i} j_2 || T^j(\chi) || \chi_{1i} j_1 \rangle \\
 &= (-1)^{j_2 + j + Q_1} \begin{pmatrix} j_2 j j_1 \\ Q_2 q - Q_1 \end{pmatrix} \langle \chi_{2i} j_2 || T^j(\chi) || \chi_{1i} j_1 \rangle
 \end{aligned} \tag{19}$$

with the help of the familiar Clebsch-Gordan and Wigner coefficients of SU(2). In fact, from (15) it can be deduced that

$$\begin{aligned}
 \langle \chi_{2i} j_2 Q_2 | T_q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle &= \\
 &= \sum_{q' Q'_1 Q'_2} \overline{D_{Q_1 Q'_1}^{j_1}(\mu)} D_{qq'}^j(\mu) D_{Q_2 Q'_2}^{j_2}(\mu) \langle \chi_{2i} j_2 Q'_2 | T_{q'}^j(\chi) | \chi_{1i} j_1 Q'_1 \rangle
 \end{aligned}$$

If the right-hand side is averaged over SU(2), it results that

$$\begin{aligned}
 \langle \chi_{2i} j_2 Q_2 | T_q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle &= (2j_1 + 1)^{-1} (j_2 Q_2 i j_1 j_1 Q_1) \\
 &\times \sum_{q' Q'_1 Q'_2} (j_1 Q'_1 j_2 Q'_2 i j_1 j_1 Q'_1) \langle \chi_{2i} j_2 Q'_2 | T_{q'}^j(\chi) | \chi_{1i} j_1 Q'_1 \rangle
 \end{aligned}$$

This proves the justification of (19) and the inverse

$$\begin{aligned}
 \langle \chi_{2i} j_2 || T^j(\chi) || \chi_{1i} j_1 \rangle &= \\
 &= (-1)^{2j} (2j_1 + 1)^{-\frac{1}{2}} \sum_{q Q_1 Q_2} (j_1 Q_1 j_2 Q_2 i j_1 j_1 Q_1) \langle \chi_{2i} j_2 Q_2 | T_q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle
 \end{aligned} \tag{20}$$

In order to find the reduced matrix elements of the vector operators, we apply the operators (11) to (14) to the basis vectors $f_Q^J(z)$, $[L]$, (3-83), (3-84) and use (18) and (19). This gives

$$\underline{\nu = \mu = 0}$$

$$\begin{aligned} \langle \chi_{2i} j \| T^0 \| \chi_{1i} j \rangle &= (2j+1)^{\frac{1}{2}} \gamma_{00}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j \rangle &= -\frac{1}{2} m_1 \left[\frac{2j+1}{j(j+1)} \right]^{\frac{1}{2}} \gamma_{00}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j-1 \rangle &= + \left[\frac{1}{j} (j-\frac{1}{2} m_1)(j+\frac{1}{2} m_1) \right]^{\frac{1}{2}} \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j+1 \rangle &= - \left[\frac{1}{j+1} (j-\frac{1}{2} m_1+1)(j+\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \end{aligned} \quad (21)$$

$$\underline{\nu = 1, \mu = 0}$$

$$\begin{aligned} \langle \chi_{2i} j \| T^0 \| \chi_{1i} j \rangle &= (2j+1)^{\frac{1}{2}} \gamma_{10}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j \rangle &= -\frac{1}{2} \rho_1 \left[\frac{2j+1}{j(j+1)} \right]^{\frac{1}{2}} \gamma_{10}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j-1 \rangle &= + (j-\frac{1}{2} \rho_1) \left[\frac{1}{j} (j+\frac{1}{2} m_1)(j+\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j+1 \rangle &= - (j+\frac{1}{2} \rho_1+1) \\ &\quad \times \left[\frac{1}{j+1} (j-\frac{1}{2} m_1)(j-\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \end{aligned} \quad (22)$$

$$\underline{\nu = 0, \mu = 1}$$

$$\begin{aligned} \langle \chi_{2i} j \| T^0 \| \chi_{1i} j \rangle &= (2j+1)^{\frac{1}{2}} \gamma_{01}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j \rangle &= + \frac{1}{2} \rho_1 \left[\frac{2j+1}{j(j+1)} \right]^{\frac{1}{2}} \gamma_{01}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j-1 \rangle &= - (j-\frac{1}{2} \rho_1) \left[\frac{1}{j} (j-\frac{1}{2} m_1)(j-\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j+1 \rangle &= + (j+\frac{1}{2} \rho_1+1) \\ &\quad \times \left[\frac{1}{j+1} (j+\frac{1}{2} m_1)(j+\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \end{aligned} \quad (23)$$

$$\underline{\nu = \mu = 1}$$

$$\begin{aligned} \langle \chi_{2i} j \| T^0 \| \chi_{1i} j \rangle &= (2j+1)^{\frac{1}{2}} \gamma_{\nu\mu}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j \rangle &= +\frac{1}{2} m_1 \left[\frac{2j+1}{j(j+1)} \right]^{\frac{1}{2}} \gamma_{\nu\mu}^j(\chi_1) \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j-1 \rangle &= -(j-\frac{1}{2} \varrho_1)(j-\frac{1}{2} \varrho_1+1) \\ &\quad \times \left[\frac{1}{j} (j+\frac{1}{2} m_1)(j-\frac{1}{2} m_1) \right]^{\frac{1}{2}} \\ \sqrt{3} \langle \chi_{2i} j \| T^1 \| \chi_{1i} j+1 \rangle &= +(j+\frac{1}{2} \varrho_1)(j+\frac{1}{2} \varrho_1+1) \\ &\quad \times \left[\frac{1}{j+1} (j+\frac{1}{2} m_1+1)(j-\frac{1}{2} m_1+1) \right]^{\frac{1}{2}} \end{aligned} \quad (24)$$

with the coefficients $\gamma_{\nu\mu}^j(\chi)$ defined by [(I), (3-24)]. For the interested reader who wants to check these results, we mention that it suffices to apply the operators T_0^0 and T_0^1 to the functions $f_J^j(z)$.

