CM-P00057847

Ref. TH. 933

ASYMPTOTIC CONTRIBUTIONS OF REGGE CUTS TO FORWARD AMPLITUDESI: THEORY

F. J. Yndurain
CERN - Geneva

A B S T R A C T

We consider the contributions to high energy forward elastic scattering amplitudes of possible cuts in the complex angular momentum plane (Regge cuts). After a survey of the necessity of their incorporation, a very general conjecture is made about their structure (viz., positivity of the jumps). It is then shown that this assumption is sufficient to prove the important property that the absorptive parts of the corresponding scattering amplitudes are non-oscillating at high energy. Hence we are able to compute finite energy sum rules and dispersion relations. The asymptotic behaviours of these are then analyzed both when the leading ℓ plane singularities contain cuts, and when they do not. It turns out that these two possibilities lead to drastically different predictions, which are then used to propose experimental tests of the existence of Regge cuts, showing that such tests are feasible with the help of presently available data; their relevance is discussed.

At the end of the article we have inserted an Appendix containing some of the results of these tests, due to Rubio and the present author. A completed and improved survey of them will be presented separately.

1. INTRODUCTION

It seems now well established, both from the theoretical and the experimental point of view, that the singularity structure of the complex angular momentum plane (ℓ plane) is more complicated than what we would expect of a straightforward Regge pole picture holds. As to the first, it is known that the perturbation theory predicts that cuts in the ℓ plane (Regge cuts) should appear when considering repeated pole exchange. (It is known that the cuts found by Amati, Fubini and Stanghellini¹⁾ cancel out, but cuts essentially equivalent appear nevertheless, if one considers somewhat more complicated diagrams, as shown by Mandelstam¹⁾.) On the other hand, terms that can be assimilated to Regge cuts have been shown to arise in some phenomenological models, notably when considering absorptive and/or unitary corrections to pure Regge pole processes^{2),3)}.

With respect to the experimental situation, a pure pole picture encounters (among others) the following difficulties: first, for forward elastic scattering, the rate of approach to the asymptotic Pomeranchuk limit³⁾ of total cross-sections is so slow that we have, in order to fit the observed data, to introduce poles^{4),5)} whose intercepts differ from what we would obtain by extrapolating t channel resonances; and the same is true if the value of the ratio of real to imaginary parts of collision amplitudes is considered. Second, the variations of the shape of the diffractive peaks in πN , NN and $\bar{N}N$ elastic and charge exchange scattering are so different that the characteristics of the relevant Regge poles change dramatically from one to the other⁶⁾. Thus, for example, the slope of the Pomeranchuk (vacuum) trajectory is orders of magnitude smaller than that of the ρ . Third, polarization phenomena seem to be completely resistant to an explanation in terms of Regge poles alone. However, answers to these questions appear to be somewhat improved by adding, to the standard Regge picture, what essentially amounts to corrections due to cuts. Thus, for forward scattering, it has been shown^{7),8)} that fits to total cross-sections as well as to finite-energy sum rules or dispersion relations are

2.

much improved if one assumes Regge cuts to be present, whilst a good agreement off the elastic forward case is found in the analyses of Refs. ^{3),9)}, for example. Also, the phenomenological analysis of A. Bialas and E. Bialas ¹⁰⁾ who approximate scattering amplitudes by sums of the form

$$\sum_j C_j E^{\alpha_j} (\log E)^{\beta_j}, \quad (1.1)$$

(i.e., contributions of Regge poles and of simple cuts) agrees very well with experimental results.

The usual argument against Regge cuts is essentially one of simplicity. As a matter of fact, the general belief seems to be that it is hopeless (or almost) to extract relevant results in the presence of Regge cuts because, first, their existence complicates the simple Regge pole power law by introducing extra logarithmic type terms, and, second, there seems to be no unambiguous way of formulating models with Regge cut corrections (a fact exemplified by our brief survey above); this, in spite of the efforts aimed both at the theoretical "exact" computations of Regge cuts as well as to more or less heuristic estimates of their effects ^{1),2),11),12)}.

That, in a quite general situations, these considerations are correct, is seen immediately. In fact, the asymptotic contribution of a Regge cut to a two-body collision amplitude is given by an expression like

$$T(E,t) = \int d\ell E^\ell g_t(\ell),$$

where E is the lab. energy of the projectile, t the momentum transfer, and $g_t(\ell)$ is essentially the jump over the cuts. Therefore, $T(E,t)$ is (as a function of $\log E$) the Laplace transform of $g_t(\ell)$ and thus, by varying $g_t(\ell)$ we can generate almost any T we like (the only constraint being the result obtained in Ref. ¹²⁾).

However, one knows that, apart from analyticity, there are further requirements (notably unitarity) that one also has to satisfy and which, one would hope, might put some limitations on $g_t(\ell)$. As a very weak assumption we have taken the requirement that, for $t \geq 0$ and even amplitudes, the imaginary part of $g_t(\ell)$ be positive-definite. Whether taking this as a heuristic, but anyway very general hypothesis ¹³⁾, or, somewhat naively, trying to justify it by noting that the unitarity condition reads much the same for $g_t(\ell)$ as for $\text{Im} T$, we shall make it the (only) basis of our analysis here. In this situation we devote the present article to an explanation of general results concerning the asymptotic contributions of Regge cuts; it will turn out that our very weak condition is still sufficient to establish rigorously some interesting results. The predictions of our analysis will be checked against experiment in a forthcoming article by A. Rubio and the present author (Ref. ¹⁴⁾, hereafter to be denoted by II). An advance of the results one gets is collected in the Appendix.

It is to be noted that, since only positivity of $\text{Im} g_t(\ell)$ will be assumed, we shall only obtain direct predictions about $\text{Im} T(E, t)$. This is not a serious problem; in fact, we shall be able to prove that $\text{Im} T(E, t)$ is non-oscillating at infinity (as a function of E) in the sense of Ref. ¹⁵⁾, so that the asymptotic behaviour of the whole $T(E, t)$ follows by using dispersion relations as in Refs. ⁸⁾ and ¹⁵⁾. Finally, we remark that, as sketched above, positivity of $\text{Im} g_t(\ell)$ is more likely to be obtained for $t \geq 0$, so that our results will be particularly relevant for forward elastic collisions, which is the case we shall consider here.

The precise statement of our assumptions, together with some preliminary results, is presented in Section 2. Section 3 is devoted to obtain some basic results, in particular the proof of non-oscillation of $\text{Im} T$ as well as the connection with finite energy sum rules ^{16), 17)}, presenting a refinement of their predictions which allows us to use them in the presence of Regge cuts. In Section 4 we examine the possible detection of Regge cuts, in particular through finite energy sum rules

(that we henceforth abbreviate to FESR) as well as from some extra sum rules (extending the one derived in Ref. ¹⁷⁾). The main part of the paper is finished in Section 5, where we have (for the sake of completeness) described the asymptotic estimate of forward dispersion relations in the presence of Regge cuts ⁸⁾ (and without leading Regge cuts). The article is completed in Section 6 where some comments and conclusions are presented. Finally, we have gathered in an Appendix some preliminary numerical computations checking our results; this analysis, included here for information purposes, is due to A. Rubio and the present author, and a full version of it will be presented in II ¹⁴⁾.

