CM-P00058253

Archives

Ref.TH.1030-CERN

VENEZIANO'S MODEL AND FREEDMAN-WANG DAUGHTER TRAJECTORIES

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A B S T R A C T

Veneziano's formula is analyzed in the complex ℓ plane for the case of unequal mass scattering. The cancellation of unwanted singularities at $t=0$, through the built-in daughter trajectories is discussed following the original analysis of Freedman and Wang. It is then shown how this analysis can be reversed to construct a general scattering amplitude of the Veneziano type, given the leading Regge trajectory, and assuming linear trajectories and crossing symmetry.

Ref.TH.1030-CERN

14 May 1969

1. Veneziano's proposal ¹⁾ for a crossing symmetric amplitude with simple resonances and Regge behaviour, independently of its physical significance, provides an interesting and instructive example for the theory of Regge poles. Applying it in particular to the scattering of particles with unequal masses one can study on this explicit model how the built-in daughter terms, in the spirit of the original analysis of Freedman and Wang, serve to cancel unwanted singularities in the physical region. In Section 2 we derive and discuss the partial wave expansion of a typical Veneziano term, say in the t channel. Repeating the analysis of Freedman and Wang ²⁾, we then show which parts of the Veneziano daughter residues are needed to cancel those terms in the Khuri pole residues that are singular at $t=0$.

This exercise teaches us how the whole argument can be reversed and can be used to construct scattering amplitudes of the Veneziano type: given the residue of the leading Regge pole in, say, the s channel, and assuming linear trajectories and crossing symmetry, a general Veneziano amplitude, (i.e., a sum over satellites) may be derived from the requirement that the amplitude for unequal masses be analytic in the physical region at $t=0$. This is done in Section 3.

2. We consider the elastic scattering of spinless particles (e.g., $\pi\pi$ scattering) which we assume to be given by the appropriate combinations of terms of the form ³⁾

$$F(x,y) = -\beta \frac{\Gamma(1-\alpha_x)\Gamma(1-\alpha_y)}{\Gamma(1-\alpha_x-\alpha_y)} \quad (1)$$

with $(x,y) = (s,t)$ or (s,u) or (t,u) and where $\alpha_x \equiv ax+b$, a being the slope and b being the intercept of the trajectory α . To be specific we shall consider the term with $x=s$ and $y=t$, all our results are then easily extended to the remaining two terms.

We start by giving the four external particles, $A \rightarrow B \rightarrow C \rightarrow D$, different masses, $m_A \neq m_B \neq m_C \neq m_D$. As Lovelace has discussed ³⁾, this extrapolation in the external masses from their common value can only affect the constant β in Eq. (1) but not the trajectory function α_x . Later on we shall consider the limit of all four masses going to a common value (the pion mass in the example of $\pi\pi$ scattering). We construct the partial wave amplitudes of (1) in the t channel, i.e., for $t > 0$ and $s < 0$. The scattering angle in the t channel c.m. system

$$Z_t = \cos \theta_t = \frac{2s}{4p_t p'_t} + \varphi(t); \text{ with } \varphi(t) = \frac{1}{4p_t p'_t} \left(t - \Sigma + \frac{\Lambda}{t} \right)$$

$$\Sigma = m_A^2 + m_B^2 + m_C^2 + m_D^2; \quad \Lambda = (m_A^2 - m_C^2)(m_B^2 - m_D^2) \quad (2)$$

can be expressed in terms of α_s as

$$Z_t = \frac{\alpha_s}{A} + B \quad (3)$$

where we have introduced the abbreviations

$$A = 2ap_t p'_t$$

$$B = \varphi(t) - \frac{b}{A}$$

and p_t and p'_t being the initial and final c.m. momenta respectively. The partial wave amplitudes are found most easily from one of the integral representations of the beta function ⁴⁾

$$F(t, s) = -\beta \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} = \beta \alpha_t \int_0^\infty dv e^{(\alpha_s-1)v} (1-e^{-v})^{-\alpha_t-1} \quad (1')$$

