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CONSTRAINTS ON $\pi\pi$ PARTIAL WAVES
FROM POSITIVITY AND ANALYTICITY

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A B S T R A C T

We consider the constraints on the partial waves for $\pi^0\pi^0$ or $\pi\pi$ in the isospin zero channel, that follow from analyticity and positivity. We show that, if these two requirements are met, then a set of inequalities on the partial waves with $\ell \geq 2$ and energy below threshold must be satisfied. The converse is also proved showing that, if these constraints are fulfilled, then an analytic scattering amplitude with positive absorptive part can be constructed. The practical importance of these conditions is discussed showing that, for example, they imply the inequalities (pion mass = 1)

$$a_{\ell+2} < \frac{1}{16} \frac{(\ell+2)(\ell+1)}{(\ell+5/2)(\ell+3/2)} a_{\ell};$$

here a_{ℓ} are the scattering lengths,

$$a_{\ell} = \lim_{q \rightarrow 0} f_{\ell}(s) / q^{2\ell}.$$

Furthermore, these inequalities (which had essentially been also found by A. Martin) are saturated very accurately when $\ell \geq 4$. The relevance of our analysis for models of $\pi\pi$ scattering is discussed.

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1. INTRODUCTION

In recent papers ^{1),2)}, A. Martin and collaborators have shown that, by using positivity and dispersion relations, one can obtain a set of conditions on the partial waves of the $\pi\pi$ scattering amplitude which ensure full crossing symmetry. As remarked by Martin himself, these constraints are incomplete in the sense that extra consistency conditions, following again from positivity and analyticity, must also be imposed for higher waves ($\ell \geq 2$). It is precisely these conditions that we want to investigate in the present article.

To be definite : let $T(s,t)$ be the scattering amplitude for $\pi^0\pi^0$ scattering or $\pi\pi$ scattering with isospin zero in the s channel ^{*)}. Then it is known that from, e.g., axiomatic field theory, one can show that a Froissart-Gribov representation

$$f_\ell(s) = \int_{z_0(s)}^{\infty} dz A_\ell(s,z) Q_\ell(z); \quad z_0(s) = \frac{4+s}{4-s} \quad (1)$$

holds for the even waves, $\ell \geq 2$, whenever s is in the range $0 \leq s \leq 4$ (we are consistently taking units such that the pion mass = 1). Furthermore, from unitarity, it follows that $A_\ell(s,z)$, which is the absorptive part of the scattering amplitude in the t channel is positive. Now the question we want to study is the obtention of a set of constraints on the f 's such that, if they are satisfied then there exists a unique scattering amplitude T solving (1) and with the properties of positivity and analyticity, and conversely. In this context we are able to solve completely the problem, deriving a set of necessary and sufficient conditions. Each of these conditions involves an infinite number of waves; however, by suitable majorizations, we are able to reformulate them to be valid when only a finite number of waves is known. As a particular

^{*)} Or, more generally, any combination of the form

$$(1 + \lambda/3) T_s^{(0)}(s,t) + (2\lambda/3) T_s^{(2)}(s,t)$$

with $\lambda \geq -1$.

Proof

In Ref. 5), pp. 10,11, it is shown that a necessary and sufficient condition for the solution to (5) to be unique is that, if $\varphi(z)$ is a continuous function admitting the bound (7), then

$$\left. \begin{aligned} \inf_{\gamma(z^{-2}), \gamma \geq \varphi} \mu(\varphi) &= \sup_{\gamma(z^{-2}), \gamma \leq \varphi} \mu(\varphi), \\ \mu(\varphi) &\equiv \int dz A(z) \varphi(z). \end{aligned} \right\} \quad (9)$$

As is well known, any continuous function $\psi(u)$ in the interval $(0,1)$ can be approximated by two polynomials $p_i^\epsilon(u)$

$$p_1^\epsilon(u) < \psi(u) < p_2^\epsilon(u), \quad |p_2^\epsilon(u) - p_1^\epsilon(u)| < \epsilon \quad \text{on } (0,1);$$

hence, by a change of variables, we can find two "polynomials" $p_1^\epsilon(z^{-2}), p_2^\epsilon(z^{-2})$ with

$$p_1^\epsilon(z^{-2}) < \varphi(z) < p_2^\epsilon(z^{-2}), \quad |p_2^\epsilon(z^{-2}) - p_1^\epsilon(z^{-2})| < \epsilon z^{-l_0-1} \quad \text{on } (z_0, \infty). \quad \text{Therefore, we get}$$

$$|\inf \mu(\varphi) - \sup \mu(\varphi)| \leq \epsilon M_{l_0},$$

and letting $\epsilon \rightarrow 0$ the desired result, (9), follows.