By a comparison of the results (21) to (24), we find the symmetry

$$\begin{aligned} \langle \chi_{2i} j_2 \| T^j(\nu, \mu) \| \chi_{1i} j_1 \rangle &= \\ = -\beta^{j_2}(\chi_2)^{-1} \beta^{j_1}(\chi_1) \langle -\chi_{2i} j_2 \| T^j(1-\nu, 1-\mu) \| -\chi_{1i} j_1 \rangle \end{aligned} \quad (25)$$

with

$$\beta^j(\chi) = \frac{\Gamma(j+\frac{1}{2} \varrho+1)}{\Gamma(j-\frac{1}{2} \varrho+1)}$$

In other words, the operators $T_{\nu\mu}$ and $T_{1-\nu, 1-\mu}$ are related by the intertwining operator A [(L), Section 3-9] as

$$T_{\nu\mu} = -A T_{1-\nu, 1-\mu} A^{-1} \quad (26)$$

4. FOURIER TRANSFORMATIONS OF VERTEX FUNCTIONS

A vertex function which belongs to a current transforming as a representation χ of the principal series has the same covariance properties (5), (8), as the matrix element

$$\langle \chi_2, S_2 q_2 | T_{a_2}^{\chi_2} T_Q^J(\chi) T_{a_1}^{\chi_1} | \chi_1, S_1 q_1 \rangle \quad (27)$$

In (27) $T_Q^J(\chi)$ denotes a covariant operator satisfying the condition (15), and the matrix element is defined as in (16)-(18). One might therefore suggest that the Fourier expansion of the vertex function is an expansion in terms of the matrix elements (27). That this is indeed so, will be shown at the end of this Section.

We set out from Eq. (7) and insert the Fourier transformed wave function $\Phi_q(a) \in C_c^\infty(M, S)$, [(I), (2-8)]

$$\begin{aligned} \Phi_q(a) &= \frac{1}{2} \sum_{m=-2S}^{2S} \int_{-\infty}^{\infty} d\rho (p^2 + m^2) \sum_{j'q'} D_{S_q j'q'}^{\chi}(\rho) \bar{F}_{j'q'}(\chi) \\ &\equiv \frac{1}{2} \int d\chi \sum_{j'q'} D_{S_q j'q'}^{\chi}(\rho) \bar{F}_{j'q'}(\chi) \end{aligned} \quad (28)$$

This gives

$$\begin{aligned} \langle \Phi^2 | j_Q^J(\chi) | \Phi^1 \rangle &= (2S_1+1) N_1 c_1 (2S_2+1) N_2 c_2 \\ &\times \frac{1}{2} \int d\chi_2 \sum_{j''q''} \overline{\bar{F}_{j''q''}^2(\chi_2)} \frac{1}{2} \int d\chi_1 \sum_{j'q'} \bar{F}_{j'q'}^1(\chi_1) \\ &\times \int d\mu(a_1) d\mu(a_2) \overline{D_{S_2 q_2 j''q''}^{\chi_2}(a_2)} D_{S_1 q_1 j'q'}^{\chi_1}(a_1) \Gamma_Q^J(a_2, a_1 | \chi)_{q_2 q_1} \end{aligned} \quad (29)$$

We set

$$\begin{aligned} a_1 &= u_1 d(\eta_1)^{-1} a \\ a_2 &= u_2 d(\eta_2) a \end{aligned} \quad (30)$$

with $d(\eta)$ defined by

$$d(\eta) = e^{+\frac{1}{2}\eta\sigma_3}, \quad \eta \geq 0 \quad (31)$$

With the notation

$$\eta = \eta_1 + \eta_2, \quad 0 \leq \eta < \infty$$

and the Jacobian $[[L], \text{Appendix A-1d}]$, we have

$$\begin{aligned} d\mu(a_1) d\mu(a_2) &= d\mu(a_2 a_1^{-1}) d\mu(a) \\ &= \frac{1}{4\pi} d\mu(u_1) d\mu(u_2) sh^2 \eta d\eta d\mu(a) \end{aligned} \quad (32)$$

With (30) and (32), the integral over a_1 and a_2 in (29) takes the form

$$\begin{aligned} & [4\pi(2S_1+1)(2S_2+1)]^{-1} \int_0^\infty d\eta sh^2 \eta \sum_{j_2 Q_2} d_{S_2 j_2 Q_2}^{\bar{\chi}_2}(\eta_2) \sum_{j_1 Q_1} d_{S_1 j_1 Q_1}^{\chi_1}(\eta_1) \\ & \times \sum_{j' Q'} \Gamma_{Q'}^{j'}(d(\eta_2), d(\eta_1)^{-1} | \chi)_{Q_2 Q_1} \int d\mu(a) \overline{D_{j_2 Q_2 j'' Q''}^{\chi_2}(a)} \\ & \times D_{j_1 Q_1 j' Q'}^{\chi_1}(a) D_{j Q j' Q'}^{\chi}(a^{-1}) \end{aligned} \quad (33)$$

where $\bar{\chi}$ denotes the representation $(m, -\rho)$ if $\chi = (m, \rho)$. This integral over a can be expressed by the G kernel function $[[I], (1-32)]$ (we set $a \rightarrow a^{-1}$)

$$\begin{aligned} & \int d\mu(a) \overline{D_{j' Q' j_1 Q_1}^{\chi_1}(a)} D_{j'' Q'' j_2 Q_2}^{\chi_2}(a) D_{j Q j' Q'}^{\chi}(a) \\ & = [4\pi(2j+1)(2j'+1)]^{-1} \sum_{q_1 q_2 Q''} (j_1 Q_1 | j_2 Q_2 j' Q') \\ & \times (j_2 q_2 j' Q'' | j_1 q_1) (j' q_1 | j'' q_2 j Q'') (j'' q'', j Q | j' q') \\ & \times G(\bar{\chi}_1, \chi_2, \chi | j' j_1 q_1, j'' j_2 q_2, j j' Q'') \end{aligned} \quad (34)$$

Altogether we obtain

$$\begin{aligned}
 \langle \Phi^2 | j_Q^j(\chi) | \Phi' \rangle &= (8\pi)^{-2} N_1 c_1 N_2 c_2 \\
 &\times \int d\chi_1 d\chi_2 \sum_{j'q'j''q''} \overline{\mathcal{F}_{j''q''}^2(\chi_2)} (j''q''; j_Q | j'q') \mathcal{F}_{j'q'}^1(\chi_1) \\
 &\times \sum_{j_1 j_2 j'} \left\{ (2j_1+1)^{-\frac{1}{2}} \sum_{Q_1 Q_2 Q'} (j_1 Q_1 | j_2 Q_2; j' Q') \right. \\
 &\quad \times \int_0^\infty d\eta \sinh^2 \eta \, d_{S_2 j_2 Q_2}^{\bar{j}_2}(\eta_2) d_{S_1 j_1 Q_1}^{\chi_1}(-\eta_1) \Gamma_{Q'}^{j'}(\dots | \chi)_{Q_2 Q_1} \Big\} \\
 &\times \left\{ (2j'+1)^{-1} (2j_1+1)^{-\frac{1}{2}} \sum_{q_1 q_2 Q''} (j'q_1 | j''q_2; j_Q'') (j_2 q_2; j'Q'' | j_1 q_1) \right. \\
 &\quad \times G(\bar{\chi}_1, \chi_2, \chi | j' j_1 q_1; j'' j_2 q_2; j j' Q'') \Big\}
 \end{aligned} \tag{35}$$