Replacing α_s by $(Az_t - AB)$ and expanding the exponential $e^{Az_t v}$ in terms of spherical Bessel functions, one finds

$$F(t, s) = \sum_{\ell=0}^{\infty} (2\ell+1) a_{\ell}(\alpha_t) P_{\ell}\left(\frac{\alpha_s}{A} + B\right) \quad (4)$$

$$a_{\ell} = \beta \cdot \alpha_t \sqrt{\frac{\pi}{2A}} \int_0^{\infty} dv e^{-(AB+1)v} v^{-\alpha_t - \frac{3}{2}} \left(\frac{1-e^{-v}}{v}\right)^{-\alpha_t-1} I_{\ell+\frac{1}{2}}(Av)$$

The integrals over v converge provided $\alpha_{\Sigma} + (b-1) < \alpha_t < \ell$. The upper limit is due to the poles in ℓ which come in for $v \rightarrow 0$; the analytic continuation in α_t below the lower limit, in particular to negative t , can be done only after the Sommerfeld-Watson transformation of the partial wave expansion.

We expand the function appearing in the integrand in a power series around the origin

$$\left(\frac{1-e^{-v}}{v}\right)^{-\alpha_t-1} = \sum_{n=0}^{\infty} C_n(-\alpha_t-1) v^n \quad (5)$$

where the $c_n(x)$ are polynomials of degree n in x ^{*)}. Inserting this expansion in (4) each term could be integrated by making use of the well-known Laplace transform ⁴⁾

$$\int_0^{\infty} dv e^{-sv} v^{\mu} I_{\nu}(\lambda v) = \Gamma(\mu+\nu+1) (s^2 - \lambda^2)^{-\frac{\mu+1}{2}} P_{\mu}^{-\nu}\left(\frac{s}{\sqrt{s^2 - \lambda^2}}\right) \quad (6)$$

thus exhibiting the poles in the complex ℓ plane.

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*) We were unable to derive an explicit expression for $c_n(x)$, for arbitrary n and x . We give however a recursion relation for them in the Appendix, as well as the first six polynomials.

However, because of the cuts at $v = \pm 2n\pi i$ of the function $((1-e^{-v})/v)^x$, the expansion (5) converges only for $|v| < 2\pi$, and the series for a_ℓ does not converge. The correct procedure to calculate the pole terms of a_ℓ is of course to split the integral over v in a part from $v=0$ to $v=1$, which, upon inserting the expansion (5), contains the poles, and the remainder from $v=1$ to $v=\infty$ which is regular in the finite ℓ plane. Doing so and inserting the well-known power series expansion for the modified Bessel function ⁴⁾, one finds that the amplitude $F(s,t)$ has only moving poles in ℓ , at the points $\ell = \alpha_t$, $\ell = \alpha_t - 1, \dots$. The residues of these poles can be readily calculated. To do this, it is however simpler to introduce the Laplace transform (6) into (5) directly: although the series thus obtained does not converge, it does yield the correct residues of the poles. One finds

$$\beta_{(\alpha-\mu)} = \beta \cdot \alpha \sqrt{\frac{\pi}{2A}} \sum_{k=0}^{\mu} \frac{(-)^k}{k!} C_{\mu-k} (-\alpha-1) r^{\alpha+\frac{1}{2}+k-\mu} P_{-\alpha+\mu-\frac{1}{2}-k}^{(-\alpha+\mu-\frac{1}{2}-k)}\left(\frac{q}{r}\right) \quad (7)$$

where P_μ^ν denotes a Legendre function, and where we have introduced the abbreviations $q = AB+1$ and $r = \sqrt{q^2 - A^2}$ and have set $\alpha \equiv \alpha_t$, for simplicity. The first three residues are given explicitly by

$$\begin{aligned} \beta_{(\alpha)} &= \beta \cdot \alpha \frac{\sqrt{\pi}}{2} \left(\frac{A}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+\frac{3}{2})} C_0 \\ \beta_{(\alpha-1)} &= \beta \cdot \alpha \frac{\sqrt{\pi}}{2} \left(\frac{A}{2}\right)^{\alpha-1} \frac{1}{\Gamma(\alpha+\frac{1}{2})} [C_1 - C_0 q] \\ \beta_{(\alpha-2)} &= \beta \cdot \alpha \frac{\sqrt{\pi}}{2} \left(\frac{A}{2}\right)^{\alpha-2} \frac{1}{\Gamma(\alpha-\frac{1}{2})} \left[C_2 - C_1 q + \frac{1}{2} \left(q^2 + \frac{A^2}{2\alpha-1} \right) C_0 \right] \end{aligned} \quad (7')$$

