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ON SOME PROPERTIES OF THE MULTI-PARTICLE AMPLITUDE EXPRESSED
AS A FUNCTION OF GROUP THEORETICAL VARIABLES

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A B S T R A C T

A scattering process involving an arbitrary number of particles with arbitrary (not all vanishing) masses and arbitrary spins and parities is described in terms of amplitudes which are a generalization of the amplitudes introduced by Bali, Chew and Pignotti in their treatment of the multi-Regge model. The covariance properties of these functions are derived. It is proved that, if the boosts used in their definition are properly chosen in the complex Lorentz group, these amplitudes are free of kinematic singularities and constraints. The relevance of these results for the treatment of the kinematical details of the multi-Regge model is pointed out.

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1. INTRODUCTION

The multi-Regge model (MRM) ¹⁾⁻¹⁴⁾ is becoming an important tool for the phenomenological understanding of the multi-particle high energy production processes. It seems therefore useful to try to extend to the MRM all the concepts developed for the usual Regge pole model, like kinematic factors, daughters, conspirators, etc. ¹⁵⁾. As a first step in this direction, we shall generalize to the many-particle case the results obtained by Cosenza, Sciarrino and Toller in the first part of Ref. 16).

The variables and the amplitudes which are more suitable for "multi-Reggeization" have been suggested by Bali, Chew and Pignotti (BCP) ¹²⁾. Here we introduce some slightly more general variables and amplitudes. They are defined in terms of the spinor amplitudes and of some "boosts". If the boosts are chosen in the real Lorentz group (as in the BCP case), the amplitudes contain both kinematic singularities and constraints. Generalizing a result of Ref. 16), we shall prove that, if the boosts are properly chosen in the complex Lorentz group, our amplitudes are free of kinematic singularities and constraints.

The experience acquired with the usual Regge pole model suggests that the kinematical details of the MRM can be tackled in the following two different ways:

- a) to find the kinematic singularities and constraints of the BCP amplitudes (which replace here the t channel helicity amplitudes) and to work out their consequences taking into account factorization. The usual way to find the kinematic singularities and constraints of an amplitude ^{17),18)} is to compare it with another amplitude which is free of these troubles. We think that the amplitudes we shall define are very suitable for this purpose;

- b) to exploit some properties of the matrix elements of the irreducible representations of the Lorentz group, in order to build families of Regge trajectories which satisfy all the requirements of analyticity and factorization. One of these "Lorentz pole models" has been developed (for four-particle amplitudes) in Refs. 16),19) and the present paper is just a generalization to the many-particle case of the first step of this procedure. The other steps will be generalized in a forthcoming paper.

Our point of view is different from the usual approach to the problem of kinematic singularities 17),18),20),21). We too start from the assumption that the basic analytic properties are carried by a spinor function which depends on the four-momenta of the external particles and is covariant with respect to the Lorentz group. However, we do not try to express the spinor function in terms of "invariant amplitudes" which depend on the minimum number of independent variables and are not subject to covariance conditions (the centre-of-mass helicity amplitudes are of this kind). For some applications it is not necessary (and not even convenient) to use invariant amplitudes, and it is natural to use "covariant amplitudes", which depend on more independent variables than strictly necessary and satisfy some covariance conditions (not necessarily with respect to the Lorentz group).

The covariant amplitudes we shall define are useful in connection with generalized partial wave expansions of the amplitude, Refs. 22),23), which contain the multi-Regge expansion as a special case. In Ref. 25) we show that, given a process, we can define a generalized partial wave expansion for each "coupling scheme", i.e., for each tree-like diagram whose external lines correspond to the incoming or outgoing particles. In the following Sections, we shall define many covariant functions corresponding to the different possible coupling schemes.

For the various groups, group elements, Lorentz matrices, etc., we shall use the notations introduced in Ref. 16).

2. THE SCATTERING AMPLITUDE

In order to generalize the formalism developed in Ref. 16), we consider a scattering process involving $n \gg 4$ (incoming or outgoing) particles with arbitrary masses and spins. We assume that at least one of the masses is different from zero. We describe the connected part of the corresponding amplitude by means of a spinor function $\hat{M}$ continued analytically to complex values of the four-momenta $P^{(i)}$. It is defined on the $(3n-4)$ dimensional analytic set $\hat{S}$ described by

$$(P^{(i)})^2 = M_i^2, \quad i = 1, \dots, n; \quad (2.1)$$

$$\sum_{i=1}^n P^{(i)} = 0. \quad (2.2)$$

We assume that it is possible to define a "physical sheet" in a relativistically invariant way and we consider the amplitude only on this sheet.

The function $\hat{M}$ satisfies the following covariance condition ²⁴⁾

$$\begin{aligned} & \hat{M}_{m_1 \dots m_n}(P^{(1)}, \dots, P^{(n)}) = \\ & = \sum_{m'_1 \dots m'_n} \Lambda_{m_1 m'_1}^{(1)}(a^{-1}) \dots \Lambda_{m_n m'_n}^{(n)}(a^{-1}) \cdot \\ & \cdot \hat{M}_{m'_1 \dots m'_n}(L(a)P^{(1)}, \dots, L(a)P^{(n)}), \\ & a \in \tilde{A}^c, \end{aligned} \quad (2.3)$$

where the matrices $L(a)$, which operate on four-vectors, and the matrices $\bigwedge_{mm'}^{(i)}(a)$, which operate on the spinor indices, are analytic representations of the group $\tilde{A}^c$, which is the covering group of the complex Lorentz group containing the space reflection ^{16),22)}.

We call an "orbit" in $\hat{S}$ the set of all the points of $\hat{S}$ which can be obtained from one point by means of a transformation belonging to the group $\tilde{A}^c$. The covariance property (2.3) permits us to compute the function $\hat{M}$ at all the points of an orbit when it is known at one point of this orbit.

The function $\hat{M}$ is directly related to the vacuum expectation values of retarded products of local fields and therefore it is expected to have the fundamental analytic properties. As usual, we call the singularities of the function $\hat{M}$ "dynamical singularities", because we believe that they have a direct dynamical meaning.

In order to treat rigorously the analytic properties of the function $\hat{M}$, we have to recall some properties ²⁰⁾ of the analytic set $\hat{S}$. We consider the analytic subset $\hat{S}_0 \subset \hat{S}$, defined by the additional equations

$$\begin{vmatrix} P_\alpha^{(i)} & P_\beta^{(i)} \\ P_\alpha^{(k)} & P_\beta^{(k)} \end{vmatrix} = 0 ; \quad i, k = 1, \dots, n ; \quad \alpha, \beta = x, y, z, t. \quad (2.4)$$

This set is composed by all the points of $\hat{S}$ such that the corresponding four-vectors $P^{(i)}$ are all parallel (or vanishing). The set $\hat{S}_0$ is non-void only if the equality

$$\sum_{i=1}^n \pm M_i = 0 \quad (2.5)$$

holds for some choice of the signs. We consider also the analytic sets $\hat{S}_i \subset \hat{S}$, defined by the additional condition

$$P^{(i)} = 0. \quad (2.6)$$

Clearly, the set $\hat{S}_i$ is non-void only if $M_i = 0$.

The proposition (B.I) of the Appendix B says that if we exclude the points belonging to $\hat{S}_0$ or to one of the sets $\hat{S}_i$, the analytic set $\hat{S}$ has the structure of an analytic manifold, so that the definition of "function analytic at a point of $\hat{S}$ " is unambiguous. For the points belonging to $\hat{S}_0$ or to $\hat{S}_i$ we use the concept of "strong analyticity" ²⁰⁾. If at least one of the masses is not zero, from the Proposition (B.VI) of the Appendix B, we have that the union of the sets $\hat{S}_0$ and $\hat{S}_i$ has:

Property (2.I): We say that a closed subset $I \subset \hat{S}$ has the property (2.I) if, given an arbitrary open subset $D \subset \hat{S}$ and an arbitrary function defined and analytic in $D - I \cap D$, it can be continued analytically in the whole set D .

Note that from Eq. (2.3) it follows that, if $\hat{M}$ is analytic at a point, it is analytic at all the points of the corresponding orbit.

Now we want to study the general problems which arise when we describe our scattering process by means of another function T different from $\hat{M}$. We assume that this function is defined in the analytic set R and that there is a mapping φ of R into $\hat{S}$. For each set $D \subset R$, we define a set $\Phi(D) \subset \hat{S}$ which is the union of all the orbits of $\hat{S}$ which have a non-void intersection with the image $\varphi(D)$. In general, $\Phi(R)$ does not coincide with $\hat{S}$.

The function T is defined by means of a formula which gives $T(P)$, ($P \in R$), in terms of $\hat{M}(\varphi(P))$. We assume that it is possible to invert this formula and to write $\hat{M}(\varphi(P))$ as a function of $T(P)$. Therefore, if we know the function T in a set $D \subset R$, we can compute the function $\hat{M}$ in the set $\Phi(D)$ defined above.