2.2. The "Froissart-Gribov" moment problem

For convenience we shall rewrite (1) dropping the s dependence,

$$f_l = \int_{z_0}^{\infty} dz A_t(z) Q_l(z); \quad z_0 > 1, \quad l \geq l_0; \quad A_t \geq 0. \quad (10)$$

Clearly, the problem associated with (10), that we shall call Froissart-Gribov moment problem, is a special case of (4) and, therefore, the result applies:

Corollary 1

A necessary and sufficient condition for (10) to hold is that whenever $p_Q(z) \geq 0$ on (z_0, ∞) ,

$$p_Q(z) \equiv \sum_{\substack{n=l_0 \\ n=\text{even}}}^{m_0} c_n Q_n(z),$$

then

$$f(p_Q) \equiv \sum c_n f_n = \int dz A_t(z) p_Q(z) \geq 0.$$

The generalization of Theorems 2 and 3 is less simple; nevertheless, we shall be able to prove the following results:

Corollary 2

Form the quantities μ_ℓ ,

$$\mu_\ell \equiv \frac{2^{\ell+1}}{\ell! \sqrt{\pi}} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\Gamma(\ell+\nu+3/2)}{\Gamma(\nu+1)} f_{\ell+2\nu}, \quad \ell \geq l_0. \quad (11)$$

Then a solution to (10) exists if, and only if, these μ_ℓ exist and satisfy the inequalities (6). The functional

$$f(\varphi) \equiv \int_{z_0}^{\infty} dz A_t(z) \varphi(z)$$

is then defined for all φ 's which, in the interval (z_0, ∞) admit the estimate (7) or, equivalently (as $z_0 \geq 1$), the estimate

$$|\varphi(z)| \leq \text{constant} \times Q_{\ell_0}(z).$$

Proof A. Necessity

We have the representation ⁹⁾

$$z^{-l-1} = \frac{z^{l+1}}{l! \sqrt{\pi}} \sum_{v=0}^{\infty} (-1)^v \frac{\Gamma(l+v+3/2)}{\Gamma(v+1)} Q_{l+2v}(z), \quad (12)$$

which is convergent because $z_0 > 1$.

If A_t exists, then, because of Corollary 1

$$f(\varphi) \geq 0 \text{ if } \varphi \geq 0$$

Since ¹⁰⁾

$$Q_\ell(z)/Q_\ell(z_0) - Q_{\ell+m}(z)/Q_{\ell+m}(z_0) \geq 0,$$

it then follows that

$$f_{\ell+m} \leq \frac{Q_{\ell+m}(z_0)}{Q_\ell(z_0)} f_\ell. \quad (13)$$

But, for large n ,

$$Q_{\ell+m}(z_0)/Q_\ell(z_0) = O(1/z_0^m). \quad (14)$$

Hence, the expression (11) for the μ 's is convergent. It is then sufficient to plug in the representation (12) to get that

$$\delta^k \mu_\ell = f(b_{\ell,k}),$$

$$b_{\ell,k}(z) = z^{-l-1} (z_0^{-2} - z^{-2})^k.$$

As $b_{\ell,k} \geq 0$ on (z_0, ∞) , Corollary 1 tells us that $f(b_{\ell,k})$ must be so and (6) is satisfied.

B. Sufficiency

If (6) is satisfied, Theorem 2 tell us that there exists an A_t solving Eq. (5) with the desired properties. To show that this A_t also solves (10), we insert the expansion ⁹⁾, inverse to (12),

$$Q_\ell(z) = \frac{\sqrt{\pi}}{z^{\ell+1}} \frac{\ell! z^{-\ell-1}}{\Gamma(\ell/2+1)\Gamma(\ell/2+1/2)} \sum_{\mu=0}^{\infty} \frac{\Gamma(\ell/2+\mu+1)\Gamma(\ell/2+\mu+1/2)}{\mu! \Gamma(\ell+\mu+3/2)} z^{-2\mu}, \quad (15)$$