2. ASSUMPTIONS AND PRELIMINARY RESULTS

We shall here and in what follows neglect complications due to spin; in fact, for the problems with which we shall be concerned the incorporation of spin is indeed trivial. The following definitions shall be consistently used: by E we understand the lab. energy of the projectile particle in a two-body elastic collision; m and k will denote its mass and lab. momentum, respectively. The amplitude for such a collision will be denoted by

$$T(E, t) = D(E, t) + i A(E, t); \quad D = \text{Re } T, \quad A = \text{Im } T.$$

t is the momentum transfer. As we shall be concerned with forward scattering ($t=0$) we will drop the t dependence almost invariably. The notations $T_{\pm}(E)$ will be used for the direct/crossed reactions; we remark that the crossing symmetry implies ¹⁸⁾ $T_{-}(E) = T_{+}^{*}(E)$. A bar over the relevant symbol will also sometimes be used for objects referring to the "crossed" reaction (thus, $\bar{T}$ will stand for T_{-}), and we use the notations $T^{(\pm)}(E) = T(E) \pm \bar{T}(E)$ for the symmetric even/odd amplitudes.

Finally, the current mathematical notations $x = o(y)$ [resp: $x = O(y)$] when $x/y \rightarrow 0$ [resp: $|x/y|$ is bounded] will also be employed. Remark that $x = o(1)$ means that $x \rightarrow 0$, and $o(y^{\alpha+0})$ means $o(y^{\alpha+\epsilon})$ for all $\epsilon > 0$.

A. CONTRIBUTIONS OF REGGE SINGULARITIES

As it is well known ¹⁹⁾, the asymptotic contributions of Regge singularities can be written as

$$A(E, t) = \int_S d\ell E^\ell \gamma_t(\ell); \quad \gamma_t(\ell) = \text{Im} g_t(\ell), \quad (2.1)$$

where the contour S contains the ℓ plane singularities. At high energy, $\int_S$ can be written as an integral along the real line plus a background term; following current usage we shall neglect the last ²⁰⁾.

For reasons that will become obvious later, we will split $\gamma_t(\ell)$ into two terms: a discrete part, of the form

$$\sum_j f_j(t) \delta(\ell - \alpha_j) \quad (\text{pure Regge poles})$$

plus a "continuous" part given by the sum of the jumps over the cuts, of the form

$$\sum_{j'} \beta_{j'}^c(\ell, t) \theta(\ell - \alpha_{j'}^c).$$

The α_i^c are the branch points. In the case where some of the poles coincide with the branch points, their contribution will be included in the second (cut) term. Therefore, in the first term, $\sum_j f_j \delta(\ell - \alpha_j)$, the sum runs only over isolated poles; and we allow for possible δ function singularities in the β_i^c located at the branch points (i.e., at $\ell = \alpha_i^c$), that will appear when Regge poles coincide with branch points (or accumulation points of branch points) of Regge cuts. Accordingly, (2.1) becomes

Theorem 2.4 (Ref. ¹⁷⁾): If we define

$$\eta(E) = \frac{EA(E)}{\int_c^E dE' A(E')} - \alpha - 1, \quad (2.6)$$

Then $\eta(E) \rightarrow 0$ as $E \rightarrow \infty$, and the same is true for higher moments [i.e., replacing $A(E)$ by $E^n A(E)$]. This will be recognized as a generalized FESR; we remark that it follows from non-oscillation alone ^{17), 21)}.

Theorem 2.5 (Ref. ¹⁷⁾, Appendix, and Ref. ²¹⁾): Solving (2.6) we can show that $A(E)$ can be represented as

$$A(E) = (\text{Constant}) \times [1 + \eta(E)] E^\alpha \exp \int_c^E dE' \eta(E') / E'. \quad (2.7)$$

From theorems 2.1 to 2.5 we see that $A(E)$ is essentially of a power form, a fact that sometimes will be made apparent by using the notation $A(E) \sim E^\alpha$. However, as exemplified in Refs. ¹⁵⁾ and ¹⁷⁾ such A 's allow for behaviours that differ in detail from a simple power behaviour [a characteristic example being a behaviour like (1.1)]. We then reserve the notation $A(E) \simeq f(E)$ for exact asymptotic behaviour, viz., we say that $A \simeq f$ if

$$\lim_{E \rightarrow \infty} A(E)/f(E) = 1. \quad (2.8)$$

For instance, if $A(E) = (1 + 1/\log E) E^\alpha (\log E)^\beta$, then $A \sim E^\alpha$, $A \simeq E^\alpha (\log E)^\beta$ but not $A \simeq E^\alpha$. It is precisely this difference in detail that permits us to deal with Regge cuts, whilst the results quoted above are sufficient to ensure that the basic asymptotic properties (signature, high-energy theorems, etc.) are still satisfied ²²⁾.

3. DETAILED BEHAVIOUR OF FESR

Before entering into the actual discussion of FESR, we shall make a trivial remark that will nevertheless help simplifying the notation. From the results just described, it is clear that, if $A(E)$ is non-oscillating and $A(E) \sim E^\alpha$, then also $A(E)E^{\alpha'}$ is non-oscillating and $A(E)E^{\alpha'} \sim E^{\alpha+\alpha'}$. This permits us, in the place of a Regge contribution of the form

$$\int_0^\infty d\ell E^{-\ell} \beta_1(\ell) = A_1(E)$$

to study a contribution like

$$\int_0^\infty d\ell E^{-\ell} \beta(\ell) = E^\alpha A_1(E) \equiv A(E); \quad \beta(\ell) = \beta_1(\ell - \alpha)$$

Therefore, we may reduce our analysis to branch points and/or poles located at $\ell = 0$.

A. PROOF OF NON-OSCILLATION

The proof is based on the two following results:

Lemma 3.1 (Ref. ²³), Vol. 1, p. 482, Theorem 4): If $\beta(\ell)$ satisfies the positivity assumption then, for all $\epsilon > 0$,

$$\limsup_{E \rightarrow \infty} \frac{1}{\log E} \int_0^\epsilon d\ell E^{-\ell} \beta(\ell) = 0. \quad (3.1)$$

As a matter of fact, this result holds even when β is not positive but still does not vanish identically. For the implications of this result for the shape of the diffraction peak, see Ref. ¹²).