The Sommerfeld-Watson transformation can be readily performed, the functions a_ℓ being sufficiently well behaved at infinity in the complex ℓ plane. Furthermore, the Regge background

*) These expressions have been derived and discussed independently by Amati and Alessandrini ⁵⁾, as well as by Fivel and Mitter ⁶⁾.

integral can be shifted to the left of $\ell = -\frac{1}{2}$ up to say $\ell = -L$, by introducing Q_ℓ functions in the well-known fashion ⁷⁾. This extension is however only possible if the imaginary part of α_s tends to infinity when $s \rightarrow \infty$, as is well known from the asymptotic expansion of the beta function. We shall not go into the details of these estimates which are easily worked out from our formulae and the asymptotic representations of the Legendre functions. We thus have

$$F(t, s) = -\pi \sum_{\mu=0}^{M=[\alpha+\frac{1}{2}]} \beta_{(\alpha_t-\mu)} (2\alpha_t-2\mu+1) \frac{P_{\alpha_t-\mu}(z_t)}{\sin \pi(\alpha_t-\mu)} + \text{backgr. int.} \quad (8)$$

or the corresponding expression in $Q-\alpha-\mu-1$ with the background integral shifted to the left.

The same expressions hold for the term $F(t, u)$, when analyzed in the t channel, with α_s replaced by α_u . The signature factor comes in through the combinations $F(t, s) \pm F(t, u)$ which appear in the isospin states of $\pi\pi$ scattering: the arguments of the Legendre functions are $z_t = \alpha_s/A+B$ for $F(t, s)$; and $\tilde{z}_t = \alpha_u/A+C$ for $F(t, u)$ with $C = B - \frac{a\lambda}{A \cdot t}$, whereas the residues are the same for both. One then verifies $\tilde{z}_t = -z_t$ which shows the assertion.

A similar analysis of the third amplitude $F(s, u)$ (always in the t channel) leads to the fixed poles in the angular momentum plane discussed by Amati and Alessandrini ⁵⁾. As one easily shows this term develops its own (positive) signature factor.

As the next step, we consider the Khuri pole expansion of $F(t, s)$ in the variable s

$$F(t, s) = \sum_{n=0}^{\infty} b(n, \alpha_t) s^n \quad (9)$$

Repeating the arguments of Freedman and Wang ²⁾ one can compute the residues of the Khuri poles from the Regge representation (8). One finds in a straightforward way (omitting for a moment the index on α_t , for simplicity)

$$b(\nu, \alpha) = \sum_{\mu=0}^{M=[\alpha+L]} \frac{2\beta(\alpha-\mu)\Gamma(\alpha+\frac{3}{2}-\mu)}{\sqrt{\pi}(p_t p'_t)^{\alpha-\mu}\Gamma(\alpha+1-\mu)} \left\{ \frac{1}{\nu-(\alpha-\mu)} + \frac{2p_t p'_t \varphi(t)(\alpha-\mu)}{\nu-(\alpha-\mu-1)} + \right. \\ \left. + \frac{2(p_t p'_t)^2(\alpha-\mu)(\alpha-\mu-1)}{\nu-(\alpha-\mu-2)} \cdot \frac{(2\alpha-1-2\mu)\varphi^2(t)-1}{2\alpha-1-2\mu} + \dots \right\} + \tilde{b}(\nu, \alpha) \quad (10)$$

where $\tilde{b}(\nu, \alpha_t)$ is regular in ν for $\nu > -L$. The general term in the curly brackets is omitted for the sake of clarity but is easily written down using an explicit representation for the Q_L functions [cf., Ref. 2), Appendix]. Performing now a Sommerfeld-Watson transformation on the power series (8) one finds

$$F(t, s) = -\sqrt{\pi} \sum_{\mu} \frac{2\beta(\alpha-\mu)\Gamma(\alpha+\frac{3}{2}-\mu)}{(p_t p'_t)^{\alpha-\mu}\Gamma(\alpha+1-\mu) \sin \pi(\alpha-\mu)} \left\{ (-s)^{\alpha-\mu} - \right. \\ \left. - 2p_t p'_t \varphi(t)(\alpha-\mu)(-s)^{\alpha-\mu-1} + 2(p_t p'_t)^2(\alpha-\mu)(\alpha-\mu-1) \frac{(2\alpha-1-2\mu)\varphi^2-1}{2\alpha-1-2\mu} + \dots \right\} \\ + \text{background integral} \quad (11)$$