to get that the f_ℓ can be recovered as

$$f_\ell = \frac{\sqrt{\pi}}{z^{\ell+1}} \frac{\ell!}{\Gamma(\ell/2+1)\Gamma(\ell/2+1/2)} \sum_{\mu=0}^{\infty} \frac{\Gamma(\ell/2+\mu+1)\Gamma(\ell/2+\mu+1/2)}{\mu! \Gamma(\ell+\mu+3/2)} \mu_{\ell+2\mu} \quad (16)$$

where this expression is again convergent because, since $z_0^{-2n} - z_0^{-2n} \geq 0$, we have that

$$\mu_{\ell+2m} \leq O(1/z_0^{2m}) \mu_\ell.$$

Finally, we have:

Corollary 3

If a solution to (10) exists then it is unique. The proof is a simple consequence of Theorem 3 and the fact that we have just shown that the problems (10), (5) are equivalent.

2.3. The associated Hausdorff problem

Given a Froissart-Gribov problem, (10), we construct its associated Hausdorff problem,

$$f_\ell = \int_{\mu_0}^{\infty} d\mu B(\mu) \mu^{-\ell-1}; \quad \ell \geq \ell_0, \quad \mu_0 = z_0 + \sqrt{z_0^2 - 1} > 1. \quad (17)$$

Of course, B. will be different from A. We have

Theorem 4

The associated Hausdorff moment problem has a solution if and only if,

$$\begin{aligned} \delta^0 f_\ell &= f_\ell > 0 \\ \delta^1 f_\ell &= \mu_0^{-2} f_\ell - f_{\ell+2} \geq 0 \\ &\dots \\ \delta^k f_\ell &= \sum_{j=0}^k \binom{k}{j} (-1)^j \mu_0^{2(j-k)} f_{\ell+2j} \geq 0 \\ &\dots \end{aligned} \tag{18}$$

In particular, these conditions are satisfied if the original Froissart-Gribov problem had a solution ¹¹⁾, so that (18) is a necessary (but not sufficient!) condition for (10) to be true.

Proof

From the representation

$$Q_\ell(z) = \int_0^\infty dt [u(z,t)]^{-\ell-1}, \quad u(z,t) = z + (z^2-1)^{1/2} \cosh t,$$

and the fact that, when $z \geq z_0$, $u(z,t) \geq \mu_0$, it follows that the functions

$$\begin{aligned} \beta_{\ell,k}(z) &= \sum_{j=0}^k \binom{k}{j} (-1)^j \mu_0^{2(j-k)} Q_{\ell+2j}(z) = \\ &= \int_0^\infty dt [u(z,t)]^{-\ell-1} \{ \mu_0^{-2} - [u(z,t)]^{-2} \}^k \end{aligned}$$

are positive. Furthermore, they clearly satisfy the bound (7), so that, by Corollary 1, we should have

$$f(\beta_{\ell,k}) = \delta^k f_{\ell} \geq 0,$$

and the second part of the Theorem is proved. The first part is obvious, as (Theorem 2) Eqs. (18) are nothing but the necessary and sufficient conditions for (17) to have a solution.

3. CONDITIONS ON THE PARTIAL WAVES FROM POSITIVITY. ANALYTICITY

3.1 Positivity

From the preceding analysis it is quite clear that the results of Corollaries 1 to 3 will apply to (1) with $\ell_0 = 2$, when $0 \leq s \leq 4$, providing the desired set of necessary and sufficient constraints, namely, Eqs. (6) with the μ_{ℓ} 's given by (11). To what extent do Eqs. (6) imply positivity of $T(s,t)$? [The converse, of course, is well known; cf., e.g., Ref. 1)]. Let us use the fact, to be proved below, subsection 3.3, that, if (6) is satisfied, then the scattering amplitude

$$\left. \begin{aligned} T(s,t) &= f_0(s) + \sum_{\ell=\text{even} \geq 2}^{\infty} (2\ell+1) f_{\ell}(s) P_{\ell}(\cos \theta_s); \\ \cos \theta_s &= 1 + t/2(s-4) \end{aligned} \right\} \quad (19)$$

is defined and can be extended to be analytic in the cut $\cos \theta_s$ plane. If we then make the assumption ¹²⁾ of crossing symmetry, we will get that the partial wave expansion of A_t in the t channel,

$$\left. \begin{aligned} A_t(s,t) &= \sum_{\ell=\text{even} \geq 0} (2\ell+1) \text{Im} f_{\ell}(t) P_{\ell}(\cos \theta_t); \\ \cos \theta_t &= 1 + s/2(t-4) \end{aligned} \right\} \quad (20)$$

is convergent for $\cos \theta_t$ in the Martin ellipse ¹³⁾. Furthermore, it is positive for all t and $0 \leq s \leq 4$. Now, we see that the available information is not sufficient to conclude that T has positivity, as this necessitates to have not only $A_t \geq 0$, but all the $\text{Im} f_{\ell}(t) \geq 0$ which is clearly a stronger requirement ¹⁴⁾.