Lemma 3.2 (Ref. 24): Let $\varphi_E(\ell)$ be any function (or distribution) vanishing for $\ell < 0$. Then, $\varphi_E(\ell) \rightarrow (\text{constant}) \times \delta(\ell)$ iff, for any ϵ small enough, and all $N > \epsilon$,

$$\lim_{E \rightarrow \infty} \int_0^\epsilon d\ell' \varphi_E(\ell') = 1 \quad (3.2a)$$

$$\lim_{E \rightarrow \infty} \int_\epsilon^N d\ell' \varphi_E(\ell') = 0 \quad (3.2b)$$

(Cf. also Ref. 17).)

We shall now prove that $A(E) = \int_0^\infty d\ell E^{-\ell} \beta(\ell)$, where $\beta(\ell)$ satisfies the positivity assumption, is non-oscillating. For this, form the function

$$\varphi_E(\ell) = \begin{cases} E^{-\ell} \beta(\ell) / \int_0^\infty d\ell' E^{-\ell'} \beta(\ell'), & \ell \geq 0 \\ 0, & \ell < 0. \end{cases} \quad (3.3)$$

Now, $A(E)$ is non-oscillating if $\varphi_E(\ell) \rightarrow (\text{constant}) \times \delta(\ell)$. In fact, the condition (2.3) may be written as

$$\int_0^\infty d\ell \varphi_E(\ell) u^\ell \rightarrow (\text{constant}) \times u^0 \equiv \psi(u).$$

In order to show clearly the mechanism at work, we shall split the proof of $\varphi_E \rightarrow \delta$ into steps. (i) Because of Lemma 3.2, it suffices to show that, if $\epsilon > 0$ is small enough,

$$I_E^\epsilon = \frac{\int_\epsilon^\infty d\ell E^{-\ell} \beta(\ell)}{\int_0^\epsilon d\ell E^{-\ell} \beta(\ell)} \rightarrow 0, \quad (3.4)$$

Since then one has, for instance,

$$\begin{aligned} \left| \int_{\epsilon}^N d\ell \varphi_E(\ell) \right| &= \frac{\left| \int_{\epsilon}^N d\ell E^{-\ell} \beta(\ell) \right|}{\left| \int_0^{\epsilon} d\ell' E^{-\ell'} \beta(\ell') - \int_{\epsilon}^N d\ell' E^{-\ell'} \beta(\ell') \right|} \leq \\ &\leq I_E^{\epsilon} / (1 - I_E^{\epsilon}) \rightarrow 0, \end{aligned}$$

i.e., (3.2b) is satisfied [the proof of (3.2a) would be quite similar].

(ii) Note that the positivity assumption allows us to take that $\beta(\ell) \geq 0$ for $\ell \leq \epsilon$. Now, $\int_0^{\epsilon} d\ell E^{-\ell} \beta(\ell)$ is decreasing, since its derivative is

$$(-1/E) \int_0^{\epsilon} d\ell \ell E^{-\ell} \beta(\ell) \leq 0.$$

Therefore, the sup lim of Lemma 3.1 becomes an ordinary limit, and we get

$$\lim_{E \rightarrow \infty} (\log E)^{-1} \log \int_0^{\epsilon} d\ell E^{-\ell} \beta(\ell) = 0.$$

Then, for any δ and all $E > E_{\delta}$,

$$\int_0^{\epsilon} d\ell E^{-\ell} \beta(\ell) > E^{-\delta}.$$

We shall choose $\delta < \epsilon/4$. (iii) On the other hand, it is known ^(3.5a)₂₃₎ that

$$E^{\epsilon - \epsilon'} \int_{\epsilon}^{\infty} d\ell E^{-\ell} |\beta(\ell)|, \text{ any } \epsilon' > 0, \quad (3.5b)$$

is bounded ²⁵⁾. Thus, for E large enough, and choosing $\epsilon' = \epsilon/2$,

$$\int_{\epsilon}^{\infty} d\ell E^{-\ell} |\beta(\ell)| < E^{-\epsilon/2} \quad (3.7)$$

Combining (3.5a, b) the desired result follows, as

$$I_E^{\epsilon} < \int_{\epsilon}^{\infty} / \int_{\epsilon}^{\infty} < E^{-\epsilon/2} \rightarrow 0. \quad \text{QED.}$$

We have shown:

Theorem 3.1. If $A_c(E)$ is the asymptotic contribution of a Regge cut with branch point at $-\alpha_c$ (plus perhaps a pole at $-\alpha_c$), then $A_c(E) \sim E^{-\alpha_c}$ and it is non-oscillating.

This result had been conjectured in Ref. ¹⁷⁾, where it was proved for a special class of β 's. The analysis presented here provides the most general proof of this conjecture (in fact, relaxing the positivity condition it is not difficult to construct counterexamples ¹⁷⁾).

Because of this result, Theorem 2.4 allows us to conclude that the $\eta(E)$ of (2.6) tends to zero, and the FESR is satisfied. What is more interesting is that the way in which the limit is attained depends critically on whether or not leading Regge cuts are present. We are now going to discuss this problem.

B. FESR WITHOUT LEADING CUTS

We suppose here that the leading ℓ plane singularity is an isolated pole, followed by what it makes little difference whether it be another pole or a cut. In this situation, we may write $A(E)$ as $CE^{\alpha+b(E)}$, and we shall assume for the explicit analysis that $C=1$, $\alpha=0$, and $b(E) \sim E^{-\alpha_c}$, with $\alpha_c < 1$ (the case $\alpha_c \geq 1$ is indeed simpler to handle).

We start with a rough estimate of the contribution of b to η ; this will prove useful later. We have:

Lemma 3.3. The contribution of $b(E)$ to $\eta(E)$ in (2.6) is of the form

$$(\text{constant}) \times \text{Max} \{ O(E^{-\alpha_c + \epsilon}), O(1/E) \}, \text{ any } \epsilon > 0.$$

Proof: From (2.6), (3.6), and taking for simplicity $c=0$,

$$\eta(E) = \frac{b(E) - (1/E) \int_0^E dE' b(E')}{1 + (1/E) \int_0^E dE' b(E')} \quad (3.7)$$

However, $b(E)$ is non-oscillating (Theorem 3.1); therefore, Theorem 2.2 tells us that, for any $\epsilon > 0$, and all E large enough, $E > E_\epsilon$,

$$b(E) < E^{-\alpha_c + \epsilon}$$

Defining the constant

$$C_\epsilon = \left| \int_0^{E_\epsilon} dE' b(E') \right|$$

we get

$$\left| \int_0^E dE' b(E') \right| < C_\epsilon + \frac{1}{1 + E^{-\alpha_c}} E^{-\alpha_c + \epsilon + 1},$$

and hence,

$$\eta(E) = \frac{O(E^{-\alpha_c + \epsilon}) - O(1/E) O(E^{-\alpha_c + \epsilon + 1})}{1 + O(1/E) O(E^{-\alpha_c + \epsilon + 1})},$$

Q.E.D.