The Khuri background integral is convergent and regular at $t=0$. Obviously, for unequal masses, all terms in the square brackets of Eq. (11) but the first one are singular at $t=0$, since $p_t, p'_t \sim 1/t$ for $t \rightarrow 0$. Consider now the limit where the four external masses tend to a common value m_0 (e.g., the pion mass). Following the argument of de Alfaro, Kuo, Rebbi and Rossetti ⁸⁾ we would require that the resulting amplitude should be independent of the way in which the common limit external masses $\rightarrow m_0$,

and $t \rightarrow 0$ is taken ^{*}). This, however, is only possible if the terms singular at $t=0$ in Eq. (11) all cancel out. That this is indeed the case can be verified by explicit calculation: with the aid of the residues (7), the second term in the square bracket in (11) for the leading Regge pole and the first term for the first daughter combine to

$$\begin{aligned} & \frac{2\beta_{(\alpha-1)} \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi} (p_t p'_t)^{\alpha-1} \Gamma(\alpha) \sin \pi(\alpha-1)} - \frac{2\beta_{(\alpha)} \Gamma(\alpha + \frac{3}{2})}{\sqrt{\pi} (p_t p'_t)^{\alpha} \Gamma(\alpha+1) \sin(\pi\alpha)} 2 p_t p'_t \varphi(t) \cdot \alpha = \\ & = \frac{\alpha a^{\alpha-1} \beta}{\Gamma(\alpha) \sin \pi(\alpha-1)} [C_1(-\alpha-1) + (A \cdot \varphi - q) C_0] = \frac{\alpha a^{\alpha-1} \cdot \beta}{\Gamma(\alpha) \sin \pi(\alpha-1)} [C_1 - (1-b) C_0] \end{aligned}$$

which is indeed regular at $t=0$. Similarly, for the next term one has

$$\begin{aligned} & \frac{2\beta_{(\alpha-2)} \Gamma(\alpha - \frac{1}{2})}{\sqrt{\pi} (p_t p'_t)^{\alpha-2} \Gamma(\alpha-1) \sin \pi(\alpha-2)} - \frac{2\beta_{(\alpha-1)} \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi} (p_t p'_t)^{\alpha-1} \Gamma(\alpha) \sin \pi(\alpha-1)} \cdot 2 p_t p'_t \varphi(t) (\alpha-1) \\ & + \frac{2\beta_{(\alpha)} \Gamma(\alpha + \frac{3}{2})}{\sqrt{\pi} (p_t p'_t)^{\alpha} \Gamma(\alpha+1) \sin(\pi\alpha)} 2 (p_t p'_t)^2 \alpha (\alpha-1) \frac{(2\alpha-1)\varphi^2 - 1}{2\alpha-1} = \\ & = \frac{\alpha a^{\alpha-2} \beta}{\Gamma(\alpha-1) \sin \pi(\alpha-2)} [C_2(-\alpha-1) - (1-b) C_1(-\alpha-1) + \frac{1}{2} (1-b)^2 C_0] \end{aligned}$$

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^{*}) This is a physical requirement generalized from Feynman diagrams and leads to a physical insight into the phenomena of daughters and conspiracies.

etc. This cancellation holds for any term in the square brackets of Eq. (11); we shall however not write down the somewhat lengthy general expression.

As expected, the Veneziano daughter terms in Eq. (8) thus have the right reduced residues to cancel all terms in $1/t$. Note that this cancellation is independent of the functions $c_n(-\alpha-1)$ - a fact which we shall make use of in the following Section. We notice in parentheses that the same regrouping of terms in the expression (10) yields the pole terms of the function $b(\nu, \alpha_t)$ of Eq. (9):

$$b_{\text{pole terms}}(\nu, \alpha) = \alpha \cdot \beta \sum_{\mu=0}^M \frac{1}{\nu - (\alpha - \mu)} \frac{a^{\alpha - \mu}}{\Gamma(\alpha - \mu + 1)} \sum_{k=0}^{\mu} \frac{(-)^k}{k!} C_{\mu-k} \cdot (1-b)^k$$

3. In this Section we shall show how a general scattering amplitude of the Veneziano type can be constructed, for linear trajectories, from the knowledge of the leading Regge residue and from the physical requirement that the amplitude for pairwise equal masses be independent of the way on which the transition from the unequal mass to the equal mass case, and for $t \rightarrow 0$, is done. This requirement implies, as is well known, the existence of unit-spaced daughter trajectories whose reduced residues have definite singularities at $t=0$. When regrouped as in Eq. (10) these daughter poles will yield the pole terms of the Khuri functions $b(\nu, \alpha_t)$.