We call an orbit of R the inverse image (under φ) of an orbit of $\hat{S}$. It is clear that the function T can be computed at all the points of an orbit if it is known at one of these points. This means that T has to satisfy a set of conditions which we call "the covariance conditions of T ". We say that a subset of R is "invariant" if it can be represented as a union of orbits.

Now we give the following definitions

Definition (2.II): We say that the function T is "free of kinematic singularities" when, for any choice of the function $\hat{M}$ [satisfying the condition (2.3)], if the function $\hat{M}$ is analytic in an open set $\hat{D} \subset \hat{S}$, the corresponding function T is analytic in the set $\varphi^{-1}(\hat{D})$, i.e., in the inverse image of $\hat{D}$. In simpler words, the function T contains only the dynamical singularities already present in $\hat{M}$.

Definition (2.III): We say that the function T is "free of kinematic constraints" when, for any choice of the function T (satisfying its covariance conditions) and of the open invariant set $\hat{D} \subset \hat{S}$, if T is analytic in the set $\varphi^{-1}(\hat{D})$, the corresponding function $\hat{M}$ is analytic and uniquely determined in the set $\hat{D}$.

We see that if T is free of kinematic singularities and constraints, there is a one-to-one correspondence between the covariant functions $\hat{M}$ defined and analytic in the invariant open set $\hat{D}$ and the covariant functions T defined and analytic in the set $\varphi^{-1}(\hat{D})$ (which is also invariant).

We shall use the following

Proposition (2.IV): The function T is free of kinematic constraints if the following conditions are satisfied:

- a) if T is analytic at a point P , $\hat{M}$ is analytic at the point $\varphi(P)$;
- b) the set $\hat{S} - \hat{\Phi}(R)$ has the property (2.I) defined above.

Proof: From the condition a), we have that, if T is analytic in $\varphi^{-1}(\hat{D})$, the function $\hat{M}$ is defined and analytic in the set

$$\varphi[\varphi^{-1}(\hat{D})] = \hat{D} \cap \varphi(R). \quad (2.7)$$

From Eq. (2.3) and the fact that $\hat{D}$ is invariant, we have that $\hat{M}$ is defined and analytic also in the set

$$\hat{D} \cap \Phi(R) = \hat{D} - \hat{D} \cap [\hat{S} - \Phi(R)], \quad (2.8)$$

and the condition b) tells us that $\hat{M}$ can be continued analytically in the whole set $\hat{D}$.

From the discussion given above, we also have:

Proposition (2.V): Assume that the function T is free of kinematic singularities and satisfies the condition a) of the preceding Proposition. Then, if it is analytic at a point, it is analytic at all the points of the corresponding orbit.

Now we consider a special case.

We define the n four-vectors $Q^{(i)}$ with the property

$$(Q^{(i)})^2 = M_i^2, \quad Q^{(i)} \neq 0, \quad i = 1, \dots, n, \quad (2.9)$$

and the function

$$\begin{aligned} M_{m_1 \dots m_n}(a_1, \dots, a_n) &= \\ &= \sum_{m'_1 \dots m'_n} \Lambda_{m_1 m'_1}^{(1)}(a_1^{-1}) \dots \Lambda_{m_n m'_n}^{(n)}(a_n^{-1}) \cdot \\ &\cdot \hat{M}_{m'_1 \dots m'_n}(L(a_1)Q^{(1)}, \dots, L(a_n)Q^{(n)}). \end{aligned} \quad (2.10)$$

This function is defined on the $(6n-4)$ dimensional analytic set S composed by the points $\{a_i\}$ with the property

$$\sum_{i=1}^n L(a_i) Q^{(i)} = 0, \quad a_i \in \tilde{A}^c. \quad (2.11)$$

From Eqs. (2.3) and (2.10) we obtain easily the invariance property

$$\begin{aligned} M_{m_1 \dots m_n}(a a_1, \dots a a_n) &= \\ &= M_{m_1 \dots m_n}(a_1, \dots a_n), \end{aligned} \quad (2.12)$$

and n covariance properties

$$\begin{aligned} M_{m_1 \dots m_i \dots m_n}(a_1, \dots a_i h, \dots a_n) &= \\ &= \sum_{m'_i} \Lambda_{m_i m'_i}^{(i)}(h^{-1}) M_{m_1 \dots m'_i \dots m_n}(a_1, \dots a_i, \dots a_n), \\ h &\in H^{(i)c}, \quad i = 1, \dots n. \end{aligned} \quad (2.13)$$

The complex little groups $H^{(i)c}$ are defined by

$$H^{(i)c} = \left\{ h : h \in \tilde{A}^c, L(h) Q^{(i)} = Q^{(i)} \right\}. \quad (2.14)$$

Now we have to apply to the function M the general discussion given above. It is clear that the function M and the set S stand for the function T and the set R of the general discussion. The mapping φ of S into $\hat{S}$ is given by

$$L(a_i) Q^{(i)} = P^{(i)}, \quad i = 1, \dots, n. \quad (2.15)$$

The set $\Phi(S)$ is given by

$$\Phi(S) = \hat{S} - \bigcup_i \hat{S}_i. \quad (2.16)$$

It is easy to show that an orbit in S is the set of all the points which can be obtained from a point of S by means of transformations of the type

$$\begin{aligned} \{a_i\} &\rightarrow \{a a_i h_i\}, \\ a &\in \tilde{A}^c, \quad h_i \in H^{(i)c}. \end{aligned} \quad (2.17)$$

It is clear from the definition that the function M is free of kinematic singularities. In order to show that it is also free of kinematic constraints, we have to invert Eq. (2.10). We consider a point $\{a_i^0\} \in S$ and the corresponding point $\{P^{(i)0}\} \in \hat{S}$. In a neighbourhood of this point (which cannot belong to the sets $\hat{S}_i$) we may use the Proposition (A.I) of the Appendix A in order to define the elements

$$a_i = a_i^0 (d_{P'})^{-1} d_{Q^{(i)}}, \quad P' = L[(a_i^0)^{-1}] P^{(i)}, \quad (2.18)$$

which are analytic functions of $P^{(i)}$, satisfy Eq. (2.15) and coincide with the elements a_i^0 for $P^{(i)} = P^{(i)0}$. Then we write

$$\begin{aligned} \hat{M}_{m_1 \dots m_n}(P^{(1)}, \dots, P^{(n)}) &= \\ &= \sum_{m'_1 \dots m'_n} \Lambda_{m_1 m'_1}^{(1)}(a_1) \dots \Lambda_{m_n m'_n}^{(n)}(a_n) \cdot \\ &\cdot M_{m'_1 \dots m'_n}(a_1, \dots, a_n). \end{aligned} \quad (2.19)$$

Note that the covariance conditions (2.13) are necessary in order to assure that the right-hand side does not depend on the choice of the elements a_i .

From Eq. (2.19) and from the analyticity of the elements (2.18), we see that if M is analytic at a point $P \in S$, the function $\hat{M}$ is analytic at the point $\varphi(P)$. From the results of the Appendix B we have that the set

$$\hat{S} - \Phi(S) = \bigcup_i \hat{S}_i \quad (2.20)$$

has the property (2.I). Then, the conditions of Proposition (2.IV) are satisfied and M is free of kinematic constraints.

Summarizing our results, we have

Proposition (2.VI): The function M defined by Eq. (2.10) is free of kinematic singularities according to the definition (2.II) and has the property described in the Proposition (2.V). If at least one of the masses is not zero it is also free of kinematic constraints according to the definition (2.III). Its covariance conditions are given by Eqs. (2.12) and (2.13).

In order to complete our discussion, we have to take into account the fact that not all the components (in the spinor space) of the functions defined above are physically relevant for the process we are considering. In general, the covariance condition (2.3) mixes the physical and the unphysical components of the function $\hat{M}$.

One of the advantages of the function M is that it is possible to consider its physical components separately. This is due to the fact that only transformations belonging to the little groups $H^{(i)c}$ act on its spinor indices in Eq. (2.13) and the representations $\bigwedge^{(i)}$, when restricted to $H^{i(c)}$, are in general reducible. If $M_i > 0$, the real little group $H^{(i)}$ is compact and, choosing a suitable basis, it is always possible to write the matrix

$\Lambda_{mm'}^{(i)}(h)$, $h \in H^{(i)c}$ in a square-block form ²⁵⁾. One of the square blocks acts on the physical components which are never mixed with the unphysical ones.