Corollary 3.1. The result is clearly also true for any $A(E) = a(E) + b(E)$, with $a(E) \sim E^0$ but not necessarily $a(E) \simeq E^0$ (the same proof applies).

The exact estimate of $\eta(E)$ will now be given. Starting from (3.7), which is exact, we remark that the preceding Lemma 3.3 has shown that both $b(E)$, and $(1/E) \int^E dE' b(E')$ tend to zero; therefore, asymptotically,

$$\eta(E) \simeq b(E) + \frac{1}{E} \int^E dE' b(E').$$

But $b(E)$ is non-oscillating; therefore, it satisfies a FESR, so that writing (2.6) for it,

$$\left[\int_0^E dE' b(E') \right] / E b(E) \rightarrow 1/(1-\alpha_c),$$

i.e., $\eta(E) \simeq b(E) [\alpha_c/(1-\alpha_c)]$. Quite generally:

Theorem 3.2. If

$$A(E) = C E^\alpha + b(E), \quad b(E) = \int_{\alpha_c}^{\infty} d\ell E^{-\ell} \beta(\ell)$$

i.e., A is of the form "dominating pure Regge pole plus corrections (either from poles or cuts)", then $\eta(E)$ is non-oscillating and has the asymptotic value

$$\eta(E) \simeq \left(\frac{-\alpha_c + \alpha}{1 - \alpha_c - \alpha} \right) b(E) / C E^\alpha \sim E^{-\alpha_c - \alpha}, \quad (3.8a)$$

if $-\alpha_c + \alpha > -1$ and

$$\eta(E) \simeq (\text{constant}) \times (1/E) \quad (3.8b)$$

if $-\alpha_c + \alpha > 1$.

C. FESR WITH LEADING CUTS

In this case, $A(E) = a(E) + b(E)$, where $a(E) \sim E^\alpha$ and $b(E) \sim E^{-\alpha_c}$, $\alpha' > -\alpha_c$. (In explicit proofs we take as always $\alpha = 0$.) The fact that the leading singularity contains cuts, and is not an isolated pole, is displayed by writing

$$a(E) = \int_0^\infty d\ell E^{-\ell} \beta(\ell), \quad \int_0^\ell d\ell' \ell' \beta(\ell') > 0 \quad (3.9)$$

for all ℓ small enough. (If we had an isolated pole the ℓ' factor in the second integral would kill the corresponding δ function, and $\int^\ell \equiv 0$.) We remark that this situation contains both cases where we have either a cut alone or a pole plus a cut; this will be further discussed in Section D below.

The analysis is now somewhat more involved, and again we shall split it into successive steps. (i) Because of Corollary 3.1 (and Lemma 3.3) we know that the contribution of $b(E)$ to $\eta(E)$ is $o(E^{-\alpha_c+0})$. As we shall show in a moment that $\eta(E) \sim E^0$, it follows that we can, when evaluating asymptotically η , drop altogether the details of the terms in b . Hence, Eq. (2.6) becomes,

$$\eta(E) = \frac{E a(E)}{\int_0^E dE' a(E')} - 1 + o(E^{-\alpha_c+0}). \quad (3.10a)$$

(ii) Let us choose $\alpha_c < \gamma < 1$ (as before we assume $\alpha_c < 1$ for simplicity). Then, we can substitute $a(E)$ in (3.9) by

$$\int_0^\gamma d\ell E^{-\ell} \beta(\ell).$$

In fact, we are thereby committing an error of $o(E^{-\gamma+0})$, which can be absorbed into the $o(E^{-\alpha_c+0})$ term in (3.10a). Therefore, we can write

$$\eta(E) = \frac{E \int_0^\gamma d\ell E^{-\ell} \beta(\ell)}{\int_0^E dE' \int_0^\gamma d\ell' E^{-\ell'} \beta(\ell')} - 1 + o(E^{-\alpha_c+0}) \quad (3.10b)$$

(iii) Since the integration ranges in (3.10b) are compact, and the integrands positive-definite, we can, for all E , interchange orders of integration. The dE' integration is then readily performed yielding

$$\begin{aligned} \eta(E) &= \frac{\int_0^{\gamma} d\ell E^{-\ell} \beta(\ell)}{\int_0^{\gamma} d\ell' E^{-\ell'} \beta(\ell') (1-\ell')^{-1}} - 1 + o(E^{-\alpha_c+0}) = \\ &= \frac{\int_0^{\gamma} d\ell E^{-\ell} \beta(\ell)}{\int_0^{\gamma} d\ell' E^{-\ell'} \beta(\ell') + \int_0^{\gamma} d\ell' E^{-\ell'} \beta(\ell') + \int_0^{\gamma} d\ell' E^{-\ell'} \beta(\ell') O(\ell'^2)} \\ &\quad - 1 + o(E^{-\alpha_c+0}), \end{aligned} \tag{3.10c}$$

where in the last step we have used the fact that, inside the range of integration, $0 \leq \ell' < 1$ so that a Maclaurin expansion, $1/(1-\ell') = 1 + \ell' + O(\ell'^2)$ holds. (iv) Now, we want to show that, for any N ,

$$\lim_{E \rightarrow \infty} F_E^N = 0; \quad F_E^N = \frac{\int_0^{\gamma} d\ell E^{-\ell} \ell^{N+1} \beta(\ell)}{\int_0^{\gamma} d\ell' E^{-\ell'} \ell'^N \beta(\ell')} \tag{3.10d}$$

In fact: given any ϵ , we have, using by now familiar arguments,

$$\int_{\epsilon}^{\gamma} d\ell E^{-\ell} \ell^m \beta(\ell) \leq E^{-\epsilon/2}$$

for all $E \geq E_{\epsilon}$. We then take ϵ so small that $\beta(\ell) \geq 0$ for $\ell \leq \epsilon$, what is possible by virtue of the positivity assumption. Then, we can apply the mean value theorem to the $\int_0^{\epsilon}$ part which, together with the above estimate for $\int_{\epsilon}^{\gamma}$ leads us to

$$\begin{aligned} F_E^N &= \frac{[1 + O(E^{-\epsilon/2})] \int_0^{\epsilon} d\ell \beta(\ell) \ell^{N+1} E^{-\ell}}{[1 + O(E^{-\epsilon/2})] \int_0^{\epsilon} d\ell \beta(\ell) \ell^N E^{-\ell}} = \\ &= [1 + O(E^{-\epsilon/2})] \ell_{\epsilon} \end{aligned}$$

with $0 \leq \ell_{\epsilon} \leq \epsilon$. Therefore, taking E large enough, $O(E^{-\epsilon/\ell}) < \epsilon$ and thus $|\frac{N}{E}| < 2\epsilon$, which finishes the proof of (3.10d). (v) The analysis is now easily completed. Because of (3.10d), the term containing $O(\ell'^2)$ in the denominator of the right-hand side of (3.10c) is negligible against the term containing ℓ' . Consequently,