Of course what happens is that (1) does not incorporate the full content of positivity which also demands that $\partial^n A_t / \partial s^n \geq 0$. This problem is currently under investigation.

3.2 Analysis of the conditions

At first sight conditions (6) do not appear very convenient because (11) makes an infinite number of waves involved in each inequality. Fortunately, however, it is possible, using again the positivity of A_t , to terminate effectively the sums (11) after a finite number of terms giving a precise estimate of the error committed. This error is, in the region of physical interest (s near threshold), very small so that the fact that only a finite number of waves is retained is not worse than, say, the uncertainty produced by experimental errors in the known waves.

To be precise, we consider the physical case $\ell_0 = 2$, suppose that only the first waves up to $\ell = N$ are known, and that we want to use inequalities (6) for $\ell + k \leq L$, $L \leq N$, i.e., involving μ_ℓ 's up to $\ell = L$. Let us then define, from (11),

$$\mu_\ell^N(s) \equiv \frac{2^{\ell+1}}{\ell! \sqrt{\pi}} \sum_{\nu=0}^{(N-\ell)/2} (-1)^\nu \frac{\Gamma(\ell+\nu+3/2)}{\Gamma(\nu+1)} f_{\ell+2\nu}(s), \quad \ell \geq 2 \quad (21a)$$

$$R_\ell^{N,L}(s) \equiv \mu_\ell(s) - \mu_\ell^N(s) = \frac{2^{\ell+1}}{\ell! \sqrt{\pi}} \sum_{\nu=1+(N-\ell)/2}^{\infty} (-1)^\nu \frac{\Gamma(\ell+\nu+3/2)}{\Gamma(\nu+1)} f_{\ell+2\nu}(s). \quad (21b)$$

Then, as the series defining R is alternate, it is clear that a good estimate will be given for it as soon as its terms become decreasing in absolute magnitude, since then,

$$\left. \begin{aligned} R_\ell^{N,L}(s) &= (-1)^{1+(N-\ell)/2} B_\ell^{N,L}(s) \\ 0 < B_\ell^{N,L}(s) &< \frac{2^{\ell+1}}{\ell! \sqrt{\pi}} \frac{\Gamma([N+\ell]/2+5/2)}{\Gamma([N-\ell]/2+2)} f_{N+2}(s), \end{aligned} \right\} \quad (22)$$

i.e., the sign and a simple bound for R will be known. This situation will be encountered as soon as

$$\frac{\Gamma([N+l]/2 + 5/2)}{\Gamma([N-l]/2 + 2)} f_{N+2}^{(j)} \leq \frac{\Gamma([N+l]/2 + 3/2)}{\Gamma([N-l]/2 + 1)} f_N^{(j)}.$$

We want this to be valid for all $l \leq L$. The most defavourable situation is when $l = L$ so that the inequality above will be satisfied if

$$\frac{f_{N+2}^{(j)}}{f_N^{(j)}} \leq \frac{N-L+2}{N+L+3}. \quad (23)$$

We now need a bound for f_{N+2}/f_N . From (13), we have

$$f_{N+2}^{(j)}/f_N^{(j)} \leq Q_{N+2}(z_0)/Q_N(z_0),$$

and, on the other hand, using the formula

$$z_0 Q_m(z_0) - Q_{m+1}(z_0) = \frac{z_0^2 - 1}{m+1} \sum_{n=1}^{\infty} (2n+1) Q_n(z_0),$$

we see that

$$z_0 Q_m(z_0)/Q_{m+1}(z_0) = 1 + \frac{2m+3}{m+1} (z_0^2 - 1) + \text{rest};$$

therefore, since the rest is positive [and $O(1/z_0)$],

$$\frac{Q_{m+1}(z_0)}{Q_m(z_0)} < \frac{z_0}{1 + (z_0^2 - 1)(2 + [m+1]^{-1})} < \frac{z_0}{2z_0^2 - 1}.$$