In many cases, knowledge of j_0^0 is sufficient because of the covariance (8). Instead of (35) we get the simpler formula

$$\begin{aligned}
 \langle \Phi^2 | j_0^0(\chi) | \Phi' \rangle &= (8\pi)^{-2} N_1 c_1 N_2 c_2 \\
 &\times \int d\chi_1 d\chi_2 \sum_{j_1 q_1} \overline{\mathcal{F}_{j_1 q_1}^2(\chi_2)} \mathcal{F}_{j_1 q_1}^1(\chi_1) \\
 &\times \sum_{j_1 j_2 j'} \left\{ \dots \right\} \\
 &\times \left\{ (2j'+1)^{-1} (2j_1+1)^{-\frac{1}{2}} \sum_{q_1'} (j_2 q_1' | j_0 | j_1 q_1') \right. \\
 &\quad \times G(\bar{\chi}_1, \chi_2, \chi | j j_1 q_1'; j j_2 q_1'; 0 j_0) \Big\}
 \end{aligned} \tag{36}$$

The first curly bracket remained unaltered.

The angles η_1 and η_2 are not completely determined by (30). In addition, we may submit them to a condition which selects the Lorentz system in which the form factors are Fourier analyzed. The following possibilities might be of interest :

- 1) $\eta_1 = 0, \eta_2 = \eta$: in this system particle 1 is at rest;
- 2) $\eta_1 = \eta, \eta_2 = 0$: in this system particle 2 is at rest;
- 3) $\eta_1 = \eta_2 = \frac{1}{2} \eta$: in this system particles 1 and 2 have equal velocities of opposite direction;
- 4) $M_1 \text{sh} \eta_1 = M_2 \text{sh} \eta_2$: this is the brick-wall system;
- 5) $M_1 \text{ch} \eta_1 = M_2 \text{ch} \eta_2$: in this system the little group of the momentum transfer is $SU(1,1)$, see Toller ²⁾.

If $M_1 = M_2 = M$, systems 3)-5) coincide. If $\eta_1 = 0$ or $\eta_2 = 0$, Eqs. (35) and (36) simplify somewhat due to

$$d_{s_2, j_2, Q_2}^{\bar{\chi}_2}(0) = \delta_{s_2, j_2}, \quad d_{s_1, j_1, Q_1}^{\chi_1}(0) = \delta_{s_1, j_1}$$

In addition, we note that the vertex function

$$\Gamma_Q^J(d(\eta_2), d(\eta_1)^{-1} | \chi)_{Q_2 Q_1}$$

vanishes unless

$$Q_2 + Q - Q_1 = 0$$

An expansion of this vertex function into a series of multipole form factors ⁹⁾ makes this condition explicit.

As promised, we show now that the result (35) can be regarded as a Fourier expansion obtained with the functions (27). This matrix element (27) can itself be expanded as

$$\sum_{j_2, Q_2, j_2, Q_2} D_{s_2, j_2, Q_2}^{\chi_2}(a_2) D_{j_1, Q_1, s_1, Q_1}^{\chi_1}(a_1^{-1}) \langle \chi_2, j_2, Q_2 | T_Q^J(\chi) | \chi_1, j_1, Q_1 \rangle$$

The covariant operator $T_Q^J(\chi)$ is expressible by Naimark's kernel N , [I], Section 1-2]. Fixing the proportionality factor conveniently, we have

$$\begin{aligned}
 & \langle \chi_{2i} j_2 Q_2 | T_Q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle^* = \\
 & = (-1)^{\frac{1}{2} m_1 - Q_1} \int D\bar{z}_1 D\bar{z} D\bar{z}_2 N(-\chi_1, \chi, \chi_2 | z_1, z, z_2) \\
 & \quad \times f_{Q_2}^{j_2}(z_2)^{\chi_2} f_Q^j(z)^{\chi} f_{-Q_1}^{j_1}(z_1)^{-\chi_1}
 \end{aligned} \tag{37}$$

where * denotes the complex conjugate if all three variables g , g_1, g_2 are on the real axis, but the analytic continuation of this complex conjugate in the general case. The functions f_q^j are elements of the canonical basis $[(I), (1-7)]$ or $[(L), (3-83), (3-84)]$. With the notation $[(I), (1-21)]$ of the Clebsch-Gordan coefficients of $SL(2, \mathbb{C})$ we have therefore

$$\begin{aligned}
 & \langle \chi_{2i} j_2 Q_2 | T_Q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle^* = \\
 & = (\chi_1 j_1 Q_1 | \chi j Q | \chi_2 j_2 Q_2)
 \end{aligned} \tag{38}$$

and with $[(I), (1-22)]$

$$\begin{aligned}
 & \langle \chi_{2i} j_2 Q_2 | T_Q^j(\chi) | \chi_{1i} j_1 Q_1 \rangle = \\
 & = (\chi j Q | \chi_2 j_2 Q_2 | \chi_1 j_1 Q_1)
 \end{aligned} \tag{39}$$

Recalling $[(I), (1-28)]$, we inspect the relation

$$\begin{aligned}
 & \int d\mu(a) \overline{D_{j'q'j_1Q_1}^{\chi_1}(a)} D_{j''q''j_2Q_2}^{\chi_2}(a) D_{jQj'Q'}^{\chi}(a) \\
 & = 2 \langle \chi_{2i} j_2 Q_2 | T_{Q'}^{j'}(\chi) | \chi_{1i} j_1 Q_1 \rangle^* \\
 & \quad \times \langle \chi_{2i} j''q'' | T_Q^j(\chi) | \chi_{1i} j'q' \rangle
 \end{aligned} \tag{40}$$

which may replace formula (34). This yields finally

$$\begin{aligned}
 \langle \Phi^2 | j_Q^j(x) | \Phi^1 \rangle &= (8\pi)^{-1} N_1 c_1 N_2 c_2 \int d\chi_1 d\chi_2 \\
 &\times \sum_{j_1' j_2' q''} \overline{\mathcal{F}_{j_2' q''}^2(\chi_2)} \langle \chi_2, j_2' q'' | T_Q^j(x) | \chi_1, j_1' q' \rangle \mathcal{F}_{j_1' q'}^1(\chi_1) \\
 &\times \sum_{j_1 j_2 j'} \sum_{Q, Q_2 Q'} \langle \chi_2, j_2 Q_2 | T_{Q'}^{j'}(x) | \chi_1, j_1 Q_1 \rangle^* \\
 &\times \int_0^\infty d\eta \, \eta^2 d_{s_2 j_2 Q_2}^{\bar{\chi}_2}(\eta_2) d_{s_1 j_1 Q_1}^{\chi_1}(-\eta_1) \Gamma_{Q'}^{j'}(\chi)_{Q_2 Q_1}
 \end{aligned} \tag{41}$$

The alternative form (41) of (35) illustrates our assertion about the expansion of vertex functions in terms of the matrix elements of covariant operators (27).