$$\eta(E) = \frac{\int_0^{\gamma} d\ell E^{-\ell} \beta(\ell)}{\int_0^{\gamma} d\ell' \beta(\ell') E^{-\ell'} + [1+o(1)] \int_0^{\gamma} d\ell' \ell' E^{-\ell'} \beta(\ell')} - 1 + o(E^{-\alpha+0})$$

i.e.,

$$\begin{aligned} \eta(E) &= -[1+o(1)][\eta_1(E)/\eta_0(E)], \\ \eta(E) &\equiv \int_0^{\gamma} d\ell \ell^{\alpha} \beta(\ell) E^{-\ell}; \end{aligned} \tag{3.11}$$

we have arrived at a decomposition of $\eta(E)$ in the form of a product of functions, $-[1+o(1)]$, $\eta_1(E)$ and $[\eta_0(E)]^{-1}$ that are non-oscillating and $\sim E^0$ (the last two because of Theorem 3.1). In summary, we have proved:

Theorem 3.3. If the leading Regge singularity contains a cut, then the $\eta(E)$ of the FFSR (2.6) is non-oscillating and $\sim E^0$, i.e., decreases logarithmically.

The announced difference in the form of the vanishing of $\eta(E)$ is thus made apparent by comparing Theorems 3.2 and 3.3.

D. DISCUSSION AND EXAMPLES

There are three points on which we want to elaborate further. First, what kind of behaviours lead to, and are compatible with, the result of Theorem 3.3. In order to see this, let us suppose that we approximate the actual cuts by a cut of the root type. Specifically, suppose that $\beta(\ell) \propto (\ell - \alpha_c)^V$; then, it is easy to check that $A_c(E) \propto E^{\alpha_c} (\log E)^{-V}$. The substitution into the FESR (2.6) then shows that $\eta(E) \simeq 1/\log E$ (this result is in fact more general; see below). Therefore, in the presence of Regge cuts we not only expect slow "logarithmic" variations of $A_c(E)$, but also slow, "logarithmic" fulfilment of the FESR, a fact that is intuitively clear.

Second, one may ask which results would be modified if a Regge pole sits, or does not sit, on the branch point of the cut. The situation is as follows: if we have only a cut then (in the simple example considered above) we expect $A(E) \simeq E^{\alpha_c} (\log E)^{-V}$, whereas if, in addition, there is a pole at $\ell = \alpha_c$, then $A(E) \simeq E^{\alpha_c} [1 + (\log E)^{-V}]$. However, it may be directly checked that $\eta(E)$ is $1/(\log E)$ in the first case, and $1/(\log E)^V$ in the second: whilst we can establish that "the pole sits on the cut", the absence of a pole is impossible to establish in general, as one may always have $V = 1$, and both behaviours would then coincide. In the last case direct fits would, of course, be necessary in addition to asymptotic FESR estimates.

Third, the result quoted above when no pole is present can, in fact, be generalized to reasonable β 's (essentially differentiable β 's for which $[\partial \beta / \partial \ell] / \beta \rightarrow 0$ as $\ell \rightarrow \alpha_c$). However, the precise behaviour $\eta(E) \simeq 1/\log E$ does not obtain in general, as it is seen by constructing explicit counterexamples.

4. DETECTION OF REGGE CUTS

The results of the preceding Section provide us with a powerful tool for detecting asymptotic contributions of Regge cuts. For this, we only have to check the "order" of $\eta(E)$, i.e., if $\eta(E) \sim E^\lambda$, to ascertain whether $\lambda < 0$ (no cuts) or $\lambda = 0$ (cuts present). The estimates of λ , that may be obtained combining non-oscillation of η with the results of Section 2, may be obtained directly or indirectly, as is going to be described.

A. DIRECT METHODS

The following formulae hold (the number in brackets after each of them refers to the relevant theorem in Section 2);

$$\lambda = \lim_{E \rightarrow \infty} \log \eta(E) / \log E \quad (2.2)$$

$$\lambda = \frac{1}{\log u} \lim_{E \rightarrow \infty} \log [\eta(uE) / \eta(E)], \text{ any } u \quad (2.1)$$

$$\lambda = \lim_{E \rightarrow \infty} \frac{E^{n+1} \eta(E)}{\int_c^E dE' E'^n \eta(E')} - 1 - n, \text{ any } n \quad (2.4)$$

(4.1)

Apart from this, a further result may be obtained:

Theorem 4.1. If Regge cuts are present, then

$$\psi_E(u) \simeq u^{\eta(E)+\alpha}, \quad \psi_E(u) \equiv \frac{A(uE)}{A(E)},$$

$$A(E) \sim E^\alpha.$$

(4.2)

Proof: From (2.7) it follows that

$$\begin{aligned}\psi_E(u) &= \frac{1+\eta(uE)}{1+\eta(E)} u^\alpha \exp \int_E^{uE} dE' \frac{\eta(E')}{E'} = \\ &= [1+o(1)] u^\alpha \exp \left\{ \eta(E) \int_1^u dv \frac{\eta(vE)}{v\eta(E)} \right\}\end{aligned}$$

after change of variables, $E' = vE$. Since η is non-oscillating, Theorems 2.1 and 2.3 tell us respectively that $\eta(vE)/\eta(E) \rightarrow v^\lambda = 1$ (as $\lambda = 0$ due to the presence of Regge cuts) and that we have sufficient majorizations to apply Lebesgue's theorem, thereby interchanging limits and integration. Thus,

$$\begin{aligned}\psi_E(u) &\simeq [1+o(1)] u^\alpha \exp \left\{ \eta(E) \int_1^u dv/v \right\} \simeq \\ &\simeq [1+o(1)] u^{\alpha+\eta(E)} \simeq u^{\alpha+\eta(E)},\end{aligned}$$

Q.E.D. Whereas the use of this result for direct computations is dubious, it will nevertheless be crucial in analyzing the asymptotics of infinite range sum rules (see below) and, mainly, of dispersion integrals. (Ref. ⁸⁾ and Section 5.)