To do this, we first consider the simpler case of an amplitude of the type $\Gamma(1-\alpha_x)\Gamma(1-\alpha_y)/\Gamma(2-\alpha_x-\alpha_y)$, that was originally proposed by Veneziano for the reaction $\pi\pi \rightarrow \pi\omega$, and shall then briefly indicate a similar construction for say $\pi\pi$ elastic scattering.

This construction is somewhat simplified if we consider the Khuri expansion in the variable α_s instead of the variable s :

$$F(t, s) = \sum_{n=0}^{\infty} a(n, \alpha_t) \alpha_s^n \quad (12)$$

where $a(\nu, \alpha_t)$, for the example studied in Section 2, would be

$$a(\nu, \alpha) = \beta \cdot \alpha \sum_{\mu=0}^M \frac{1}{\nu - (\alpha - \mu)} \frac{1}{\Gamma(\alpha - \mu + 1)} \sum_{k=0}^{\mu} C_{\mu-k} (-\alpha - 1) \frac{(-)^k}{k!}$$

Suppose we are given the residue of the leading Regge pole at $\ell = \alpha_t - 1$ in the s channel

$$\beta_{(\alpha_t-1)} = \beta \frac{\sqrt{\pi}}{2} (p_t p'_t)^{\alpha_t-1} \frac{a^{\alpha_t-1}}{\Gamma(\alpha_t + \frac{1}{2})} = \beta \frac{\sqrt{\pi}}{2} \left(\frac{A}{2}\right)^{\alpha_t-1} \frac{1}{\Gamma(\alpha_t + \frac{1}{2})} \quad (13)$$

with linear trajectories $\alpha_x = ax + b$; $x = t$ or s , and β being an over-all coupling constant. This single Regge pole gives rise to a series of Khuri poles at $\nu = \alpha_t - 1, \alpha_t - 2, \dots$ for the function $a(\nu, \alpha_t)$ in the expansion (12), which for the example studied reads (writing again $\alpha = \alpha_t$, for simplicity)

$$a(\nu, \alpha) = \frac{2\beta_{(\alpha-1)} \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi} (ap_t p'_t)^{\alpha-1} \Gamma(\alpha)} \left\{ \frac{1}{\nu - (\alpha - 1)} + \frac{2ap_t p'_t (\alpha - 1) (\varphi - \frac{b}{A})}{\nu - (\alpha - 2)} + \right. \\ \left. + \frac{2a^2 (p_t p'_t)^2 (\alpha - 1) (\alpha - 2)}{\nu - (\alpha - 3)} \cdot \frac{(2\alpha - 3) (\varphi - \frac{b}{A})^2 - 1}{2\alpha - 3} + \dots \right\} \quad (14)$$

For unequal masses (more precisely for $m_A \neq m_C$ and $m_B \neq m_D$) the second, third, ... terms in the square brackets of Eq. (14) are increasingly singular at $t=0$ and would give rise, after the Sommerfeld-Watson transformation of (12), to singularities of the scattering amplitude in the physical domain. We demand again that

the scattering amplitude for the physical values of the external masses (i.e., in the Veneziano example $m_A = m_C = m_\pi = m_B$; $m_D = m_\omega$) be independent, for $t \rightarrow 0$, of the way on which the external masses tend to their physical values. As is well known, this means that there must exist a series of daughter trajectories $\alpha_\mu = \alpha_t - 1 - \mu$ whose residues at $t=0$ are arranged such as to cancel all unwanted singularities in the scattering amplitude. Specifically, the first daughter residue $\beta(\alpha-2)$ must cancel the terms in $1/t$ of the second term in the square brackets of (14)

$$\frac{2\beta(\alpha-2)\Gamma(\alpha-\frac{1}{2})}{\sqrt{\pi}(ap_t p'_t)^{\alpha-2}\Gamma(\alpha-1)} + \frac{\beta}{\Gamma(\alpha)} 2ap_t p'_t (\alpha-1)(\varphi - \frac{b}{A}) \stackrel{!}{=} \text{regular at } t=0$$

hence $\beta(\alpha-2)$ must contain the singular term $2ap_t p'_t \varphi = A\varphi$ plus, possibly, terms which are regular at $t=0$. We have the condition, using as before the abbreviation $q \equiv AB+1 = A\varphi + 1-b$

$$\beta(\alpha-2) \stackrel{!}{=} \frac{\sqrt{\pi}}{2} (ap_t p'_t)^{\alpha-2} \frac{1}{\Gamma(\alpha-\frac{1}{2})} [-q\beta + g_1(\alpha)]$$

Here we denote the (unknown) part regular at $t=0$ by $g_1(\alpha)$, which will in general be a function of $\alpha = \alpha_t$.