5. THE ANALYTIC CONTINUATION IN χ

We consider the result (36) for the special representation

$$\chi = (0, \rho)$$

and continue it analytically in ρ till $\rho = -4i$. In this limit the representation becomes reducible. The irreducible sub-representation spanned by the vectors $J=0$ and $J=1$ is equivalent with the vector representation, as is shown explicitly in Appendix 2. We obtain the limit of (36) by evaluating the distribution theoretic limit of

$$\sum_q (j_2 q_1, j_0 | j_1 q) G(\bar{\chi}_1, \chi_2, \chi | j_1 q_1, j_2 q_1, 0 | 0)$$

just as we did in (I). We shall have to show later that this procedure is permitted by establishing the right analyticity properties of the functions to which this distribution is to be applied.

In the limit the integration contour is pinched by four pairs of poles. Each of these pairs is connected with a covariant four-vector operator and yields a delta function. The remainder of the distribution vanishes. To prove this latter assertion, we can rely on the same arguments as those used in [I], Section 3-2]. After tedious algebraic calculations, we obtain with the notations of Section 3,

$$\begin{aligned}
 & \lim_{\vartheta \rightarrow -4i} \sum_q (\gamma_2 q, \gamma_0 | \gamma_1 q) G(\bar{\chi}_1, \chi_2, \chi | j \gamma_1 q, j \gamma_2 q, 0 | 0) \\
 &= 8\pi (-1)^{j+1} (2j+1)(2j+1)(2j_1+1)^{\frac{1}{2}} [(m_1^2 + p_1^2)(m_2^2 + p_2^2)]^{-1} \\
 & \times \sum_{\nu\mu} \gamma_{\nu\mu}^j(\chi_1) \langle \bar{\chi}_2 | \gamma_2 \| T^j(1-\mu, 1-\nu) \| \bar{\chi}_1 | \gamma_1 \rangle \\
 & \times \delta_{m_1, m_2+2(\mu-\nu)} \delta(p_1, -p_2-2i(\nu+\mu-1)) \\
 & + \text{terms with } m_1, p_1 \text{ replaced by } -m_1, -p_1
 \end{aligned} \tag{42}$$

In (42), χ_1 and χ_2 are connected by $\chi_1 = (\chi_2)_{\nu\mu}$, Section 3. Because of the symmetry in χ_1 of the integrand (36), the terms not explicitly written in (42) yield the same contribution as those given. In this fashion, analytic continuation of (36) leads to the decomposition of the vector current

$$\begin{aligned}
 \langle \Phi^2 | j_0^0 | \Phi^1 \rangle &= (4\pi)^{-1} N_1 C_1 N_2 C_2 \\
 & \times \sum_{\nu\mu} \sum_m \int_{-\infty}^{+\infty} d\vartheta \sum_{jq} \overline{F_{jq}^2}(\chi) \gamma_{\nu\mu}^j(\chi_{\nu\mu}) F_{jq}^1(\chi_{\nu\mu}) \\
 & \times \sum_{j_1 j_2} (-1)^{j+1} (2j+1) \langle \bar{\chi}_1 | \gamma_2 \| T^j(1-\mu, 1-\nu) \| \bar{\chi}_{\nu\mu} | \gamma_1 \rangle \\
 & \times \sum_{Q_1 Q_2 Q} (2j_1+1)^{-\frac{1}{2}} (\gamma_1 Q_1 | \gamma_2 Q_2 | \gamma Q) \\
 & \times \int_0^\infty d\eta \eta^2 \eta d_{s_2 \gamma_2 Q_2}^{\bar{\chi}}(\eta_2) d_{s_1 \gamma_1 Q_1}^{\chi_{\nu\mu}}(-\eta_1) \Gamma_Q^j(\dots)_{Q_2 Q_1}
 \end{aligned} \tag{43}$$

The similarity of (43) with (41) can be elucidated further if we use (19) to write (note that J is an integer)

$$(2J+1)^{-\frac{1}{2}} (J_1 Q_1 | J_2 Q_2; J Q) \langle \bar{\chi}_i | J_2 || T^J(1-\mu, 1-\nu) || \bar{\chi}_{\nu\mu} | J_1 \rangle$$

$$= \langle \bar{\chi}_i | J_2 Q_2 | T_Q^J(1-\mu, 1-\nu) | \bar{\chi}_{\nu\mu} | J_1 Q_1 \rangle$$

By the replacement

$$\sum_{j_1 q_1} \overline{\mathcal{F}_{j_1 q_1}^2(\chi)} \gamma_{\nu\mu}^j(\chi_{\nu\mu}) \mathcal{F}_{j_1 q_1}^1(\chi_{\nu\mu}) \rightarrow$$

$$\sum_{j_1 q_1, j_2 q_2} \overline{\mathcal{F}_{j_2 q_2}^2(\chi)} \langle \chi_i | j_2 q_2 | T_Q^J(\nu, \mu) | \chi_{\nu\mu} | j_1 q_1 \rangle \mathcal{F}_{j_1 q_1}^1(\chi_{\nu\mu})$$

we can guess the corresponding decomposition of the matrix element

$$\langle \Phi^2 | j_Q^J | \Phi' \rangle$$

If we want to eliminate superfluous equivalent representations of the integral (43), i.e., integrate only over the positive real ϱ axis, we obtain only an additional factor of two. It is easy to check that (43) goes into [I], (3-27) if we choose the vertex function as an appropriate delta function and take into account that

$$(-1)^{\nu+\mu} \gamma_{1-\nu, 1-\mu}^S(\chi) = \gamma_{1-\mu, 1-\nu}^S(\bar{\chi}_{\nu\mu})$$

$$= \langle \bar{\chi}_i | S q | T_0^0(1-\mu, 1-\nu) | \bar{\chi}_{\nu\mu} | S q \rangle$$