If $M_i = 0$, the situation is more complicated, because the real little group $H^{(i)}$ is not compact and a reducible representation of $H^{(i)c}$ is not in general "completely reducible" ²⁵⁾. Choosing a suitable basis, it is always possible to write the representation matrices in the block form

$$\Lambda^{(i)}(h) = \begin{pmatrix} \Lambda_{GG} & \Lambda_{GP} & \Lambda_{GO} \\ 0 & \Lambda_{PP} & \Lambda_{PO} \\ 0 & 0 & \Lambda_{OO} \end{pmatrix}, \quad h \in H^{(i)c}. \quad (2.21)$$

The amplitude has to be considered as a one-column matrix of the kind

$$M = \begin{pmatrix} M_G \\ M_P \\ 0 \end{pmatrix}, \quad (2.22)$$

where M_P represents the set of the physical components. We have to assume that the matrices Λ_{PP} , which represent the part of the matrices $\Lambda^{(i)}$ which acts on the physical components, form a unitary representation of the real little group $H^{(i)}$. The components of the last set have to vanish [as shown in Eq. (2.22)], otherwise, if $\Lambda_{PO} \neq 0$, a transformation of the kind (2.21) would mix them to the physical components. This is just the gauge invariance condition. Note that this condition is invariant under transformations of the kind (2.21). The components denoted by M_G are never mixed to the physical components by a transformation of the kind (2.21). They are the components which can be changed by a gauge transformation.

In conclusion, we may consider only the physical components of the function M and the sum which appears in the covariance equation (2.13) is automatically restricted to physical values of the indices. There is still a delicate point to be clarified. We have assumed that all the components of the function M have some analytic property and the analytic properties of the unphysical components could give rise to some constraints involving the physical components. This is certainly not the case for particles with non-vanishing mass, because then we can always consistently assume that the non-physical components vanish identically. The same is true for zero mass particles if there are no components of the kind M_G described above. This is the case treated in Ref. ²⁶⁾. If there are components of the kind M_G and $\bigwedge_{GP} \neq 0$, the assumption that they vanish contradicts Eq. (2.13) and a more detailed discussion is needed. This is the case of electrodynamics if the photon is described by a four-vector field. We shall discuss these difficulties elsewhere and in the following we do not drop the components M_G from our equations.

3. COUPLING SCHEMES AND GROUP THEORETICAL VARIABLES

According to the arguments given in Ref. 23), we describe a "coupling scheme" by means of a simply connected (tree) diagram, whose external lines correspond to the n particles which appear in the process we are considering. It contains $(n-3)$ internal lines and $(n-2)$ vertices. We label the lines of the coupling scheme by means of numbers from 1 to $(2n-3)$. The first n numbers correspond to the external lines.

It is convenient to choose for each line a conventional orientation (indicated by an arrow). Coupling schemes which differ only for the orientation of their lines are not essentially different, but they give rise to different conventions in the definition of the group theoretical variables. In the following, we assume that all the external lines are "incoming". If the number i labels a line (with its conventional orientation), we indicate by $\bar{i}$ the same line with the opposite orientation. Sometimes in the following a letter may stand for a number with a bar; then it is clear that the letter with bar stands for the same number without bar.

A vertex of the coupling scheme can be identified by means of the symbols which describe the three lines which form the vertex, oriented towards the vertex itself (if one of the lines is conventionally oriented in the opposite direction, a bar has to be used).

An incoming four-momentum $P^{(i)}$ ($i=1,\dots,n$) is associated to each external line. It is convenient to define also the four-momenta $P^{(i)}$ ($i=n+1,\dots,2n-3$) which are transmitted along the internal lines. If we use the obvious convention

$$P^{(\bar{i})} = -P^{(i)}, \quad (3.1)$$

these four-vectors are completely defined by the conditions which impose the four-momentum conservation at each vertex. For the vertex (ijk) the condition has the form

$$P^{(i)} + P^{(j)} + P^{(k)} = 0. \quad (3.2)$$

If we sum all the conditions of the form (3.2) and use the convention (3.1), we see that the four-vectors corresponding to the internal lines cancel each other and we obtain the condition (2.2), which is equivalent to Eq. (2.11).

Generalizing the procedure of Ref. 16), we define the variables

$$W_i = (P^{(i)})^2, \quad i = 1, \dots, (2n-3); \quad (3.3)$$

clearly it is

$$W_i = M_i^2, \quad i = 1, \dots, n. \quad (3.4)$$

Also for each internal line we choose a standard four-vector $Q^{(i)}$, which depends only on W_i and has the property

$$(Q^{(i)})^2 = W_i. \quad (3.5)$$

For $i \leq n$, this equation is just Eq. (2.9). Also for $i > n$ we define, by means of Eq. (2.14), the complex little group $H^{(i)c}$, which depends on the corresponding variable W_i .

For each vertex (ijk) , we define three elements b_i , b_j and b_k of the complex Lorentz group, each one depending on three variables W_i , W_j and W_k , and with the property

$$L(b_i)Q^{(i)} + L(b_j)Q^{(j)} + L(b_k)Q^{(k)} = 0. \quad (3.6)$$

The criteria to be used for the choice of the four-vectors $Q^{(i)}$ and of the elements b_i will be discussed later.

Due to the invariance condition (2.12), the amplitude M depends only on the ratios $(a_j)^{-1} a_k$. We want to express these elements in terms of the variables W_i and of the group elements

$$h_i \in H^{(i)c}, \quad i = 1, \dots, (2n-3). \quad (3.7)$$

We use the convention

$$h_{\bar{i}} t = (h_i t)^{-1}, \quad (3.8)$$

where $L(t)$ inverts all the space-time co-ordinates ¹⁶⁾. We consider the sequence of internal lines, labelled by the symbols $r, \dots, s$, which connect the external lines k and j (going from k to j). These lines are oriented in the direction which goes from k to j and, if the conventional orientation is different, the corresponding symbol must contain a bar. Then we put

$$\begin{aligned} (a_j)^{-1} a_k &= [(h_j t)^{-1} (b_j)^{-1}] \cdot [b_s h_s t (b_{\bar{s}})^{-1}] \dots \\ &\dots [b_r h_r t (b_{\bar{r}})^{-1}] \cdot [b_k h_k t]. \end{aligned} \quad (3.9)$$

Before using Eq. (3.9) we must take care of the following problems:

a) Equation (3.9) must be consistent, i.e., the identity

$$[(a_j)^{-1} a_k] \cdot [(a_k)^{-1} a_i] = [(a_j)^{-1} a_i] \quad (3.10)$$

must be automatically satisfied if we substitute into it expressions similar to the left-hand side of Eq. (3.9). This can easily be checked using the convention (3.8).

- b) The condition (2.11), which expresses the conservation of four-momentum, follows automatically from Eq. (3.9). In order to show this, we fix arbitrarily the element a_1 and express all the other elements a_i by means of Eq. (3.9). Then we orient all the internal lines in the direction which goes towards the external line 1. If i is an arbitrary line ($i \neq 1$) we consider the sequence of internal lines $r, \dots, s$ which connects the lines i and 1 (going from i to 1) and define the following four-vector

$$P^{(i)} = -L \left\{ [a_1(h_1 t)(b_1)^{-1}] \cdot [b_s h_s t(b_{\bar{s}})^{-1}] \dots \right. \\ \left. \dots [b_r h_r t(b_{\bar{r}})^{-1}] b_i \right\} Q^{(i)}. \quad (3.11)$$

From Eqs. (3.9) and (3.7), we see that, if i represents an external line, Eq. (3.11) gives exactly the four-momentum connected with this line [see Eq. (2.15)]. If we can show that the condition (3.2) is satisfied at each vertex, then Eq. (2.2) follows and our proof is complete. We consider the vertex $(ik\bar{j})$. If we put

$$c_j = [a_1(h_1 t)(b_1)^{-1}] \cdot [b_s h_s t(b_{\bar{s}})^{-1}] \dots \\ \dots [b_r h_r t(b_{\bar{r}})^{-1}] \cdot [b_j h_j t(b_{\bar{j}})^{-1}], \quad (3.12)$$

from Eq. (3.11) we have, using also Eq. (3.7),

$$\begin{cases} P^{(i)} = -L(c_j b_i) Q^{(i)}, \\ P^{(k)} = -L(c_j b_k) Q^{(k)}, \\ P^{(\bar{j})} = -P^{(j)} = -L(c_j b_{\bar{j}}) Q^{(j)}, \end{cases} \quad (3.13)$$

and from Eq. (3.6)

$$p^{(i)} + p^{(k)} + p^{(\bar{j})} = 0, \quad (3.14)$$

which is just Eq. (3.2), if we take into account the different orientation convention.

- c) If we fix the elements a_i (and the external masses), in such a way that they satisfy the condition (2.11), we may use Eqs. (2.15) and (3.2) to compute all the four-vectors $p^{(i)}$ and from Eq. (3.3) we obtain the variables W_i . Then the elements b_i and $b_{\bar{i}}$ are also fixed and we may consider Eq. (3.9) (for all the values of j and k) as a system of equations for the unknowns h_i . It is sufficient to consider $j=1$ and $k=2, \dots, n$, because the other equations are not independent. It is clear that if this system of equations cannot be solved under sufficiently general conditions, the new variables W_i and h_i we have introduced are not suitable for the description of the scattering amplitude.