B. INDIRECT METHODS

As shown in Ref. ¹⁷⁾, if $A(E)$ is non-oscillating and $\sim E^\alpha$,

$$\lim_{E \rightarrow \infty} \frac{\int_0^\infty dE' e^{-E'/E} A(E')}{EA(E)} \rightarrow \Gamma(1+\alpha). \quad (4.3)$$

On the other hand, if $A(E) = 1+b(E)$ (leading pole), the left-hand side in (4.3) may be explicitly computed to obtain (here $\alpha=0$),

$$\begin{aligned}\frac{\int_0^\infty dE' e^{-E'/E} A(E')}{EA(E)} &= \Gamma(1) + \{ \Gamma(1+\alpha_c) - \Gamma(1) \} b(E) + \\ &\quad + o(b(E)).\end{aligned} \quad (4.4)$$

If the leading term contains a cut, however, we have $A(E) = a(E) + b(E)$, and

$$\frac{\int_c^\infty dE' e^{-E'/E} A(E')}{E A(E)} = \int_{c/E}^\infty du e^{-u} \frac{A(uE)}{A(E)},$$

and, substituting Eq. (4.2),

$$\frac{\int_c^\infty dE' e^{-E'/E} A(E')}{E A(E)} = \Gamma(1 + \eta(E)) + o(\eta(E)). \quad (4.5)$$

what is interesting here is that the right-hand sides in (4.4) and (4.5) can be calculated directly from experimental data; and so are the $\int_c^E$ of the left-hand side integrals. Therefore, we may use (4.4) and (4.5) for self-consistent tests of explicit extrapolations (power-type for pure Regge, logarithmic when cuts are included) beyond the region of experimentally accessible energies.

Of course, the form of the functional fits is to some extent arbitrary. A possible line of reasoning would, however, be as follows: on a fairly large energy range, variations slower than a logarithm are impossible to detect. Thus, it is plausible to take the functional form $E^\alpha (a + [\log E + b]^{-\delta})$ for a , i.e., to approximate the actual ℓ plane singularities by a pole at α plus a cut of the root type, $\theta(\ell - \alpha) \times (\ell - \alpha)^\delta$; and the same for extra terms in the expansion of A (viz., b , etc.). This is, for instance, the extrapolation used by A. Bialas and E. Bialas¹⁰⁾.

To finish this Section we want to point out a last possible way of detecting Regge cuts. Regge parameters have been obtained with the help of so-called "continuous moment" sum rules. In the same spirit it might be possible to detect Regge cuts (at least in a rough way) replacing the continuous moments, E^μ , by functions of the form $E^\mu (a + [\log E + b]^{-\delta})$. In this context, we argue that the "theoretical" value of α in the FESR (2.6) is unity (essentially constant cross-sections). Thus, one may argue that most of the error of the FESR,

$\eta(E)$ is due to cut effects, i.e., one may obtain the parameters a , b , δ , by minimizing the functions

$$\eta_{a,b,\delta}^{\nu}(E) = \frac{A(E) E^{\mu+1} [a + (\log E + b)^{-\delta}]}{\int_c^E dE' A(E') E'^{\mu} [a + (\log E' + b)^{-\delta}]} - \mu - 1, \quad (4.6)$$

where the minimum must hold for all μ 's and large E 's.

5. ASYMPTOTIC DISPERSION RELATIONS

As it was the case before for sum rules, we shall see that also forward elastic dispersion relations show completely different asymptotic behaviours depending on whether the leading term in A contains or not Regge cuts. It is interesting to remark that this difference appears when $A \sim E^n$, $n = \text{integer}$ ¹⁵⁾. This is the reason why the usual Regge analysis is insensitive to cuts for e.g., charge-exchange scattering ^{15), 26)} where $A^{(-)}(E) \sim E^{0.55}$.

The results we are going to describe are more or less known, but we reproduce them here for the sake of completeness.

A. NO LEADING CUTS

The forward absorptive part may then be written as ²⁷⁾

$$\begin{aligned} A^{(+)}(E) &= 2\sigma E + c(E) & ; c(E) &\sim E^{\gamma} \\ A^{(-)}(E) &= b(E) & ; b(E) &\sim E^{\beta}, \end{aligned} \quad (5.1)$$

and here as before the detailed form of b , c is irrelevant. (β , $\gamma \neq \text{integer}$.) We shall assume the validity of the Goldberger sum rule ²⁸⁾

$$I = (\text{pole terms}) + (\text{scattering length terms}) + \int_0^\infty dE [A_+(E) - A_-(E)]/k^2 = 0 \quad (5.2)$$

We shall not justify this assumption here, but refer to the work of Goldberger ²⁸⁾. (Cf., also Refs. ⁸⁾, ¹⁵⁾ and ²⁹⁾.) The dispersion relation we will consider is of the form

$$\begin{aligned} \frac{D(E)}{k^2} &= \frac{1}{\pi} \text{P.P.} \int_m^\infty \frac{dE'}{k'^2(E'^2 - E^2)} \{E' A^{(+)}(E') + E A^{(-)}(E')\} + \\ &\quad + \int_\Lambda, \\ \int_\Lambda &\equiv \int_\Lambda dz \frac{z T^{(+)}(z) + E T^{(-)}(z)}{(z^2 - m^2)(z^2 - E^2)}, \end{aligned} \quad (5.3)$$

where Λ is a contour around the (finite) Bros-Epstein-Glaser domain ¹⁸⁾ of (possible) non-analyticity of $T(z)/(z^2 - m^2)$; the reason for this is that we will consistently work to order $1/E$, and the high energy contribution of $\int_\Lambda$ is precisely ¹⁵⁾ $O(1/E)$. Finally, the following remark should be made: because of the Pomeranchuk theorem ⁴⁾, ¹⁵⁾, ³⁰⁾ (that we will assume to hold in the version of Ref. ¹⁵⁾), $A_+(E)/A_-(E) \rightarrow 1$. Therefore, at high energy, it is irrelevant to consider A_+ or A_- or $A^{(\pm)}$.

Whatever we choose, from (5.3) we obtain, after substitution of (5.1) and some manipulations

$$\begin{aligned} \alpha(E) &\equiv \alpha_+(E) + \alpha_-(E) \simeq \frac{(\cos \pi \gamma + 1) c(E)}{(\sin \pi \gamma) 2\sigma E} \\ \alpha_+(E) - \alpha_-(E) &\simeq \frac{(1 - \cos \pi \beta) b(E)}{(\sin \pi \beta) \sigma E}; \\ \alpha_\pm(E) &\equiv \text{Re } T_\pm(E) / \text{Im } T_\pm(E). \end{aligned} \quad (5.4)$$

Proof: The term σE contributes $O(1/E)$ to the $\alpha_{\pm}$; it may thus be neglected. The rest is nothing but Eq. (4.3) of Ref. 15).