We still have to establish that the limiting process leading to (43) is justified. If premise one (Section 2) is fulfilled, the vector Φ^1 has to be selected from the space $C_c^\infty(M_1, S_1)$ and $\mathcal{F}_{j_1 q_1}^1(\chi)$ is entire analytic in ϱ . On the other hand, we choose Φ^2 from $L^2(M_2, S_2)$ such that $\mathcal{F}_{j_2 q_2}^2(\chi)$ is in general only defined on the real ϱ axis and is square integrable in ϱ . From premise one and the Plancherel theorem for $SL(2, C)$ we know that the sums over J_1, J_2, J and Q_1, Q_2, Q and the integral over η can be interpreted in some fashion as an L^2 limit which we denote $\mathcal{F}_{j_1 q_1}^1(\chi)'$ and which is square integrable on the real ϱ axis. Consequently, the integral and sums in

$$\langle \Phi^2 | j_0^0 | \Phi' \rangle = (4\pi)^{-1} N_1 C_1 N_2 C_2 \sum_m \int_{-\infty}^{\infty} d\varphi \sum_{jq} \overline{\mathcal{F}_{jq}^2(\chi)} \mathcal{F}_{jq}^1(\chi)'$$

converge absolutely. The L^2 limit $\mathcal{F}_{jq}^1(\chi)'$ must in fact be independent of the manner in which we choose the relation between η_1 and η_2 . We want to illustrate this by investigating the conditions on the electromagnetic form factors of the nucleon, which we proved to be sufficient for the validity of premise one (Section 2).

Whatever the relation between η_1 and η_2 has been assumed, the square of the momentum transfer can be expressed as

$$q^2 = M_1^2 + M_2^2 - 2M_1 M_2 \cosh \eta, \quad \eta = \eta_1 + \eta_2 \quad (44)$$

In the brick-wall system $\eta_1 = \eta_2 = \frac{1}{2}$ (equal masses!), we have

$$\Gamma_0^0(\dots)_{\pm\frac{1}{2}, \pm\frac{1}{2}} = e\pi^{\frac{1}{2}} (F_1 + \frac{q^2}{4M^2} \mu F_2)$$

$$\Gamma_0^1(\dots)_{\pm\frac{1}{2}, \pm\frac{1}{2}} = 0$$

(from current conservation or from parity invariance, time reversal invariance and the Hermiticity of the current density operator) and

$$\Gamma_{\pm 1}^1(\dots)_{\mp\frac{1}{2}, \pm\frac{1}{2}} = e\left(\frac{2\pi}{3}\right)^{\frac{1}{2}} \left[-\frac{q^2}{4M^2}\right]^{\frac{1}{2}} (F_1 + \mu F_2)$$

We denote (Section 2)

$$\alpha = \frac{3}{2} + \varepsilon, \quad \varepsilon > 0$$

From [(I), (1-37)] we obtain the estimates

$$\begin{aligned} & |d_{s_2, j_2, Q_2}^{\bar{\chi}}(\tfrac{1}{2}\eta) d_{s_1, j_1, Q_1}^{\chi_{\nu\mu}}(-\tfrac{1}{2}\eta)| \\ & \leq \text{const} \begin{cases} \exp \eta (-\tfrac{1}{2} + \tfrac{1}{2} |Jm\varphi|), & Q_1 = Q_2 \\ \exp \eta (-1 + \tfrac{1}{2} |Jm\varphi|), & |Q_1 - Q_2| = 1 \end{cases} \end{aligned}$$

They imply that the integrals over η converge absolutely and define a holomorphic function of ρ in the strip $|\operatorname{Im} \rho| < 2\varepsilon$.

Next we set $\eta_1 = 0$. We have

$$\Gamma_Q^J(d(\eta), c)_{Q_2 Q_1} = \sum_{j'} d_{jj'Q}^{(0, -4i)}(-\tfrac{1}{2}\eta) \times \Gamma_Q^{j'}(d(\tfrac{1}{2}\eta), d(\tfrac{1}{2}\eta)^{-1})_{Q_2 Q_1}$$

with the functions

$$d_{000}^{(0, -4i)}(-\tfrac{1}{2}\eta) = \cosh \tfrac{1}{2}\eta, \quad d_{100}^{(0, -4i)}(-\tfrac{1}{2}\eta) = \tfrac{1}{\sqrt{3}} \sinh \tfrac{1}{2}\eta, \quad d_{11\pm 1}^{(0, -4i)}(-\tfrac{1}{2}\eta) = 1$$

obtained by actual evaluation of $[[I], (1-37)]$. This gives the estimate

$$\overline{\lim}_{\eta \rightarrow \infty} |(\cosh \eta)^{1+\varepsilon} \Gamma_Q^J(d(\eta), c)_{Q_2 Q_1}| < \infty$$

for all J and Q . This estimate and the inequality

$$|d_{S_2, \eta, Q_2}^{\bar{x}}(\eta)| \leq \text{const} \exp \eta (-1 + \tfrac{1}{2} |\operatorname{Im} \rho|)$$

yield the same strip of convergence and holomorphy $|\operatorname{Im} \rho| < 2\varepsilon$ as before.

Finally we have to comment on the possibility that premise one is not valid, but only premise zero holds. In this case we choose also Φ^2 of $C_c^\infty(M_2, S_2)$ such that the Fourier transform $\mathcal{F}_{jq}^2(\chi)^*$ of the wave function is entire analytic (remember that $*$ denotes the analytic continuation of the complex conjugate off the real axis). In this case we may try to proceed as follows. We regularize the form factors, in particular this may involve a cut-off at $q^2 = -\omega$, such that the regularization depends analytically on a parameter σ . In a certain domain, premise one is fulfilled and we can give the Fourier transforms in the usual fashion. If these Fourier transforms are analytic in σ and ρ , we continue in σ until the regularization is removed. In the course of this continuation, singularities in the ρ

plane move towards the real axis and force us to deform the integration contour. Since $\mathcal{F}_{jq}^2(\chi)^*$ is entire analytic, we are allowed to do so. If the form factors obey dispersion relations, the vertex function can always be regularized by a factor

$$|a_2 \bar{a}_1|^{2\sigma}, \quad |a|^2 = \text{Tr}(a a^\dagger)$$

which does not destroy the covariance properties (5) and (8) of the vertex function.