The bound is very good for large values of n and z_0 [its accuracy is $\sim 1/(n+1)z_0$] which, of course, are the interesting ones. Then, we get

$$\frac{f_{m+2}^{(j)}}{f_m^{(j)}} < \left[\frac{z_0}{2z_0^2 - 1} \right]^2, \quad (24)$$

so that to satisfy (23) it is sufficient to have

$$\frac{z_0^2}{(2z_0^2-1)^2} \leq \frac{N-L+2}{N+L+3} \quad (25)$$

Next, we compute $B^{N,L}$. Of course, the interesting quantity is its maximum contribution relative to the last non-neglected term in (21), viz., the quantity

$$\epsilon_{N,L}(s) = \max_{l \leq L} \frac{B_l^{N,L}(s)}{2^{l+1} \Gamma([N+l]/2+3/2) f_N(s) / l! \sqrt{\pi} \Gamma([N-l]/2+1)},$$

because, then, one will have $\mu_l = \mu_l'^N$, where $\mu_l'^N$ is given by (21a) with $f_N(s)$ substituted by

$$f_N(s) + (-1)^{1+(N-l)/2} \epsilon_{N,L,l}(s), \quad 0 \leq \epsilon_{N,L,l}(s) < \epsilon_{N,L}(s),$$

that is to say, we have only increased the "experimental" error in the last known wave by the amount $\leq \epsilon_{N,L}$. An explicit bound on $\epsilon_{N,L}$ is then obtained directly from (22) and (24):

$$\epsilon_{N,L}(s) \leq \frac{N+L+3}{N-L+2} \left[\frac{z_0}{2z_0^2-1} \right]^2 \quad (26)$$

To get an idea of the practical meaning of these results, suppose that we want to use as many of the constraints (6) as possible, i.e., $L=N$, and take $s \geq 2$. Then, (25) is satisfied if $L \leq 30$. If, furthermore, $s \geq 3$, the relative error is $\epsilon_{L,L}(s) < 17\%$. As seen in this example, in the upper half of the range $0 \leq s \leq 4$, full use can be made of the conditions (6) up to $l+k \simeq 30$, and the approximation made by replacing μ_l by $\mu_l'^N$ is not worse than an error of $\leq 17\%$ in the last wave, even if we consider the extreme situation where the first thirty waves are known and one wants to use all of them. Clearly, and at least in the physically interesting region, Eqs. (6) in the approximation (21a) are as good for practical purposes as if they only involved the first N waves.

3.3. Analyticity

Let us consider the physical situation $\ell_0 = 2$, for definiteness.

Define the function k_2 as

$$\left. \begin{aligned} k_2(u, \xi) &\equiv \left(u^2 \frac{\partial}{\partial u} + \frac{u}{2}\right) [k(u, \xi) + k(-u, \xi) - 2], \\ k(u, \xi) &\equiv (1 - 2u\xi + u^2)^{-1/2} \end{aligned} \right\} \quad (27)$$

By direct inspection we may check that $k_2(1/z, \cos \theta_s)$ is analytic in $\cos \theta_s$, and satisfies the bound (7) with $\ell_0 = 2$, when $\cos \theta_s$ is in the cut complex plane. From this we shall prove:

Theorem 5

If the f_ℓ are solutions of a Froissart-Gribov moment problem, i.e., if Eqs. (6) are satisfied, then the amplitude $T(s, t)$, as given by (19), exists and can be analytically continued into the cut t plane; its absorptive part is A_t , which is positive. More generally, analyticity (but perhaps not positivity) will still be obtained even if the f_ℓ are not solutions of a Froissart-Gribov moment problem, but only the constraints (18) are satisfied.

Remarks

A. The result of the theorem shows that, as was to be expected, analyticity is weaker than positivity, even in its rudimentary form $A_t \geq 0$.

B. Conditions (18) do not constitute a unique set, although they seem to us the most convenient. Thus, Common¹¹⁾, has obtained a set of determinantal inequalities that also provide a sufficient set of conditions.