B. LEADING CUTS

Here we write $A^{(\pm)}$ in the form

$$\begin{aligned} A^{(+)}(E) &= E a(E) \\ A^{(-)}(E) &= b(E) \end{aligned}$$

(5.5)

b , as before (5.1). The contributions of b may be handled as before; we are thus left with that of $Ea(E)$. The problem here is caused by the fact that the contribution of the last term, $Ea(E)$, does not vanish as fast as before, but, as shown in Ref. ⁸⁾, (Theorem), yields a term of the form

$$\frac{1 - \cos \pi \eta_{\pm}(E)}{\sin \pi \eta_{\pm}(E)}$$

Putting everything together, we are finally led to the conclusion

$$\alpha(E) = \alpha_+(E) + \alpha_-(E) \simeq \frac{1 - \cos \pi \eta_+(E)}{\sin \pi \eta_+(E)} + \frac{1 - \cos \pi \eta_-(E)}{\sin \pi \eta_-(E)}$$

$$\alpha_+(E) - \alpha_-(E) \simeq \frac{(1 - \cos \pi \beta) b(E)}{(\sin \pi \beta) E a(E)},$$

(5.6)

and the difference with (5.4) is obvious: $\alpha_+ + \alpha_-$ decreases only logarithmically now.

6. DISCUSSION AND COMMENTS

The present article has two aims: first, to show that Regge cuts are not as hopeless as they may seem to be at first sight. In fact, we have proved that, under very general conditions, their contributions can be handled (at least in the forward direction) and lead to unambiguous predictions. Second, we wanted to ascertain up to which point one can decide, from the present experimental data, whether Regge cuts exist or not. In this context, our analysis provides a set of predictions, valid when Regge cuts are present, that differ sharply from those of a pure Regge pole model. These predictions are expressible in terms of measurable quantities and can, therefore, be tested with the help of the very precise presently available data ^{4),31)}. The detailed analysis will be done II; an insight will be found in the Appendix. We now want to elaborate a bit further in the relevance of our analysis, restricting our discussion to the forward elastic case ³²⁾.

It is clear that we can never be sure of having a cut or a set of closely spaced poles, a fact easily understood if we recall that the contribution of a cut is of the form $\int d\ell E^{-\ell} \beta(\ell)$ and note that any integral can be approximated arbitrarily well by a finite sum. However, we feel that, if the data can be fitted equally well either with a single cut, or with several poles whose parameters are somewhat displaced from what one would expect on the basis of a t channel resonance analysis, then the first alternative is favoured.

It is, on the other hand, possible that neither pure Reggeistic predictions nor the poles-plus-cut ones agree with experiment. (This does not seem to be the actual situation; cf., Refs. ^{7),8)} and especially II.) Although this situation would be difficult to understand, it seems that one should interpret it as due to a breakdown of the assumptions put forward here. In this context, we remark that the necessary analyticity domains have not yet been proved from more fundamental assumptions (either field-theoretic or S matrix theory), and the foundations of our positivity hypothesis are shaky, if not purely heuristic.

ACKNOWLEDGEMENTS

The author is grateful to Professor D. Amati for a very stimulating discussion. Comments and suggestions of Professors A. Martin and P. Pascual and Drs. G. Cohen-Tannoudji, C.B. Chiu, R. Finkelstein and J.A. Rubio are also acknowledged.

A P P E N D I X

COMPARISON WITH EXPERIMENTAL DATA

This Appendix has been written in collaboration with J.A. Rubio. A detailed expanded and improved version of it will be published separately. In this Appendix only πp collisions have been analyzed; the extension to pp and $\bar{p}p$ ones will be carried out in II.

A. DIRECT FITS

We have fitted the values of σ_{total} for $\pi^{\pm}p$, given in Ref. ³¹⁾, using both pure pole and cut parametrizations. The fits are done with the standard minimization program of the TC Division at CERN (Minuit).

In what respects the pure Regge pole fit, we have repeated the fits of the Lindenbaum group ^{4),31)}, obtaining a χ^2 of ~ 3.9 . It is to be noted that absolute minima different from the values of Ref. ³¹⁾ exist if one fits separately σ_+ and σ_- ; however, these minima were not considered significant as here the Pomeranchuk theorem is only satisfied up to $\gtrsim 1$ standard deviation and the values of the ρ , P' intercepts are much displaced off their current values.

Fits with Regge cuts are of two kinds: pure leading cut and cut plus pole. We have not introduced cuts in the term corresponding to $\sigma_+ - \sigma_-$; in fact, the cut correction is a small correction, so that, since $\sigma_+ - \sigma_- \rightarrow 0$, we may expect it to be negligible. Therefore, we fit $\sigma_+ - \sigma_- = 2Bk^{-C}$, obtaining, of course, the same values for B and C as in Ref. ³¹⁾. The fit with a pure cut then is made with the ansatz $\sigma_+ + \sigma_- = 2\sigma_{\infty}(a + \log k)^{\delta}$. We obtain a χ^2 of ~ 2.4 , and the values of the parameters are ³³⁾
 $\sigma_{\infty} = 26.17 \pm 0.59$, $a = -1.13 \pm 0.19$, $\delta = -0.11 \pm 0.015$. We remark that the number of parameters is also five here, as we do not have to take into account the P' .

If we introduce a new parameter in the fit, viz.,

$$\sigma_+ + \sigma_- = 2 \sigma_\infty [c + (a + \log k)^\delta]$$

corresponding to a (Pomeranchuk) pole in the Regge cut, the χ^2 does not seem to improve, and the value of the new parameter, d , is compatible with zero. We think that this may be interpreted as suggesting that the leading Regge singularity is a pure cut. However, we are aware of the fact that the good χ^2 obtained makes it difficult to draw unambiguous conclusions.

B. SUM RULES

- i) In Table I we have presented a computation of the FESR (2.6). No attempt has been made to compare with the pure Regge pole predictions (3.8); in fact, whilst the contribution of (3.8) is $\sim 3 \times 10^{-4}$, the low energy contribution is orders of magnitude larger. Of course, this is a reflection of the well-known fact that pure Regge pole predictions are, at least in the ~ 20 GeV region, in clear disagreement with FESR.
- ii) Considering that part $\int_c^E$ (that can be computed with the experimental data of Table I) of the sum rule (4.3), it may be seen that the difference between its exact values and the value obtained entering the η 's as determined in Table I, the difference is $\sim 1\%$. Thus, we may expect (4.3) to be a good test of extrapolations. In this spirit we have compared Eqs. (4.3) and (4.5), computing the right-hand side "experimentally", the $\int_c^E$ part of the left-hand side also experimentally, and the $\int_E^\infty$ part of the left-hand side using the cut extrapolation obtained in part A. of this Appendix. The fulfilment of the sum rule is excellent and generally of 1% . Precise values are, however, not very meaningful, as small oscillations (presumably due to experimental errors) change the exact value of the difference, typically, from 0.01 to 0.0001. This is to be expected, given the smallness of these numbers.

Again here the pure pole formula, Eq. (4.4) breaks down. Of course, this was to be expected and is due to the same mechanism causing the breakdown of the FESR (2.6).