6. ASYMPTOTIC EXPANSIONS OF FORM FACTORS

We start from our formula (43) and introduce the short-hand notation

$$\begin{aligned} M_{\nu\mu}(S_2, S_1, \chi) &= \\ &= \sum_{j_1 j_2} (-1)^j (2j+1) \langle \bar{\chi}_j j_2 \| T^j(1-\mu, 1-\nu) \| \bar{\chi}_{\nu\mu} j_1 \rangle \\ &\times \sum_{Q_1 Q_2 Q} (2j_1+1)^{-\frac{1}{2}} (j_1 Q_1 j_2 Q_2 j Q) \\ &\times \int_0^\infty d\eta \, \eta^2 d\bar{\chi}_{S_2 j_2 Q_2}(\eta_2) d\bar{\chi}_{S_1 j_1 Q_1}(\eta_1) \Gamma_Q^j(\dots)_{Q_2 Q_1} \end{aligned} \quad (45)$$

With this notation, (43) can be brought into the concise form

$$\begin{aligned} \langle \Phi^2 | j_Q^j | \Phi^1 \rangle &= \\ &= - (4\pi)^{-1} N_1 C_1 N_2 C_2 \sum_{\nu\mu} \sum_m \int_{-\infty}^{+\infty} d\rho \, M_{\nu\mu}(S_2, S_1, \chi) \\ &\times \sum_{j_1 q_1 j_2 q_2} \overline{\mathcal{F}_{j_2 q_2}^2(\chi)} \langle \chi_{j_1 j_2 q_2} | T_Q^j(\nu, \mu) | \chi_{\nu\mu} j_1 q_1 \rangle \mathcal{F}_{j_1 q_1}^1(\chi_{\nu\mu}) \end{aligned} \quad (46)$$

Inserting the inverse Fourier transforms [I], (2-6) of the wave functions

$$\mathcal{F}_{jq}(\chi) = \int \Phi_{q'}(a) \overline{D_{S_{q'} j q}^{\chi}(a)} d\mu(a) \quad (\text{general})$$

and comparing with (7) leads us to suggest the following representation

$$\begin{aligned} \Gamma_Q^J(a_2, a_1)_{q_2 q_1} = \\ = - [4\pi (2S_1+1)(2S_2+1)]^{-1} \sum_{\nu\mu} \sum_m \int_{-\infty}^{+\infty} d\rho M_{\nu\mu}(S_2, S_1 | \chi) \quad (47) \\ \times \langle \chi, S_2 q_2 | T_{a_2}^\chi T_Q^J(\nu, \mu) T_{a_1}^{\chi_{\nu\mu}} | \chi_{\nu\mu}, S_1 q_1 \rangle \end{aligned}$$

of the vertex function. A priori, this expansion (47) converges only in a sense of distributions, but we would like to know when it converges in an L^2 sense at least.

In order to illustrate this problem, we consider the Hilbert space $L^2(R_1)$ of square integrable functions on the real line. Convolutions by means of a kernel $K(x-x')$ present a bounded linear operator in this Hilbert space, if $K(x)$ is integrable (in the proper sense), but also in some more general cases like $K(x) = \delta(x)$. The assumption that the convolution generates a bounded linear operator corresponds to our premise one. Since the convolution commutes with translations on the real line, we can apply Schur's lemma and obtain with it a Fourier decomposition of $K(x)$ whenever this kernel generates a bounded operator. The inverse Fourier transformation of this kernel does not converge in an L^2 sense unless the class of admitted kernels is much more restricted than only by boundedness of the operator. We may, for example, assume that $K(x)$ is integrable and square integrable. We shall, however, not try to specify such sufficient condition for the vertex function, but assume that (47) converges in an L^2 sense if this is desired.

The matrix element (27) involved in (47) is symmetric as

$$\begin{aligned} \langle \chi, S_2 q_2 | T_{a_2}^\chi T_Q^J(\nu, \mu) T_{a_1}^{\chi_{\nu\mu}} | \chi_{\nu\mu}, S_1 q_1 \rangle = \\ = - \beta^{S_1}(\chi_{\nu\mu}) \beta^{S_2}(\chi)^{-1} \quad (48) \\ \times \langle -\chi, S_2 q_2 | T_{a_2}^{-\chi} T_Q^J(1-\nu, 1-\mu) T_{a_1}^{(-\chi)_{1-\nu, 1-\mu}} | (-\chi)_{1-\nu, 1-\mu}, S_1 q_1 \rangle \end{aligned}$$

as can be inspected from (26), [(I), (1-11)], and the relation

$$-\chi_{\nu\mu} = (-\chi)_{1-\nu, 1-\mu}$$

The function $M_{\nu\mu}(S_2, S_1 | \chi)$ obeys the dual symmetry relation

$$\begin{aligned} M_{1-\nu, 1-\mu}(S_2, S_1 | -\chi) &= \\ &= -\beta^{S_1}(\chi_{\nu\mu}) \beta^{S_2}(\chi)^{-1} M_{\nu\mu}(S_2, S_1 | \chi) \end{aligned} \quad (49)$$

as can be shown by means of (26) and the definition (45). We assume now that the functions $M_{\nu\mu}(S_2, S_1 | \chi)$ are meromorphic in $\wp$ with poles of first order at positions whose imaginary parts accumulate at most at infinity. These poles are correlated by (49), to each pole in $M_{\nu\mu}$ we have a mirror pole in the mirror function $M_{1-\nu, 1-\mu}$. We need therefore to consider the poles in one half plane only, say $\text{Im } \wp \geq 0$. We order these poles such that

$$\text{Im } \wp_n \geq \text{Im } \wp_{n-1}, \quad n=1, 2, 3, \dots$$

We set for example

$$a_1 = e, \quad a_2 = d(\eta)$$

With the decomposition [(I), (4-56), (4-57)]

$$d_{S_2, T_2, q_2}^{\chi}(\eta) = e_{S_2, T_2, q_2}^{\chi}(\eta) + \beta^{S_2}(\chi)^{-1} \beta^{T_2}(\chi) e_{S_2, T_2, q}^{-\chi}(\eta) \quad (50)$$

and taking account of the symmetries (48), (49), we have

$$\begin{aligned} \Gamma_Q^J(d(\eta), e)_{T_2, q_1} &= \\ &= -[2\pi(2S_1+1)(2S_2+1)]^{-1} \sum_{\nu\mu} \sum_m \int_{-\infty}^{+\infty} d\wp M_{\nu\mu}(S_2, S_1 | \chi) \quad (51) \\ &\times \sum_{T_2} \langle \chi_1, T_2, q_2 | T_Q^J(\nu, \mu) | \chi_{\nu\mu}, S_1, q_1 \rangle e_{S_2, T_2, q_2}^{\chi}(\eta) \end{aligned}$$

In addition to the poles of the function $M_{\nu\mu}$, the e function in (51) exhibits first order poles $[(L), \text{Section } 4-5]$ one of which falls on the real axis if S_2 is an integer. In this case the integral over the real axis has to be understood as a principal value integral $[(L), \text{the discussion following Eq. (4-90)}]$. If $M_{\nu\mu}(S_2, S_1 | \chi)$ falls off sufficiently rapidly at $\text{Re } \varphi \rightarrow \pm\infty$, we can shift the integration contour upwards. In the case that the form factors obey a dispersion relation such that they are infinitely differentiable in the spacelike region considered here, we can expect that this fall-off is faster than any inverse power of φ . Under "natural" conditions $[(L), \text{Section } 4-7]$, the contributions of the poles of the e function cancel each other. In this case (51) yields the series

$$\begin{aligned} \Gamma_Q^j(d(\eta), e)_{q_2 q_1} &\cong \\ &\cong -i [(2S_1+1)(2S_2+1)]^{-1} \sum_{n=0}^{\infty} \sum_{\substack{\varphi = p_n \\ m = m_n}} \text{Res } M_{\nu\mu n}(S_2, S_1 | \chi) \quad (52) \\ &\quad \times \sum_{j_2} \langle \chi_{n1} j_2 q_2 | T_Q^j(\nu_{\mu 1} \mu_n) | (\chi_n)_{\nu \mu 1} S_1 q_1 \rangle C_{S_2 j_2 q_2}^{\chi_n}(\eta) \end{aligned}$$

As the expansion $[(I), (1-56)]$ of the e function shows, the series (52) presents an asymptotic expansion of the vertex function if $\eta \rightarrow \infty$, i.e., $q^2 \rightarrow -\infty$.