C. That the f_ℓ 's coming from (1) generate an analytic amplitude is not surprising since analyticity is one of the ingredients used to prove the Froissart-Gribov representation ^{*}). Actually, a direct proof of the first part of the Theorem could be given, essentially by replacing, in the proof below, the identity

$$\sum_{n=0}^{\infty} u^n P_n(\xi) = k(u, \xi)$$

by

$$\sum_{n=0}^{\infty} (2n+1) Q_n(u) P_n(\xi) = 1/(u-\xi).$$

However, it is interesting to find weaker "phenomenological" conditions for analyticity, viz., Eqs. (18), and to prove the converse to the Froissart-Gribov analysis (see 3.4. B. in this connection).

Proof of Theorem 5

We first prove the second half. Assume (18) to hold; due to Theorem 2, we get that the function

$$T_2(s, \cos \theta_s) = \int dz B(z) k_2(1/z, \cos \theta_s), \quad (28a)$$

is defined and analytic in the cut $\cos \theta_s$ plane. By inserting in (27), (28a) the identity

$$\sum_{n=0}^{\infty} u^n P_n(\cos \theta_s) = k(u, \cos \theta_s),$$

We see that, formally,

$$f_0(s) + T_2(s, \cos \theta_s) = \sum_{\ell=even \geq 0} (2\ell+1) f_\ell(s) P_\ell(\cos \theta_s) \equiv T(s, t). \quad (28b)$$

Since the expansion of k_2 is convergent when $\cos \theta_s$ is inside the Martin ellipse (in the t plane), the expansion

^{*}) In this connection, see also T. Kinoshita, J.J. Loeffel and A. Martin, Phys.Rev. 135, B1369 (1964).

(28b) is also convergent there, i.e., $f_0(s) + T_2(s, \cos \theta_s)$ coincides with $T(s, t)$, as given by (19), in that region, and it thus follows that it is its (unique) analytic continuation to the cut t plane.

The proof of the first part of the Theorem is now very simple: by what has just been said, T is analytic in the cut t plane; performing the usual Froissart-Gribov analysis for it, we get that

$$f_\ell(1) = \int dz A'_t(1, z) Q_\ell(z),$$

where A'_t is the absorptive part of T . That A'_t coincides with A_t then follows from Corollary 3. Q.E.D.

3.4. Oscillating functions

If we had taken the "wrong" isospin combination (what we shall denote by the superscript w) or if we are above threshold then, as we may expect oscillations of A_t^w , $\text{Re } A_t$, ρ , we shall not be able to obtain clear-cut results. For the sake of completeness, however, we shall comment briefly on the problem.

If we write generically f_ℓ^w , $\text{Re } f_\ell$, $\text{Im } f_\ell$ as h_ℓ , and A_t^w , $\text{Re } A_t$, ρ , as H , then we can decompose¹⁵⁾ $H = \tilde{\rho}_1 - \tilde{\rho}_2$, $\tilde{\rho}_i \geq 0$. Consequently, one can generalize the results obtained before just by substituting the sentence "that f_ℓ satisfy such requirements" by "that h_ℓ be the difference of two quantities satisfying such requirements". More definite statements may, however, be made about three important questions.

A. Corollary 3 remains unchanged, because the argument of Ref. 5), p. 97, Theorem 3.9 still holds.

B. If the h_ℓ are given by a Froissart-Gribov representation or, more generally, by a Hausdorff representation, then one can, in a way that is unique (up to, of course, the lowest waves)

construct an analytic scattering amplitude. This is because the proof of Theorem 5 also holds, mutatis mutandis, for oscillating H 's. In particular, we can give a simple proof of the fact that analyticity in ℓ implies analyticity in t :

Theorem

If h_λ is analytic in λ for $\text{Re } \lambda \geq \ell_0$, and $O(\exp\{-\alpha\lambda\})$, $\alpha > 0$ there, then one can construct a scattering amplitude, analytic in the cut t plane and bounded by t^{ℓ_0} .

Proof

It is sufficient to show that there exists H with

$$h_\ell = \int d\mathbf{z} \, z^{-\ell-1} H(\mathbf{z}), \quad \ell \geq \ell_0.$$

But this is so: because of the analyticity and boundedness of h_λ , it follows ¹⁶⁾ that we can write

$$h_\lambda = \int d\xi \, e^{-\lambda\xi} H_1(\xi),$$

i.e., h_λ is the Laplace transform of H_1 , and it suffices to change variables, $z = e^\xi$. Q.E.D. Thus, we see that analyticity in ℓ implies full analyticity in t .