In order to solve this system, it is convenient to consider for each vertex a system of equations of the kind (3.13) for the unknowns c_j . If all these systems can be solved, also our initial problem can be solved. In fact, we have from Eq. (3.12)

$$h_1 = (c_1 b_1)^{-1} a_1 t. \quad (3.15)$$

If $k \neq 1$ is an external line and j is the next internal line, we have from Eqs. (3.9) and (3.12)

$$h_k = (c_j b_k)^{-1} a_k t. \quad (3.16)$$

If k is an internal line and j is the next line going towards the external line 1, we have from Eq. (3.12)

$$h_k = (c_j b_k)^{-1} c_k b_{\bar{k}} t. \quad (3.17)$$

It is easy to show that the elements h_k defined by Eqs. (3.15)-(3.17) satisfy the condition (3.7), i.e., they belong to the little groups. For instance, if k is an external line, from Eqs. (2.15) and (3.13) we have

$$P^{(k)} = L(a_k) Q^{(k)} = -L(c_j b_k) Q^{(k)} \quad (3.18)$$

and comparing with Eq. (3.16)

$$L(h_k) Q^{(k)} = Q^{(k)}. \quad (3.19)$$

In a similar way we may proceed if k is an internal line. It is also easy to show, by direct substitution, that Eqs. (3.15)-(3.17) really are a solution of the Eq. (3.9).

In conclusion, we have only to study the properties of the system (3.13). We consider the eight-dimensional complex space of all the triplets of complex four-vectors whose sum vanishes. The complex Lorentz group $\tilde{A}^C$ gives rise to an equivalence relation between the points which can be connected by a complex Lorentz transformation (the same for the three four-vectors). Therefore we may decompose our space into equivalence classes called "orbits". These orbits are described and classified in the Appendix A. It is clear from the definition that the system (3.13) can be solved if and only if the triplet

$$\{ P^{(i)}, P^{(k)}, P^{(\bar{j})} \} \quad (3.20)$$

and the triplet

$$\left\{ -L(b_i)Q^{(i)}, -L(b_k)Q^{(k)}, -L(b_{\bar{j}})Q^{(j)} \right\} \quad (3.21)$$

belong to the same orbit. A necessary condition is that the squares of the four-vectors of the two triplets are the same. This condition is certainly satisfied, as it follows from Eqs. (3.3) and (3.5), which show that these squares are just W_i , W_k and W_j . In the Appendix A we show that if none of the equalities

$$\pm \sqrt{W_i} \pm \sqrt{W_k} \pm \sqrt{W_j} = 0 \quad (3.22)$$

holds, for each value of these parameters there is one and only one orbit. In this case, the system (3.13) is certainly soluble. For values of the parameters which satisfy one of the equalities (3.22), we have more than one orbit and the system (3.13) may have no solution. We shall discuss this difficulty in the following Section.

If the system (3.13) is soluble, in general its solution is not unique. In fact for each vertex we may define a "covariance group". For the vertex $(ik\bar{j})$ it is given by

$$K^{(ik\bar{j})} = \left\{ k : k \in \tilde{A}^c, L(kb_i)Q^{(i)} = L(b_i)Q^{(i)}, \right. \\ \left. L(kb_k)Q^{(k)} = L(b_k)Q^{(k)}, L(kb_{\bar{j}})Q^{(j)} = L(b_{\bar{j}})Q^{(j)} \right\} \quad (3.23)$$

It is clear that if c_j is a solution of the system (3.13), also $c_{j,k} (k \in K^{(ik\bar{j})})$ is a solution and all the solutions have this form. From Eqs. (3.15)-(3.17), we have that the right-hand side of Eq. (3.9) does not change if we perform the following transformations

$$\begin{cases} h_i \rightarrow (b_i)^{-1} k^{-1} b_i h_i , \\ h_k \rightarrow (b_k)^{-1} k^{-1} b_k h_k , \\ h_j \rightarrow h_j (b_{\bar{j}} t)^{-1} k b_{\bar{j}} t, \quad k \in K^{(i k \bar{j})} . \end{cases}$$

(3.24)

We may use Eq. (3.8) to write the last equation in a form similar to the preceding ones:

$$h_{\bar{j}} \rightarrow (b_{\bar{j}})^{-1} k^{-1} b_{\bar{j}} h_{\bar{j}} .$$

(3.25)

These transformations (and the similar ones corresponding to the other vertices) represent the only arbitrariness in the determination of the elements h_i .

4. THE AMPLITUDE AS A FUNCTION OF THE NEW VARIABLES

Now we want to describe the amplitude in terms of the variables introduced in the preceding Section. Therefore, we introduce the new function

$$\begin{aligned} \hat{T}_{m_1 \dots m_n} (W_{n+1}, \dots, W_{2n-3}, h_1, \dots, h_{2n-3}) = \\ = M_{m_1 \dots m_n} (a_1, \dots, a_n). \end{aligned} \quad (4.1)$$

In the right-hand side we may, for instance, fix the element a_1 arbitrarily and compute the others in terms of the new variables by means of Eq. (3.9). From the condition (2.12) we see that the arbitrariness of the choice of a_1 does not affect the value of the function M .

We have seen in the last Section that a transformation of the kind (3.24) does not change the ratios of the elements a_i . Therefore, from the definition (4.1) we see that the new function must satisfy some covariance conditions (one for each vertex). We call them the "vertex conditions". For the vertex $(i\bar{j}k)$ we have that the function $\hat{T}$ does not change if we perform the transformation (3.24) on its arguments h_i, h_j, h_k .

From Eq. (3.9) we see that the transformation

$$h_i \rightarrow h_i h, \quad i \leq n, \quad h \in H^{(i)c}, \quad (4.2)$$

implies the transformation

$$a_i \rightarrow a_i t^{-1} h t. \quad (4.3)$$

Therefore, from Eq. (2.13) we have another set of covariance conditions of the kind

$$\begin{aligned}
 \hat{T}_{m_1 \dots m_i \dots m_n}(W_{n+1}, \dots W_{2n-3}, h_1, \dots h_i h, \dots h_{2n-3}) &= \\
 = \sum_{m'_i} \Lambda_{m_i m'_i}^{(i)}(t^{-1} h^{-1} t) \cdot \\
 \cdot \hat{T}_{m_1 \dots m'_i \dots m_n}(W_{n+1}, \dots W_{2n-3}, h_1, \dots h_i, \dots h_{2n-3}), \\
 i \leq n, \quad h \in H^{(i)c}. \quad (4.4)
 \end{aligned}$$

For practical purposes, it is convenient to define the simpler function

$$\begin{aligned}
 T_{m_1 \dots m_n}(W_{n+1}, \dots W_{2n-3}, h_{n+1}, \dots h_{2n-3}) &= \\
 = \hat{T}_{m_1 \dots m_n}(W_{n+1}, \dots W_{2n-3}, e, \dots e, h_{n+1}, \dots h_{2n-3}) \quad (4.5)
 \end{aligned}$$

where e is the identity element. This definition can easily be inverted using the properties (4.4), and we obtain

$$\begin{aligned}
 \hat{T}_{m_1 \dots m_n}(W_{n+1}, \dots W_{2n-3}, h_1, \dots h_{2n-3}) &= \\
 = \sum_{m'_1 \dots m'_n} \Lambda_{m_1 m'_1}^{(1)}(t^{-1} h_1^{-1} t) \dots \Lambda_{m_n m'_n}^{(n)}(t^{-1} h_n^{-1} t) \cdot \\
 \cdot T_{m'_1 \dots m'_n}(W_{n+1}, \dots W_{2n-3}, h_{n+1}, \dots h_{2n-3}). \quad (4.6)
 \end{aligned}$$

The conditions (4.4) are automatically satisfied by Eq. (4.6). Then, the function T has only to satisfy some "vertex conditions". For vertices which do not contain external lines, the vertex condition simply says that the function T does not change if we perform a transformation of the kind (3.24) on its arguments. If the vertex contains one or two external lines, we have to use Eq. (4.4). For instance, if we consider the vertex (ijk) , where k is an external line, the vertex condition takes the form

$$\begin{aligned}
 & T_{m_1 \dots m_n} (W_{n+1}, \dots, W_{2n-3}, h_{n+1}, \dots, h_i, \dots, h_j, \dots, h_{2n-3}) = \\
 & = \sum_{m'_k} \bigwedge_{m_k m'_k}^{(k)} (t^{-1} b_k^{-1} k b_k t) \cdot \\
 & \cdot T_{m_1 \dots m'_k \dots m_n} (W_{n+1}, \dots, W_{2n-3}, h_{n+1}, \dots \\
 & \dots b_i^{-1} k^{-1} b_i h_i, \dots b_j^{-1} k^{-1} b_j h_j, \dots, h_{2n-3}).
 \end{aligned} \tag{4.7}$$