7. SL(2,C) AS A NON-COMPACT SYMMETRY GROUP

We do not intend to discuss the connection between our harmonic analysis of vertex functions and non-compact symmetry groups in detail, but sketch only the basic facts which illustrate the physical picture into which this analysis fits. In a model based on the group $SL(2,C)$ as a non-compact symmetry group, a typical ansatz for the vertex function of a vector current operator between two particles which are members of towers is

$$\Gamma_Q^j(a_2, a_1)_{q_2 q_1} = q \langle \chi_1 S_2 q_2 | T_{a_2}^{\chi} T_Q^j(\nu_{1\mu}) T_{a_1}^{\chi_{\nu\mu}} | \chi_{\nu\mu 1} S_1 q_1 \rangle \quad (53)$$

where g denotes a coupling constant. Particle 1 (2) is assumed to belong to an infinite tower of particles described by the representation $\chi_{\nu\mu}(\chi)$. The states of these particles at rest are represented by the vectors of the canonical basis. This assumption is physically justified if χ and $\chi_{\nu\mu}$ are both in the principal series, but nevertheless we want to admit arbitrary representations χ and $\chi_{\nu\mu}$ in (53). The known asymptotic behaviour of the matrix elements of T_a^χ in the canonical basis [I], (1-37) shows that (53) fulfills premise zero, but premise one only if χ is in the principal series.

In agreement with our remark at the end of Section 5, we can analyze the vertex (53) by the technique of analytic regularization. We multiply the vertex function with the regularizing factor

$$|a_2 a_1^{-1}|^{2\sigma}$$

If $\text{Re } \sigma$ is sufficiently small, all our formulae make sense and (47) converges. If we continue (47) till $\sigma = 0$, some poles cross the real ϑ axis (if χ is not in the principal series) or touch the real ϑ axis (if χ is in the principal series) and develop a pinch at the end point $\sigma = 0$. This pinch gives us back the original matrix element, whereas the remainder vanishes. From this discussion, we learn that only those poles of $M_{\nu\mu}(S_2, S_1 | \chi)$ can be due to an ansatz of the type (53) which cross or touch the real axis. The residues of these poles must moreover be independent of S_2 and S_1 . The poles which do not cross the real axis appear in the asymptotic expansion (52) with a contribution that falls off faster than any irreducible representation of $SL(2, \mathbb{C})$ allows. These poles resemble the Regge poles and their contribution to the Regge-Mandelstam expansion which have $\text{Re } J < -\frac{1}{2}$.

Finally, we consider the case that the vertex function is given by an ansatz of the kind (53), but χ is now to denote a unitary representation of the principal series of a simple Lie group containing $SL(2, \mathbb{C})$ as a true subgroup [say, $SU(2, 2)$]. In this case χ is a reducible representation of $SL(2, \mathbb{C})$ which decomposes into a direct integral of representations of the principal series of $SL(2, \mathbb{C})$. Premise one is satisfied and the function $M_{\nu\mu}$ is meromorphic. Its poles are ordered into sequences in which the poles are separated by even

multiples of i . This behaviour is suggested by the reduction of a representation of the principal series of $SL(2, \mathbb{C})$ on the subgroup $SU(1, 1)^{(3), (11)}$. The bigger the Lie group, the farther away lies the first pole, the stronger is the fall-off of the form factors.

A P P E N D I X

1. SINGLE-PARTICLE STATES AND COVARIANT WAVE FUNCTIONS

We choose our notations in agreement with Joos ¹⁰⁾. The manifold of four-vectors $p = (p_0, p_1, p_2, p_3)$ with

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 = p^2 = M^2 > 0, \quad p_0 > 0$$

forms an orbit $O(M_+)$ which is mapped onto itself by any element $\Lambda \in O(3,1)$

$$p \xrightarrow{\Lambda} p' : \quad p'_\mu = \Lambda_\mu^\nu p_\nu = (\Lambda p)_\mu \quad (A.1)$$

We introduce the Hermitian matrix

$$\underline{p} = p_\mu \sigma^\mu$$

where $\sigma^0 = e$ and σ^k are the Pauli matrices. Any element $a \in SL(2, \mathbb{C})$ induces an automorphism of $O(M_+)$ by

$$\underline{p} \xrightarrow{a} \underline{p}' = a \underline{p} a^\dagger \quad (A.2)$$

From (A.1) and (A.2) there follows a two-to-one homomorphism between $SL(2, \mathbb{C})$ and $O(3,1)$

$$a \sigma^\mu a^\dagger = \Lambda_\nu^\mu(a) \sigma^\nu \quad (A.3)$$

for which explicit formulae are given in ¹⁰⁾.

A special vector of $O(M_+)$ is

$$p^R = (M, 0, 0, 0)$$

We define a boost $a(p)$ in $SL(2, \mathbb{C})$ by

$$\underline{p}^R = a(p) \underline{p} a(p)^\dagger, \quad a(p) = a(p)^\dagger, \quad \text{Tr } a(p) > 0 \quad (A.4)$$

and denote the corresponding transformation of $O(3,1)$ by $\Lambda(p)$

$$p_\mu^R = \Lambda_\mu^\nu(p) p_\nu$$

In particular we have

$$\alpha(p) = e^{-\frac{1}{2}\eta\sigma_3}$$

for $p = (Mch\eta, 0, 0, Msh\eta)$.