Similarly, for the third term we must have

$$\begin{aligned} & \frac{2\beta(\alpha-3)\Gamma(\alpha-\frac{3}{2})}{\sqrt{\pi}(ap_t p'_t)^{\alpha-3}\Gamma(\alpha-2)} + \frac{1}{\Gamma(\alpha-2)} 2ap_t p'_t (\varphi - \frac{b}{A}) [-q\beta + g_1(\alpha)] + \\ & + \frac{\beta}{\Gamma(\alpha-2)} 2(ap_t p'_t)^2 \frac{(2\alpha-3)(\varphi - \frac{b}{A})^2 - 1}{2\alpha-3} \stackrel{!}{=} \text{regular at } t=0 \end{aligned}$$

and hence

$$\beta(\alpha-3) = \frac{\sqrt{\pi}}{2} (ap_t p'_t)^{\alpha-3} \frac{1}{\Gamma(\alpha-\frac{3}{2})} \left[\frac{\beta}{2} (q^2 + \frac{A^2}{2\alpha-3}) - qg_1(\alpha) + g_2(\alpha) \right]$$

where $g_2(\alpha)$ is regular at $t=0$, etc. We again do not write down the lengthy general term.

There are a finite number of such equations determining the singular parts of the residues up to the residue $\beta(\alpha_{-M})$, where M is the largest positive integer for which $\alpha_{-M} > -L$, L being the point on the negative real ℓ axis up to which the Regge-Mandelstam background integral has been shifted. When introducing these residues into Eq. (14) and defining $g_0 = \beta$, we find a series of pole terms of the Khuri function $a(\nu, \alpha_t)$

$$a(\nu, \alpha) \Big|_{\text{pole terms}} = \sum_{\mu=0}^M \frac{1}{\nu - (\alpha - \mu - 1)} \frac{1}{\Gamma(\alpha - \mu)} \sum_{k=0}^{\mu} \frac{(-)^k}{k!} g_{\mu-k}(\alpha) \quad (15)$$

with unknown functions $g_m(\alpha)$. Since M can be made arbitrarily large, the function $a(\nu, \alpha)$ has poles in ν at $\nu = \alpha - k$ with $k=1, 2, \dots$. We hence know at once that the amplitude

$$F(t, s) = \sum_{\nu=0}^{\infty} a(\nu, \alpha_t) \alpha_s^{\nu}$$

we shall eventually obtain from our construction has poles in the t channel at $\alpha_t = 1, 2, \dots, k, \dots$ whose residues are polynomials of order $k-1$ in α_s , hence in s . By construction the amplitude will have Regge behaviour in the s channel.

The series (15), when formally extended to $M = \infty$, would however not converge in the general case. Indeed, a necessary and sufficient condition for its convergence would be that the function defined by the power series

$$\sum_{m=0}^{\infty} g_m(\alpha) v^m \equiv \varphi(\alpha, v) \quad (16)$$

is an entire function of v . This will in general not be true: in the example of Section 2 for instance, where $g_m(\alpha) = c_m(\alpha) \cdot \beta$ we know that the series (16) converges only for $|v| < 2\pi$. On the other hand, the cancellation of singularities at $t=0$ determines the

function $a(\nu, \alpha)$ only up to an additive part which is entire in ν . We shall use this freedom to construct a convergent series for $a(\nu, \alpha)$ which still exhibits its poles, whose positions and residues we know from our discussion. We shall do this by subtracting suitable polynomials in ν from each term in (15), in very much the same way as the Mittag-Leffler theorem is usually proved.