Now we assume that the elements b_i and $b_{\bar{i}}$ are entire functions of the variables $\{W_i\}$. It follows from Eq. (3.6) that also the four-vectors $Q^{(i)}$ must be entire functions of W_i . For instance, we could choose

$$Q^{(i)} = \left(\frac{1}{2}(1+W_i), 0, 0, \frac{1}{2}(1-W_i) \right) \tag{4.8}$$

and, if (i, j, k) is a vertex,

$$\left\{ \begin{aligned} b_i &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \\ b_j &= \left[\begin{pmatrix} 1 & 0 \\ W_k - W_i & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right], \\ b_k &= \left[\begin{pmatrix} -1 & 0 \\ W_j & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right]. \end{aligned} \right. \tag{4.9}$$

Now we have to study the problem of kinematic singularities and kinematic constraints for the function T . The $(4n-12)$ dimensional analytic set R where the function T is defined is spanned by the variables

$$h_i, w_i, \quad i = (n+1), \dots, (2n-3), \quad (4.10)$$

subject to the constraints (3.7). It can be shown that R has the structure of a complex manifold. The Eqs. (3.9) after the substitutions

$$\begin{cases} a_1 = e, \\ h_i = e, \quad i = 1, \dots, n, \end{cases} \quad (4.11)$$

define a mapping of R into S and therefore of R into $\hat{S}$. The last is the mapping φ of the general discussion of Section 2.

From the analyticity of the elements b_i and $b_{\bar{i}}$ and from the Proposition (A.IV) of the Appendix A, we have that for each vertex, the four-vectors of Eq. (3.21) are two by two linearly independent. It follows from the discussion of Section 3 and from the classification of the orbits given in the Appendix A that the system (3.13) is soluble if and only if the triplet of four-vectors (3.20) belongs to a normal orbit, i.e., these four-vectors are two by two linearly independent.

We call $\hat{S}_{(ijk)}$ the subset of $\hat{S}$ characterized by the fact that the four-vectors (3.20) are all parallel or vanishing. [Note that these four-vectors can be expressed in terms of the four-vectors $P^{(i)}$, $i \leq n$.] It is clear from the discussion of the preceding Section that the union of the orbits of $\hat{S}$ which have no corresponding point in R is given by

$$\hat{S} - \Phi(R) = \left(\bigcup_i \hat{S}_i \right) \cup \left(\bigcup_{(i,j,k)} \hat{S}_{(i,j,k)} \right), \quad (4.12)$$

where the variables (ijk) run over the set of all the vertices of the coupling scheme.

It is clear from the chain of definitions (2.10), (4.1) and (4.5) and from the analyticity of the elements b_i , $b_{\bar{i}}$, that the function T is free of kinematic singularities. In order to show that it is also free of kinematic constraints, we use the Proposition (2.IV). We show in the Appendix B, that the set (4.12) has the property (2.I) and therefore the condition b) is satisfied. In order to prove the condition a), we consider a point $X^0 = \{w_i^0, h_i^0\} \in R$. Then $\varphi(X^0) \in \Phi(R)$ and we may use Eq. (2.18) in order to compute the elements a_i as analytic functions of $\{P^{(i)}\}$ in a neighbourhood of $\varphi(X^0)$. Then, by means of Eq. (3.12) we define the element c_j^0 which satisfies the system (3.13) at the point $\varphi(X^0)$. In order to solve the system (3.13) in a neighbourhood of this point, we consider the triplet of four-vectors

$$\{L[(c_j^0)^{-1}]P^{(i)}, L[(c_j^0)^{-1}]P^{(k)}, L[(c_j^0)^{-1}]P^{(\bar{i})}\} \quad (4.13)$$

and we use the Proposition (A.II) of the Appendix A. We call g the element of $\tilde{A}^C$ which brings the first two four-vectors (4.13) in the standard form (A.8). We call g' the element which brings the first two four-vectors (3.21) in the same standard form. In a neighbourhood of the point we are considering, these elements are analytic functions of the four-vectors (3.21) and (4.13) and therefore they are analytic functions of $\{P^{(i)}\}$. It is clear that

$$c_j = c_j^0 g^{-1} g' \quad (4.14)$$

is a solution of the system (3.13) which coincides with c_j^0 at the point $\varphi(X^0)$. We proceed in a similar way for the systems of equations connected with the other vertices. From Eqs. (3.15)-(3.17) we can also compute the elements h_i and they coincide with the elements h_i^0 at the point $\varphi(X^0)$. All these elements, as well as the variables w_i are analytic functions of $\{P^{(i)}\}$ in a neighbourhood of $\varphi(X^0)$.

In conclusion, we have built an analytic mapping Ψ of a neighbourhood of $\varphi(X^0)$ into R with the property that $\Psi[\varphi(X^0)] = X^0$ and that $\varphi[\Psi(P)]$ belongs to the same orbit as $P \in \hat{S}$. Now we may use the chain of equations (2.19), (4.1) and (4.6) in order to compute $\hat{M}(P)$ in terms of $T(\Psi(P))$. It is easy to show that, if T is analytic at the point X^0 , then $\hat{M}$ is analytic at the point $\varphi(X^0)$. Therefore also the condition a) of the Proposition (2.IV) is satisfied.

Our main result can be summarized in the following

Proposition (4.I): If all the elements b_i and $b_{\bar{i}}$ are entire functions of $\{W_i\}$, the function T is free of kinematic singularities according to the definition (2.II) and has the property described in the Proposition (2.V). If moreover at least one of the masses is non-zero, the function T is also free of kinematic constraints according to the definition (2.III). Its covariance conditions are the "vertex conditions" described above.

It is interesting to remark that all the preceding treatments can be extended to the description of off-shell amplitudes. Then the condition (2.1) has to be dropped and the function $\hat{M}$ is defined on a complex manifold (without singular points). The assumption that one of the masses is non-zero is neither meaningful nor necessary any more. The functions M and T depend on the additional variables W_i ($i \leq n$), which are not fixed any more by Eq. (3.4). Also the four-vectors $Q^{(i)}$ ($i \leq n$) depend on the variables W_i and we have to assume that they are entire functions of them.

Taking into account these slight modifications, all the preceding Definitions and Propositions can easily be extended. It follows that the functions T and M are also useful for the treatment of the dependence of the amplitudes on the external masses.

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APPENDIX A

SOME AUXILIARY RESULTS

We shall represent the complex four-vectors by means of complex 2×2 matrices in the usual way

$$P = (P_t, P_x, P_y, P_z) \rightarrow \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} P_t + P_z & P_x - iP_y \\ P_x + iP_y & P_t - P_z \end{pmatrix} =$$

$$= n, \quad P^2 = \det n. \quad (\text{A.1})$$

The complex proper Lorentz transformation $L(a, b)$ is defined by

$$L(a, b) n = a n b^{-1} \quad (\text{A.2})$$

and the parity operation $L(s)$ takes the form

$$L(s) n = \begin{pmatrix} n_{22} & -n_{12} \\ -n_{21} & n_{11} \end{pmatrix}. \quad (\text{A.3})$$

Proposition (A.1): If p^0 is a non-vanishing complex four-vector, we can always find a complex proper Lorentz transformation d_p which is an analytic function of the four-vector p in a neighbourhood of p^0 and has the property

$$L(d_p) n = \begin{pmatrix} 1 & 0 \\ 0 & \det n \end{pmatrix} = \lambda n. \quad (\text{A.4})$$

Proof: If $p_{11}^0 \neq 0$, $(p_{11})^{-1}$ is defined in a neighbourhood of p^0 and we may take

$$d_p = \left[\begin{pmatrix} (n_{11})^{-1} & 0 \\ -n_{21} & n_{11} \end{pmatrix}, \begin{pmatrix} 1 & (n_{11})^{-1} n_{12} \\ 0 & 1 \end{pmatrix} \right]. \quad (\text{A.5})$$

If $p_{11}^0 = 0$, but $p^0 \neq 0$, we can always find a fixed proper complex Lorentz transformation c such that

$$[L(c) p^0]_{11} \neq 0 \quad (\text{A.6})$$

and take

$$d_p = d_{p'} c, \quad p' = L(c) p. \quad (\text{A.7})$$

Proposition (A.II): If p^0 and q^0 are two complex four-vectors non-vanishing and linearly independent, we can always find a complex Lorentz transformation d_{pq} , which is an analytic function of the pair of complex four-vectors (p, q) in a neighbourhood of the pair (p^0, q^0) and has the following property

$$\begin{cases} L(d_{pq}) p = \begin{pmatrix} 1 & 0 \\ 0 & \det p \end{pmatrix} = x_p, \\ L(d_{pq}) q = \begin{pmatrix} 0 & 1 \\ -\det q & \det(p+q) - \det p - \det q \end{pmatrix} = y_{pq}. \end{cases} \quad (\text{A.8})$$