A state of a particle with mass M and spin S at rest is denoted

$$|p^R, q\rangle, \quad p^R \in O(M_+), \quad -S \leq q \leq S$$

States of a particle in motion with momentum p are denoted

$$|p, q\rangle = U_{\Lambda(p)^{-1}} |p^R, q\rangle \quad (A.5)$$

The spin of a particle at rest transforms classically like

$$U_R |p^R, q\rangle = \sum_{q'} D_{q'q}^S(R) |p^R, q'\rangle, \quad R \in O(3) \quad (A.6)$$

Both (A.5) and (A.6) define a representation of $O(3,1)$

$$U_\Lambda |p, q\rangle = \sum_{q'} D_{q'q}^S(R(\Lambda, p)) |\Lambda p, q'\rangle \quad (A.7)$$

with Wigner's rotation

$$R(\Lambda, p) = \Lambda(\Lambda p) \Lambda(p)^{-1} \in O(3) \quad (A.8)$$

We normalize the states as

$$\langle p', q' | p, q \rangle = 2p_0 \delta^{(3)}(p-p') \delta_{qq'} \quad (A.9)$$

which makes the (A.7) representation unitary. Similar formulae hold for $SL(2, \mathbb{C})$.

We define vectors Φ of a Hilbert space $L^2(M, S)$ by

$$\Phi = \int d\mu(p) \sum_q \Phi_q(p) |p, q\rangle \quad (A.10)$$

with the invariant measure $d\mu(p)$ on $O(M_+)$

$$d\mu(p) = (\lambda p^0)^{-1} d^3p$$

$\Phi_q(p)$ is assumed to be measurable on $O(M_+)$ and square integrable like

$$\|\Phi\|^2 = \sum_{q=-S}^S \int d\mu(p) |\Phi_q(p)|^2 < \infty \quad (A.11)$$

By (A.11) we define the scalar product in $L^2(M, S)$. The operator U_Λ acts on the "wave function" $\Phi_q(p)$ like

$$U_\Lambda \Phi_q(p) = \sum_{q'} D_{qq'}^S(R(\Lambda, \Lambda^{-1}p)) \Phi_{q'}(\Lambda^{-1}p) \quad (A.12)$$

We define covariant wave functions on $SL(2, C)$ [covariant on right cosets of $SU(2)$] by

$$\Phi_q(a(p)) \equiv \Phi_q(p) \quad (A.13)$$

and

$$\Phi_q(ua) = \sum_{q'=-S}^S D_{qq'}^S(u) \Phi_{q'}(a) \quad (A.14)$$

From (A.12)-(A.14), we have

$$U_a \Phi_q(e) = \Phi_q(a)$$

and therefore, in general

$$\begin{aligned} U_{a_1} \Phi_q(a) &= U_a [U_{a_1} \Phi_q](e) = [U_{aa_1} \Phi_q](e) \\ &= \Phi_q(aa_1) \end{aligned} \quad (A.15)$$

2. THE FOUR-VECTOR REPRESENTATION OF $SL(2, \mathbb{C})$

We denote by

$$\xi = (\xi_1, \xi_2)$$

a two-spinor which transforms like

$$\xi \rightarrow \xi' = \xi a$$

We define a homogeneous function

$$F(\xi, \bar{\xi}) = (\xi \sigma^\mu \bar{\xi}^\dagger) A_\mu \quad (\text{A.16})$$

of first order in ξ and $\bar{\xi}$. It transforms like

$$\begin{aligned} T_a F(\xi, \bar{\xi}) &= F(\xi a, \bar{\xi} \bar{a}) \\ &= (\xi \sigma^\mu \bar{\xi}^\dagger) A'_\mu \end{aligned} \quad (\text{A.17})$$

where

$$A'_\mu = \Lambda_\mu^\nu(a) A_\nu \quad (\text{A.18})$$

due to (A.3). If we introduce the notations

$$z = \xi_1 \xi_2^{-1}, \quad f(z, \bar{z}) = \xi_2^{-1} \bar{\xi}_2^{-1} F(\xi, \bar{\xi}) \quad (\text{A.19})$$

we have finally mapped the four-vectors onto polynomials of first order in both z and $\bar{z}$ which transform like

$$T_a f(z, \bar{z}) = |a_{12} z + a_{22}|^2 f(z_a, \bar{z}_a) \quad (\text{A.20})$$

with

$$z_a = \frac{a_{11} z + a_{21}}{a_{12} z + a_{22}} \quad (\text{A.21})$$

By a formula of type (A.20) (we drop $\bar{z}$ for simplicity)

$$T_a^\chi f(z) = (a_{12}z + a_{22})^{n_1-1} (\bar{a}_{12}\bar{z} + \bar{a}_{22})^{n_2-1} f(z_a)$$

$$\chi = (m, \varphi), \quad n_1 = -\frac{1}{2}m_1 + \frac{i}{2}\varphi_1, \quad n_2 = +\frac{1}{2}m_1 + \frac{i}{2}\varphi_1$$

where the functions $f(z)$ belong to a certain countably normed space, we can generate any completely irreducible representation of $SL(2, \mathbb{C})$. The vector representation belongs to

$$m = 0, \quad \varphi = -4i$$

These countably normed spaces can be spanned by a canonical basis $\{f_q^j\}$, which was given explicitly in $[[L], (3-83), (3-84)]$. In the case of the four-vector representation, it consists of the functions

$$\begin{aligned} f_0^0(z) &= \left(\frac{1}{\pi}\right)^{\frac{1}{2}} (1 + z\bar{z}) \\ f_1^1(z) &= \left(\frac{6}{\pi}\right)^{\frac{1}{2}} z \\ f_0^1(z) &= \left(\frac{3}{\pi}\right)^{\frac{1}{2}} (1 - z\bar{z}) \\ f_{-1}^1(z) &= -\left(\frac{6}{\pi}\right)^{\frac{1}{2}} \bar{z} \end{aligned} \tag{A.22}$$

We define the components of the functions $f(z)$ in this canonical basis and the vector basis by

$$\begin{aligned} f(z) &= \sum_{jq} A_q^j f_q^j(z) \\ &= \sum_{\mu} A_{\mu} f^{\mu}(z) \end{aligned} \tag{A.23}$$

From (A.16) and (A.23) we have the vector basis

$$\begin{aligned} f^0(z) &= 1 + z\bar{z} \\ f^1(z) &= z + \bar{z} \\ f^2(z) &= -i(z - \bar{z}) \\ f^3(z) &= -1 + z\bar{z} \end{aligned} \tag{A.24}$$

In this fashion we obtain the relations

$$\begin{aligned} A_0 &= \left(\frac{1}{\pi}\right)^{\frac{1}{2}} A^0_0 \\ A_3 &= -\left(\frac{3}{\pi}\right)^{\frac{1}{2}} A^1_0 \\ A_{\pm} &= \left(\frac{3}{\pi}\right)^{\frac{1}{2}} A^1_{\mp 1} \end{aligned} \tag{A.25}$$

where we used the familiar notation

$$A_{\pm} = \frac{\mp 1}{\sqrt{2}} (A_1 \pm i A_2) \tag{A.26}$$

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