In the series (15) we substitute $m = \mu - k$, and subtract a suitably chosen polynomial in ν for each m such as to make the series convergent in the limit $M \rightarrow \infty$: we then have

$$a(\nu, \alpha) = \sum_{m=0}^M \left\{ \sum_{k=0}^{M-m} g_m(\alpha) \frac{(-)^k}{k!} \frac{1}{\Gamma(\alpha-m-k)} \frac{1}{\nu-(\alpha-m-k-1)} - P_m(\nu) \right\} + \tilde{a}$$

which, when extended to $M = \infty$, is equal to

$$a(\nu, \alpha) = \sum_{m=0}^{\infty} \left(g_m(\alpha) \frac{\Gamma(\nu+m-\alpha+1)}{\Gamma(\nu+1)} - P_m(\nu) \right) + a'(\nu, \alpha) \quad (17)$$

where a well-known pole expansion of the beta function has been used ⁴⁾, and where a' is entire in ν .

If we now want to impose crossing symmetry, then the amplitude $F(t, s)$ must have the same singularity structure in the variable α_s . The series (12) thus converges only for $|\alpha_s| < 1$. Introducing an integral representation for the gamma function in (17), we can write $F(t, s)$ as

$$F(t, s) = \sum_{\nu=0}^{\infty} \frac{\alpha_s^{\nu}}{\nu!} \sum_{m=0}^{\infty} \left(\int_0^{\infty} d\nu e^{-\nu} g_m \nu^{m+\nu-\alpha_t} - P_m(\nu) \right) + \sum_{\nu=0}^{\infty} a'(\nu, \alpha_t) \alpha_s^{\nu}$$

or

$$F(t, s) = \int_0^{\infty} d\nu e^{(\alpha_s-1)\nu} \nu^{-\alpha_t} \varphi(\alpha_t, \nu) + F_r(t, s) \quad (18)$$

where we have defined

$$\varphi(\alpha_t, v) \equiv \sum_{m=0}^{\infty} g_m(\alpha_t) v^m$$

and where $F_r(t, s)$ denotes a remainder of the amplitude containing the entire parts in v of $a(v, \alpha_t)$ only.

Setting further

$$\varphi(\alpha_t, v) = \beta \left(\frac{1 - e^{-v}}{v} \right)^{-\alpha_t} \Phi(\alpha_t, v)$$

$F(t, s)$ takes the form, analogous to Eq. (1')

$$F(t, s) = \beta \int_0^{\infty} dv e^{(\alpha_s - 1)v} (1 - e^{-v})^{-\alpha_t} \Phi(\alpha_t, v) + F_r(t, s) \quad (19)$$

We now look for restrictions on the functions g_m and on the second part $F_r(t, s)$ of the amplitude (19): the restrictions on the functions $g_m(\alpha_t)$ are rather weak. They should be chosen such as to make the series $\varphi = \sum g_m v^m$ convergent for at least $|v| < 2\pi$. Furthermore, we would require them to be entire in α_t since their singularities would appear in the full scattering amplitude and would lead to an unexpected singularity structure of the scattering amplitude. Let us write the amplitude (19) as

$$F(t, s) = F_p(t, s) + F_r(t, s) \quad (20)$$

where

$$F_p(t, s) = \beta \int_0^{\infty} dv e^{(\alpha_s - 1)v} (1 - e^{-v})^{-\alpha_t} \Phi(\alpha_t, v)$$

The assumed Regge behaviour of the first part $F_p(t, s)$ in (20) is valid only if α_s is not on the real axis for s going to infinity. This is true independently of the functions g_m and may be seen most easily from the Khuri expansion (Section 2 actually provides an example for this). Hence $F_p(t, s)$ must have singularities in α_s

on the positive real axis. Since $F(t,s)$, as a function of α_t , has poles only, by crossing symmetry, these singularities in α_s must also be poles and $F_p(t,s)$ must be crossing symmetric by itself. The remainder $F_r(t,s)$ is entire in α_t and in α_s ; in order not to spoil the assumed Regge behaviour it must hence be a constant.