Proof: We consider the Lorentz transformation d_p defined in the preceding Proposition and define the new four-vectors

$$\tilde{q} = L(d_p) q, \quad \tilde{q}^0 = L(d_{p^0}) q^0. \quad (\text{A.9})$$

$\tilde{q}$ is an analytic function of (p, q) defined in a neighbourhood of (p^0, q^0) . Then we take

$$d_{pq} = \tilde{d}_{p\tilde{q}} d_p, \quad (\text{A.10})$$

where the complex Lorentz transformation $\tilde{d}_{p\tilde{q}}$ must be an analytic function of $(p, \tilde{q})$ in a neighbourhood of $(p^0, \tilde{q}^0)$, and have the properties

$$\begin{cases} L(\tilde{d}_n \tilde{q}) x_n = x_n, \\ L(\tilde{d}_n \tilde{q}) \tilde{q} = y_n q. \end{cases} \quad (\text{A.11})$$

If $\tilde{q}_{12}^0 \neq 0$, we may choose

$$\begin{aligned} \tilde{d}_n \tilde{q} = & \left[\begin{pmatrix} (\tilde{q}_{12})^{-\frac{1}{2}} & 0 \\ \tilde{q}_{11}(\tilde{q}_{12})^{-\frac{1}{2}} \det n & (\tilde{q}_{12})^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} (\tilde{q}_{12})^{-\frac{1}{2}} & 0 \\ \tilde{q}_{11}(\tilde{q}_{12})^{-\frac{1}{2}} & (\tilde{q}_{12})^{\frac{1}{2}} \end{pmatrix} \right]. \end{aligned} \quad (\text{A.12})$$

If $\tilde{q}_{12}^0 = 0$, we have to find a complex Lorentz transformation b_p , analytic function of p , in a neighbourhood of p^0 , with the properties

$$\begin{cases} L(b_p) x_n = x_n, \\ [L(b_p) \tilde{q}^0]_{12} \neq 0 \end{cases} \quad (\text{A.13})$$

and take

$$\begin{cases} \tilde{d}_n \tilde{q} = \tilde{d}_n \hat{q} b_p, \\ \hat{q} = L(b_p) \tilde{q}. \end{cases} \quad (\text{A.14})$$

If we put

$$b_p = \left[\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \det n \\ 0 & 1 \end{pmatrix} \right], \quad (\text{A.15})$$

the first Eq. (A.13) is automatically satisfied and

$$\begin{aligned} & \left[L(b p^0) \tilde{q}^0 \right]_{12} = \\ & = \tilde{q}_{12}^0 + \beta (\tilde{q}_{22}^0 - \tilde{q}_{11}^0 \det p^0) - \beta^2 \tilde{q}_{21}^0 \det p^0. \end{aligned} \quad (A.16)$$

The choice (A.15) does not work (for any value of β) only if

$$\begin{cases} \tilde{q}_{12}^0 = 0, \\ \tilde{q}_{21}^0 \det p^0 = 0, \\ \tilde{q}_{22}^0 = \tilde{q}_{11}^0 \det p^0. \end{cases} \quad (A.17)$$

If $\det p^0 \neq 0$, this means that $\tilde{q}^0$ is proportional to x_{p^0} and therefore q^0 is proportional to p^0 , which is against the hypothesis. Therefore the only case left out is

$$\det p^0 = 0, \quad \tilde{q}^0 = \begin{pmatrix} \tilde{q}_{11}^0 & 0 \\ \tilde{q}_{21}^0 & 0 \end{pmatrix}, \quad \tilde{q}_{21}^0 \neq 0. \quad (A.18)$$

Then we may take

$$b_p = \lambda \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \quad (A.19)$$

and our problem is solved.

Note that the Lorentz transformation contains the space inversion s only in the case defined by Eq. (A.18). In this case it is

$$\det p^0 = \det q^0 = \det (p^0 + q^0) = 0. \quad (A.20)$$

From the results of Proposition (A.II), we can derive important information on the structure of the orbits in the space of the pairs of complex four-vectors, with respect to the complex Lorentz group. As the linear dependence of the four-vectors is a Lorentz invariant concept, we may classify the orbits into "normal" ones, if they are composed of pairs of linearly independent four-vectors, and "abnormal" orbits, composed of pairs of proportional vectors.

We go back to the usual notation for four-vectors, and define the variables

$$\begin{cases} W_1 = P^2 = \det p, \\ W_2 = Q^2 = \det q, \\ W_3 = (P+Q)^2 = \det(p+q). \end{cases} \quad (\text{A.21})$$

Then, from Proposition (A.II), we have that for given fixed values of these parameters, all the pairs of linearly independent four-vectors can be transformed, by means of a complex Lorentz transformation, into the standard form

$$\begin{cases} P = \frac{1}{2} [(1+W_1), 0, 0, (1-W_1)] , \\ Q = \frac{1}{2} [(W_3-W_1-W_2), (1-W_2), -i(1+W_2), -(W_3-W_1-W_2)] . \end{cases} \quad (\text{A.22})$$

This means that the parameters W_i label univocally the normal orbits.

We have also seen that in general we may use only proper complex Lorentz transformations and this means that the orbits considered above are also, in general, orbits with respect to the proper complex Lorentz group. Only in the case $W_1 = W_2 = W_3 = 0$, the normal orbit considered above splits into two orbits with respect to the proper group. The corresponding representative elements are

$$a) \begin{cases} P = \frac{1}{2} (1, 0, 0, 1) , \\ Q = \frac{1}{2} (0, 1, -i, 0) , \end{cases} \quad b) \begin{cases} P = \frac{1}{2} (1, 0, 0, -1) , \\ Q = \frac{1}{2} (0, -1, i, 0) , \end{cases} \quad (A.23)$$

and one can show that it is not possible to connect these pairs of four-vectors by means of a proper complex Lorentz transformation.

It is clear that abnormal orbits exist only if one of the following equalities is satisfied

$$\pm \sqrt{W_1} \pm \sqrt{W_2} \pm \sqrt{W_3} = 0 . \quad (A.24)$$

The classification of the abnormal orbits is very simple and is left as an exercise.

Now we prove the following results

Proposition (A.III): If the four-vector P is an analytic function of W at $W=0$ and $P^2=W$, then P can never vanish.

Proof: Assume that for $W=0$, $P=0$. Then we have

$$P = O(W) , \quad W = P^2 = O(W^2) , \quad (A.25)$$

which is contradictory.

Proposition (A.IV): If the four-vectors P and Q are analytic functions of W_1 , W_2 and W_3 and satisfy Eq. (A.21), they are always linearly independent.

Proof: Assume that for $\{W_i\} = \{W_i^0\}$, P and Q take the values P^0 and Q^0 and are linearly dependent. From the preceding

Proposition we have that P^0 , Q^0 and P^0+Q^0 cannot vanish.
Therefore, we may write

$$Q^0 = \alpha P^0, \quad \alpha \neq 0, \quad \alpha+1 \neq 0. \quad (\text{A.26})$$

If we take

$$W_1 = W_1^0 + \delta, \quad W_2 = W_2^0, \quad W_3 = W_3^0, \quad (\text{A.27})$$

we have that

$$\tilde{P} = P - P^0 = O(\delta), \quad \tilde{Q} = Q - Q^0 = O(\delta). \quad (\text{A.28})$$

Then from Eq. (A.21) we have

$$\begin{cases} \delta = \tilde{P}^2 + 2(\tilde{P}, P^0), \\ 0 = \tilde{Q}^2 + 2\alpha(\tilde{Q}, P^0), \\ 0 = (\tilde{P} + \tilde{Q})^2 + 2(\alpha+1)(\tilde{P} + \tilde{Q}, P^0), \end{cases} \quad (\text{A.29})$$

and after some calculation

$$\delta = \tilde{P}^2 + \alpha^{-1} \tilde{Q}^2 - (\alpha+1)^{-1} (\tilde{P} + \tilde{Q})^2 = O(\delta^2), \quad (\text{A.30})$$

which is a contradiction.