C. ASYMPTOTIC DISPERSION RELATIONS

We have checked the dispersion relation predictions for πp scattering. Since the difference between pure poles and cuts arises in the quantity $\alpha = \text{Re } T^{(+)} / \text{Im } T^{(+)}$, it is this quantity that we consider. We present here only the asymptotic estimate, Eqs. (5.4) and (5.6); "exact" computations, using definite extrapolations will be given in II. The results are displayed, for the cut case, in Table II. Again the pure pole prediction is not shown; typically, $\alpha^{\text{pole}} = -0.008$, which shows that the low energy contributions cannot be neglected or the disagreement is violent.

It is possible to try to improve agreement between pure Regge pole predictions and experiment by using direct fits to the whole $T(E)$. However, whilst the current poles suffice to obtain reasonable agreement, the χ^2 increases (typically to values ~ 15). We defer a thorough discussion with comments on the relevance of these results until the publication of the fuller set of results in II¹⁴).

TABLE I

k_{lab}	$\eta +$	$\eta -$
16.13	-0.0556	-0.0876
16.81	-0.0554	-0.0875
17.67	-0.0555	-0.0874
18.19	-0.0553	-0.0894
18.52	-0.0553	-0.0873
18.95	-0.0552	-0.0864
19.69	-0.0549	-0.0853
21.13	-0.0542	-0.0842
22.09	-0.0544	-0.0842

Experimental numbers taken from Ref. 31). Over-all error ~ 0.0015 , obtained by shifting c in (2.6) from 400 MeV/c to 0-800 MeV/c.

TABLE II

k_{lab}	$\alpha^{\text{exper.}}$	α^{cut}
14.08	-0.318 ± 0.034	-0.228
16.01	-0.313 ± 0.030	-0.227
18.07	-0.251 ± 0.035	-0.226
20.17	-0.261 ± 0.043	-0.224

Experimental data as in Table I. Errors: ~ 0.006 ,
obtained as in Table I, and ~ 0.05 (i.e., $\sim \eta^2$)
of "theoretical" origin. Cf., Ref. ⁸).

REFERENCES AND FOOTNOTES

- 1) D. Amati, A. Stanghellini and S. Fubini, *Nuovo Cimento* 26, 896 (1962);
D. Amati, M. Cini and A. Stanghellini, *Nuovo Cimento* 30, 193 (1963);
S. Mandelstam, *Nuovo Cimento* 30, 1127 (1963) and 1148 (1963).
- 2) G. Cohen-Tannoudji, A. Morel and H. Navelet, *Nuovo Cimento* 48, 1075 (1967);
P.G.O. Freund and P.J. O'Donovan, *Phys.Rev.Letters* 20, 1329 (1968);
R.J. Ribers and M. Saunders, Imperial College preprint ICTP/67/20 (1968);
C.B. Chiu and J. Finkelstein, CERN preprints TH. 892 and 914 (1968).
- 3) I. Ya. Pomeranchuk, *Soviet Phys., JETP* 3, 306 (1956).
- 4) See, for instance, S.J. Lindenbaum in the Conference on Pion-Nucleon
Scattering, University of California, Irvine, California (1967),
and references therein.
- 5) A. Della Selva, L. Masperi and R. Odorico, *Nuovo Cimento* 55, 602 (1968).
- 6) R.J.N. Phillips and W. Rarita, *Phys.Rev.* 139, B1336 (1965) and 140, B200
(1965);
G. Höhler et al., *Phys.Letters* 20, 79 (1966).
- 7) J.A. Rubio and F.J. Yndurain, unpublished.
- 8) F.J. Yndurain, CERN preprint TH. 910 (1968).
- 9) J.D. Jackson, *Revs.Modern Phys.* 37, 484 (1965).
- 10) A. Bialas and E. Bialas, *Nuovo Cimento* 37, 1686 (1965).
- 11) V.N. Gribov, I.Ya. Pomeranchuk and K.A. Ter-Martirosian, *Phys.Rev.* 139,
B184 (1965);
J.C. Polkinghorne, *J.Math.Phys.* 6, 1960 (1965).
- 12) J.L. Gervais and F.J. Yndurain, *Phys.Rev.Letters* 20, 27 (1968).
- 13) That $\text{Im } g_t(\ell) \geq 0$ is a very weak condition may be understood by recal-
ling that what one expects intuitively of a cut is a $\sim (\ell - \alpha_c)^{\frac{1}{2}}$
behaviour, where α_c is the branch point.

- 14) J.A. Rubio and F.J. Yndurain, to be published.
- 15) J.L. Gervais and F.J. Yndurain, Phys.Rev. 167, 1289 (1968).
- 16) R. Dolen, D. Horn and C. Schmid, Phys.Rev.Letters 19, 402 (1967);
A.A. Logunov, L.D. Soloviev and A.N. Tavkhelidze, Phys.Letters 24, B181
(1967).
- 17) J.L. Gervais and F.J. Yndurain, Phys.Rev. 169, 1187 (1968).
- 18) J. Bros, H. Epstein and V. Glaser, Communs.Math.Phys. 1, 240 (1965); and
Nuovo Cimento 31, 1265 (1964).
- 19) See, for instance, M. Froissart, in "Theoretical Physics", IAEA, Vienna
(1963).
- 20) This can be proved to be possible in potential theory.
- 21) G. Karamata, Mathematica Cluj. (Rumania) 4, 38 (1930), and Bull.Soc.Math.
de France 61, 55 (1933).
- 22) Cf., References 15 and 17, where this fact is first noticed.
- 23) G. Doetsch, "Handbuch der Laplace-Transformation", Birkhauser Verlag,
Basel (1950).
- 24) I.M. Gelfand and G.E. Shilov, "Generalized Functions", Vol. 1, Academic
Press, New York (1964).
- 25) This may fail in the case where essential ℓ plane singularities exist.
We suppose that this is not the case.
- 26) Cf., e.g., Ref. 6).
- 27) We do not consider here the possibility that the cross-sections vanish
as a power of the energy as proposed by N. Cabibbo, L. Horwitz,
J.J.J. Kokkedee and Y. Ne'eman, Nuovo Cimento 45A, 275 (1966). As a
matter of fact, FESR (Refs. ^{5),7)}) as well as dispersion relation deter-
minations (4,8) seem to rule out the value $\alpha'_p = 1-0.07$ proposed here.
- 28) M.L. Goldberger, in "Dispersion Relations and Elementary Particles",
Wiley, New York (1961).

- 29) H. Lehmann, Nuclear Phys. 29, 300 (1962).
- 30) A. Martin, Nuovo Cimento 39, 704 (1965);
D. Amati, M. Fierz and V. Glaser, Phys.Rev.Letters 4, 89 (1960).
- 31) K.J. Foley et al., Phys.Rev.Letters 19, 330 (1967) and 19, 193 (1967).
- 32) Cf., the references given in the Introduction for the analysis of other situations.
- 33) The errors given here are not to be taken too seriously. In fact, the shape of the χ^2 is far from parabolic and shows strong oscillations. Thus, for example, a phenomenon similar to that of the previous fit (existence of spurious minima) may also occur.