To further determine the function ϕ we note that the s channel asymptotics demands $\phi \rightarrow 1$ for $v \rightarrow 0$. Similarly, by means of the substitution $e^{-v} = 1 - e^{-w}$ in the integral one shows that the t channel asymptotics requires $\phi \rightarrow 1$ also for $v \rightarrow \infty$. In order to make $F_p(t,s)$ crossing symmetric ϕ must be invariant under the replacement $e^{-v} \leftrightarrow 1 - e^{-v}$; this is equivalent to say that ϕ be a function $G(u)$ of the variable $u = e^{-v}(1 - e^{-v})$. We then obtain

$$F(t,s) = \beta \int_0^{\infty} dv e^{(\alpha_s - 1)v} (1 - e^{-v})^{-\alpha_t} G(e^{-v}(1 - e^{-v})) \quad (21)$$

this includes Veneziano's original formula, with $G \equiv 1$, as well as Mandelstam's proposal ⁹⁾

$$G = [1 - e^{-v}(1 - e^{-v})]^\delta \text{ with } \delta = \frac{1}{2}(\alpha_s + 2b - 2)$$

for an amplitude without even daughters.

We conclude by briefly indicating the same construction for a $\pi\pi$ amplitude (c.f., Section 2) with an assumed leading Regge residue

$$\beta(\alpha_t) = \beta \cdot \alpha_t \frac{\sqrt{\pi}}{2} (p_t p'_t)^{\alpha_t} \frac{a^{\alpha_t}}{\Gamma(\alpha_t + 3/2)} \quad (22)$$

The same technique leads to the Khuri function

$$a(v, \alpha_t) = \beta \cdot \alpha_t \sum_{\mu=0}^M \frac{1}{v - (\alpha_t - \mu)} \frac{1}{\Gamma(\alpha_t - \mu + 1)} \sum_{k=0}^{\mu} \frac{(-)^k}{k!} g_{\mu-k}(\alpha_t) \quad (23)$$

and an amplitude of the form

$$F(t,s) = \beta \cdot \alpha_t \int_0^\infty dv e^{(\alpha_s-1)v} \sum_{m=0}^\infty \left(g_m(\alpha_t) v^{m-\alpha_t-1} - p_m(v) \right) + F_r' =$$

$$= \int_0^\infty dv e^{(\alpha_s-1)v} (1-e^{-v})^{-\alpha_t-1} \phi(\alpha_t, v) + F_r(t,s)$$

which has simple poles at $\alpha_t = k$ (positive integer) whose residues are polynomials of order k in α_s , hence in s .

The requirement $\phi \rightarrow 1$ for $v \rightarrow 0$ and for $v \rightarrow \infty$, as well as of crossing symmetry in s and t is met by the choice

$$\phi(\alpha_t, v) = \alpha_t G(u) - u \frac{dG(u)}{du}; \text{ with } u \equiv e^{-v}(1-e^{-v})$$

We remark that in this case there is no simple way to cancel even daughters in the way proposed by Mandelstam.

We would like to thank J. Kupsch and W. Rühl for discussions, as well as Chan Hong-Mo for a critical reading of the manuscript and helpful remarks.

A P P E N D I X

The functions $c_n(x)$ are defined through the expansion

$$\left(\frac{1-e^{-v}}{v}\right)^x = \sum_{n=0}^{\infty} C_n(x) v^n$$

[c.f., Eq. (5)]. The c_n are clearly polynomials of order n and satisfy the recursion relation

$$C_n(x+1) = \sum_{p=0}^n \frac{(-)^p}{(p+1)!} C_{n-p}(x)$$

which is easily derived from the known expansion of $(1-e^{-v})/v$. For $x = -1$ we have

$$C_n(-1) = \frac{(-)^n}{n!} B_n$$

where the B_n are the Bernoulli numbers.

The first six polynomials are given explicitly by

$$C_0(x) = 1$$

$$C_1(x) = -\frac{x}{2}$$

$$C_2(x) = \frac{1}{2!} \left\{ \frac{x^2}{4} + \frac{x}{12} \right\}$$

$$C_3(x) = -\frac{1}{3!} \left\{ \frac{x^3}{8} + \frac{x^2}{8} \right\}$$

$$C_4(x) = \frac{1}{4!} \left\{ \frac{x^4}{16} + \frac{x^3}{8} + \frac{x^2}{48} - \frac{x}{120} \right\}$$

$$C_5(x) = -\frac{1}{5!} \left\{ \frac{x^5}{32} + \frac{5x^4}{48} + \frac{5x^3}{96} - \frac{x^2}{48} \right\}$$

$$C_6(x) = \frac{1}{6!} \left\{ \frac{x^6}{64} + \frac{5x^5}{64} + \frac{5x^4}{64} - \frac{13x^3}{576} - \frac{x^2}{96} + \frac{x}{252} \right\}$$

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