APPENDIX B

PROPERTIES OF THE ANALYTIC SET $\hat{S}$

In this Appendix it is convenient to use the Euclidean metric; therefore we put

$$P_1^{(i)} = i P_x^{(i)}, P_2^{(i)} = i P_y^{(i)}, P_3^{(i)} = i P_z^{(i)}, P_4^{(i)} = P_t^{(i)},$$

$$(P^{(i)})^2 = \sum_{\alpha=1}^4 P_{\alpha}^{(i)} . \quad (B.1)$$

We consider the $(3n-4)$ dimensional analytic set $\hat{S}$ defined by Eqs. (2.1) and (2.2) and the analytic subset $\hat{S}_0$ and $\hat{S}_i$ defined by Eqs. (2.4) and (2.6) and we prove the following

Proposition (B.I): If $P^0 = \{P^{(i)0}\}$ is a point of $\hat{S}$ which does not belong to the singular sets $\hat{S}_0$ and $\hat{S}_i$, we can choose $(3n-4)$ components of the four-vectors $P^{(i)}$ and, in a neighbourhood of P^0 , express the other $(n+4)$ components as analytic functions of them. This means that the set $\hat{S}$, outside the singular sets $\hat{S}_0$ and $\hat{S}_i$, has the structure of an analytic manifold ^{(20), (27), (28)}.

Proof: From the hypothesis we have that the $n \times 4$ matrix $(P_{\alpha}^{(i)0})$ has at least a non-singular 2×2 minor. If we put

$$\Delta(P) = \begin{vmatrix} P_1^{(1)} & P_2^{(1)} \\ P_1^{(2)} & P_2^{(2)} \end{vmatrix} , \quad (B.2)$$

we may assume that

$$\Delta(P^0) \neq 0 . \quad (B.3)$$

Moreover, each four-vector must have at least one non-vanishing component and we may assume

$$P_{\alpha i}^{(i)0} \neq 0 , \quad i = 3, \dots, n . \quad (B.4)$$

Then we choose the following $(3n-4)$ components

$$\begin{cases} P_3^{(2)}, P_4^{(2)}, \\ P_\alpha^{(i)}, \quad i=3, \dots, n, \quad \alpha \neq \alpha_i. \end{cases} \quad (\text{B.5})$$

Clearly we have

$$P_{\alpha_i}^{(i)} = \left[M_i^2 - \sum_{\alpha \neq \alpha_i} (P_\alpha^{(i)})^2 \right]^{\frac{1}{2}}, \quad i=3, \dots, n, \quad (\text{B.6})$$

$$\begin{cases} P_3^{(1)} = - \sum_{i=2}^n P_3^{(i)}, \\ P_4^{(1)} = - \sum_{i=2}^n P_4^{(i)}, \end{cases} \quad (\text{B.7})$$

and we have only to compute the components which appear in the determinant (B.2), using the equations

$$\begin{cases} P_1^{(1)} + P_1^{(2)} = \varphi_1, \\ P_2^{(1)} + P_2^{(2)} = \varphi_2, \\ (P_1^{(1)})^2 + (P_2^{(1)})^2 = \varphi_3, \\ (P_1^{(2)})^2 + (P_2^{(2)})^2 = \varphi_4, \end{cases} \quad (\text{B.8})$$

where

$$\begin{cases} \varphi_1 = - \sum_{i=3}^n P_1^{(i)}, \\ \varphi_2 = - \sum_{i=3}^n P_2^{(i)}, \\ \varphi_3 = M_1^2 - (P_3^{(1)})^2 - (P_4^{(1)})^2, \\ \varphi_4 = M_2^2 - (P_3^{(2)})^2 - (P_4^{(2)})^2. \end{cases} \quad (\text{B.9})$$

We may replace the last equation of (B.8) by its linear consequence

$$\varphi_1 p_1^{(1)} + \varphi_2 p_2^{(1)} = \frac{1}{2} (\varphi_1^2 + \varphi_2^2 + \varphi_3 - \varphi_4), \quad (\text{B.10})$$

and we obtain a system of second degree. Its solutions are given by

$$p_1^{(1)} = \left[\varphi_2 \Delta + \frac{1}{2} (\varphi_1^2 + \varphi_2^2 + \varphi_3 - \varphi_4) \varphi_1 \right] \cdot [\varphi_1^2 + \varphi_2^2]^{-1} \quad (\text{B.11})$$

and similar formulas for the other unknowns. The quantity Δ defined by Eq. (B.2) can be obtained from

$$\Delta^2 = \varphi_3 \varphi_4 - \frac{1}{4} (\varphi_1^2 + \varphi_2^2 - \varphi_3 - \varphi_4)^2. \quad (\text{B.12})$$

Using Eqs. (B.6), (B.7), (B.9), (B.11) and (B.12), we can write the solutions of the system (B.8) as meromorphic functions of the variables (B.5) in a neighbourhood of P^0 . It is easy to show that the relevant solution is also analytic in a neighbourhood of P^0 and our result is proved.

Now we have to deal with the singular points which belong to the sets $\hat{S}_0$ and $\hat{S}_i$. We quote here two well-known propositions of the theory of the functions of many complex variables ^{27), 28)}.

Proposition (B.II): Let D be a domain in the n -dimensional complex space and X be a subset of D defined by $g=0$, where g is analytic in D and non-identically vanishing. If a function f is analytic and bounded in $D-X$, it can always be continued analytically in the whole domain D .

Proposition (B.III): Let D be a domain in the n -dimensional complex space and $X \subset D$ be a manifold of real dimension equal or less than $(2n-3)$. Then, if a function f is analytic in $D-X$, it can always be analytically continued in the whole domain D .

We consider the $(3n-4)$ dimensional analytic set Z defined in the space of the $(4n-6)$ complex variables

$$\begin{cases} P_3^{(2)}, P_4^{(2)}, \\ P_\alpha^{(i)}, \quad i=3, \dots, n, \quad \alpha=1, \dots, 4, \end{cases} \quad (\text{B.13})$$

by the $(n-2)$ equations

$$(P^{(i)})^2 = M_i^2, \quad i=3, \dots, n. \quad (\text{B.14})$$

We indicate by Q_P the projection on Z of the point $P \in \hat{S}$, and we call Z_0, Z_i the projections on Z of the sets $\hat{S}_0, \hat{S}_i$ respectively.

Proposition (B.IV) ²⁹⁾: If P^0 is a point of $\hat{S}$ with the property

$$(P_1^{(1)0} + P_1^{(2)0})^2 + (P_2^{(1)0} + P_2^{(2)0})^2 \neq 0, \quad (\text{B.15})$$

and $f(P)$ is a function defined and analytic in a neighbourhood of $P^0 \in \hat{S}$, deprived of its points belonging to the singular sets $\hat{S}_0$ and $\hat{S}_i$, it can always be written in the form

$$f(P) = F^+(Q_P) + \Delta(P) F^-(Q_P), \quad (\text{B.16})$$

where $\Delta(P)$ is given by Eq. (B.2) and the functions $F^+(Q)$ and $F^-(Q)$ are analytic in a neighbourhood of $Q^0 = Q_{P^0} \in Z$ deprived of the points belonging to the sets $Z_3, Z_4, \dots, Z_n$.

Note that if P^0 is not the point $P^{(i)0} = 0$, the condition (B.15) can always be satisfied by means of a permutation of the indices i and of the four-vector indices.

Proof: If $\Delta(P^0) \neq 0$, the Proposition is trivial. In fact, we may put $F^-(Q) = 0$ and use the procedure described in the preceding

proof, in order to write all the components $P^{(i)}$ as analytic functions of the variables (B.13).

Then we consider the case $\Delta(P^0) = 0$. From Eqs. (B.7), (B.9), (B.11) and (B.12) we see that each point $Q \in Z$ corresponds to two points P_Q^+ and P_Q^- of the set $\hat{S}$, which differ for the choice of the sign of Δ . From the condition (B.15), we have that these two points are analytic functions of Q in a neighbourhood of Q^0 deprived of the points where $\Delta^2 = 0$. Then we put

$$\begin{cases} F^+(Q) = \frac{1}{2} [f(P_Q^+) + f(P_Q^-)], \\ F^-(Q) = \frac{1}{2} [f(P_Q^+) - f(P_Q^-)] \cdot [\Delta(P_Q^+)]^{-1}. \end{cases} \quad (B.17)$$

These functions do not depend on the sign of Δ and therefore are functions of Q alone. In a neighbourhood of Q^0 deprived of the points where $\Delta = 0$ and of the points belonging to the set

$$Z_0 \cup Z_1 \cup \dots \cup Z_n \quad (B.18)$$

these functions are analytic. Moreover, it is possible to show that they are bounded in a neighbourhood of all the points which do not belong to the set (B.18) (also if $\Delta = 0$). The first function is also continuous at these points. It follows from the Proposition (B.II) that the functions (B.17) are analytic at these points (the second has to be defined in a suitable way for $\Delta = 0$).

It is clear that Eq. (B.16) follows from Eq. (B.17). We only have to prove that the functions (B.17) can be continued analytically at all the points of Z_0, Z_1, Z_2 which do not belong to $Z_3, \dots, Z_n$. In a point of this kind, for some values of α_i , we must have

$$P_{\alpha_i}^{(i)} \neq 0, \quad i = 3, \dots, n, \quad (B.19)$$

and in a neighbourhood of this point we may parametrize the set Z by means of the $(3n-4)$ variables

$$\begin{cases} P_3^{(2)}, P_4^{(2)}, \\ P_{\alpha}^{(2)}, \alpha \neq \alpha_i, i = 3, \dots, n. \end{cases} \quad (\text{B.20})$$

At the points of the sets Z_1 and Z_2 we may determine the components $P_3^{(2)}$ and $P_4^{(2)}$ in terms of the other variables (B.20).

Therefore the images of these sets in the space of the variables (B.20) are contained in $(3n-6)$ dimensional complex manifolds and we may use the Proposition (B.III) in order to continue analytically the functions (B.17) at the points of Z_1 or Z_2 which do not belong to Z_0 . At the points of Z_0 we have

$$\begin{cases} P_{\beta}^{(3)} = P_{\beta}^{(4)} P_{\alpha_4}^{(3)} (P_{\alpha_4}^{(4)})^{-1}, \beta \neq \alpha_4, \beta \neq \alpha_3, \\ P_{\gamma}^{(3)} = P_{\gamma}^{(4)} P_{\alpha_4}^{(3)} (P_{\alpha_4}^{(4)})^{-1}, \gamma \neq \beta, \gamma \neq \alpha_4, \gamma \neq \alpha_3. \end{cases} \quad (\text{B.21})$$

Therefore, the image of Z_0 in the set of the complex variables (B.20) is contained in a $(3n-6)$ dimensional complex manifold and we can use again the Proposition (B.III) in order to obtain the analytic continuation.

The second step is given by the following

Proposition (B.V): Consider the analytic set defined in the space of the complex four-vectors by the equation

$$P^2 = M^2. \quad (\text{B.22})$$

If $F(P)$ is a function defined and analytic in a neighbourhood of a point P^0 deprived of the singular point $P=0$, it is always possible to write this function, in a neighbourhood of P^0 in the form

$$F(P) = \Phi^+(P_2, P_3, P_4) + P_1 \Phi^-(P_2, P_3, P_4) , \quad (B.23)$$

where the functions Φ^+ and Φ^- are analytic in a neighbourhood of P^0 .

Proof: If $P^0 \neq 0$, the Proposition is trivial and we may take $\Phi^- = 0$. If $P^0 = 0$, we define

$$\begin{cases} \Phi^+(P_2, P_3, P_4) = \frac{1}{2} [F(P_1, P_2, P_3, P_4) + F(-P_1, P_2, P_3, P_4)] , \\ \Phi^-(P_2, P_3, P_4) = \frac{1}{2} (P_1)^{-1} [F(P_1, P_2, P_3, P_4) - F(-P_1, P_2, P_3, P_4)] . \end{cases} \quad (B.24)$$

These functions do not depend on the sign of P_1 , therefore they depend only on $(P_1)^2$, which can be expressed in terms of the other three variables. It is always possible to find a neighbourhood of $P^0 = 0$ such that the functions (B.24) are analytic in this neighbourhood deprived of the points where $P_1 = 0$. Moreover, one can show that these functions are bounded in a neighbourhood of the points $P \neq 0$ (also if $P_1 = 0$). Then we may use the Proposition (B.II), in order to continue the functions (B.24) at these points. Only the point $P = 0$ is left out and we may use the Proposition (B.III), in order to obtain also the analytic continuation at this isolated point.

Combining the propositions (B.IV) and (B.V) we obtain the following final result:

Proposition (B.VI): Consider the point $P^0 \in \hat{S}$ and assume that at least one of the four-vectors $P^{(i)0}$ does not vanish. If $f(P)$ is a function defined and analytic in a neighbourhood of P^0 deprived of the points belonging to the singular sets $\hat{S}_0$ and $\hat{S}_i$, we can find a neighbourhood of P^0 where this function can be written in the form

$$f(P) = \sum_{\nu} \Gamma_{\nu}(P) \Phi_{\nu}(P). \quad (B.25)$$

The functions $\Gamma_{\nu}(P)$ are polynomials in all the components $P_{\alpha}^{(i)}$ and the functions $\Phi_{\nu}(P)$ depend analytically on a set of $(3n-4)$ properly chosen components.

The equation (B.25) defines an analytic continuation of the function $f(P)$ on the singular sets $\hat{S}_0$ and $\hat{S}_i$. Note that the function (B.25) can be considered as a restriction on the set $\hat{S}$ of a function of all the variables $P_{\alpha}^{(i)}$, analytic at the point P^0 . This means, by definition ²⁰⁾, that the function $f(P)$ is "strongly analytic" at the point P^0 if the hypotheses of our theorem are satisfied.

If one of the masses is not zero, the corresponding four-momentum cannot vanish and we have proved that the union of the sets $\hat{S}_i$ and $\hat{S}_0$ has the property (2.I).

Now we have to prove the property (2.I) for some other subsets of $\hat{S}$. We begin with the following

Proposition (B.VII): If the closed sets $X_1, X_2, \dots, X_m$ (m finite) have the property (2.I), also their union has the property (2.I). The proof is elementary.

Proposition (B.VIII): We consider a subset σ of the set composed by the first n integers and the analytic subset $Z_{\sigma} \subset \hat{S}$ defined by

$$\sum_{i \in \sigma} P^{(i)} = 0 \quad (B.26)$$

This set has the property (2.I).

Proof: We consider a point $P^0 \in Z_\sigma$ but not belonging to the sets $\hat{S}_0$ or $\hat{S}_i$, and a function $f(P)$ analytic in a neighbourhood of P^0 deprived of the points belonging to Z_σ . In a neighbourhood of P^0 we may use the parametrization introduced in the Proposition (B.I). It is easy to show that the condition (B.26) can always be used in order to eliminate two of the variables (B.5). Therefore we have that the image of Z_σ in the space of the complex variables (B.5) is a $(3n-6)$ dimensional complex manifold and we may use the Proposition (B.III) in order to continue analytically $f(P)$ on it. Now we have only to continue analytically $f(P)$ in some points of Z_σ which belong also to one of the sets $\hat{S}_0$ or $\hat{S}_i$, but this is always possible owing to the Proposition (B.VI).

Proposition (B.IX): We decompose the set of the first n integers into three disjoint sets σ , σ' , σ'' and we define the set $Y \subset \hat{S}$ where the three four-vectors

$$\begin{cases} A = \sum_{i \in \sigma} p^{(i)} \\ A' = \sum_{i \in \sigma'} p^{(i)} \\ A'' = \sum_{i \in \sigma''} p^{(i)} \end{cases}, \quad (B.27)$$

are two by two linearly dependent. The set Y has the property (2.I).

Proof: Given a function $f(P)$ analytic in a domain $D \subset \hat{S}$ deprived of the points belonging to Y , we have only to continue it at the points of Y which do not belong to $\hat{S}_0$, $\hat{S}_i$, Z_σ , $Z_{\sigma'}$, $Z_{\sigma''}$. The continuation in the other points of Y can be obtained by means of the preceding Propositions. We consider a point P^0 of this kind. It is easy to show that at this point there are certainly two four-vectors linearly independent and such that their indices belong to the same set, e.g., σ . We may assume that these four-vectors are just $p^{(1)0}$ and $p^{(2)0}$ and use, in a neighbourhood of P^0 , the parametrization introduced in Proposition (B.I). We assume also that $n \in \sigma''$. We may always

find a relativistic index $\bar{\alpha}$ such that $A_{\bar{\alpha}}^i \neq 0$ in a neighbourhood of P^0 . Then for all the points of Y we may write

$$\begin{cases} P_{\beta}^{(n)} = - \sum_{\substack{i \in \sigma'' \\ i \neq n}} P_{\beta}^{(i)} + A_{\bar{\alpha}}'' A'_{\beta} (A'_{\bar{\alpha}})^{-1}, & \beta \neq \bar{\alpha}, \beta \neq \alpha_n, \\ P_{\gamma}^{(n)} = - \sum_{\substack{i \in \sigma'' \\ i \neq n}} P_{\gamma}^{(i)} + A_{\bar{\alpha}}'' A'_{\gamma} (A'_{\bar{\alpha}})^{-1}, & \gamma \neq \beta, \gamma \neq \bar{\alpha}, \gamma \neq \alpha_n. \end{cases} \quad (B.28)$$

This means that the image of Y in the complex space of the variables (B.5) is contained in a $(3n-6)$ dimensional complex manifold and we may obtain our analytic continuation by means of the Proposition (B.III).

Note that the sets $\hat{S}_{ijk}$ which appear in Eq. (4.12) are of the kind considered by the present proposition. From the Propositions (B.VI), (B.VII) and (B.IX) it follows immediately that the set (4.12) has the property (2.I).

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