C. If one has detailed information on the continuity properties of H , then more definite results might be obtained, as the methods of Ref. 5), p. 97 ff. could be applied. We shall not pursue this problem further, though.

4. APPLICATIONS AND COMMENTS

4.1. Scattering lengths

As follows from the analysis of subsection 3.2, the most favourable situation is when $q=0$, as then we can neglect all but

the first term in (11) without committing any error. In fact, if the scattering lengths are given by (2), Eqs. (11), (6) give the constraints

$$\begin{aligned}
 \delta^0 a_\ell &\equiv a_\ell \geq 0 \\
 \delta^1 a_\ell &\equiv \frac{2^{\ell+1} \Gamma(\ell+3/2)}{\Gamma(\ell+1)} a_\ell - 4 \frac{2^{\ell+3} \Gamma(\ell+2+3/2)}{\Gamma(\ell+3)} a_{\ell+2} \geq 0 \\
 \delta^k a_\ell &\equiv \sum_{j=0}^k \binom{k}{j} (-1)^j 2^{\ell+2k+4j+1} \frac{\Gamma(\ell+2j+3/2)}{\Gamma(\ell+2j+1)} a_{\ell+2j} \geq 0 \\
 &\dots\dots\dots
 \end{aligned}
 \tag{29}$$

For $\delta^1 a_\ell$ we get that ¹⁷⁾

$$a_{\ell+2} - \frac{1}{16} \frac{(\ell+2)(\ell+1)}{(\ell+3/2)(\ell+5/2)} a_\ell \geq 0,$$

i.e., the result announced in the Introduction, Eq. (3). Similarly, decomposing $\delta^2 a_\ell \propto \delta^1 a_\ell - \delta^1 a_{\ell+2}$, (28) gives that

$$\begin{aligned}
 &|a_{\ell+2} - 16 \frac{(\ell+2+3/2)(\ell+3+3/2)}{(\ell+3)(\ell+4)} a_{\ell+4}| \leq \\
 &\leq \frac{1}{16} \frac{(\ell+1)(\ell+2)}{(\ell+3/2)(\ell+5/2)} |a_\ell - 16 \frac{(\ell+3/2)(\ell+5/2)}{(\ell+1)(\ell+2)} a_{\ell+2}|,
 \end{aligned}$$

i.e., the constraints (3) are practically saturated when $\ell \geq 4$.

4.2 Comments on $\pi\pi$ parametrizations

The results we have just obtained indicate why the low energy $\pi\pi$ scattering amplitude is sensitive only to a few parameters, something that had already been made clear by some independent work ^{1),2),18)}: once the S and D waves are fixed the remaining waves are very small and can be estimated with little uncertainty.

Since, on the other hand, it has been shown ^{1),2)} that the S wave fixes with good accuracy the D wave, this may be taken as an explanation of the fact that any model that gives a_0 and neglects terms above quadratic is likely to represent a good approximation to the low energy $\pi\pi$ scattering amplitude. These conditions are obviously met by Weinberg's linear parametrization ¹⁹⁾, and also by the one term Veneziano model ²⁰⁾ [in fact, for the last, direct computation shows that $a_{\ell+2} < (2 \times 10^{-4}) a_{\ell}$] and thus our results may be considered as a justification of these approximations.

4.3. Discussion

There are two points to which we would like to devote a few words. First, one may question the usefulness of our constraints on the grounds that they are only valid (except at $s=4$) for unphysical energies. This is, of course, not the case because, due to the analyticity properties of the f_{ℓ} , their values for $s < 4$ are determined by their value on the physical region. As a matter of fact, actual partial wave analyses are based on dispersion relations which determine the f_{ℓ} 's below as well as above threshold and on which our constraints could be imposed as consistency conditions ²¹⁾. Second, although we have shown that the problems caused by the fact that (11) contains all waves can be tackled, it would still be nice to have conditions involving only finite numbers of waves. This might be done by generalizing the representation of Lemma 1. An obvious candidate for such generalization is the set of "binomials"

$$\sum_{j=0}^k \binom{k}{j} (-1)^j Q_{\ell+2j}(z) / Q_{\ell+2j}(z_0).$$

In this context, it would be necessary to show that they are positive if $z \geq z_0$, something which is true for $k=0,1$ and for ℓ or z_0 large and when z_0 is close to 1. To prove it in general one would have to show, for example, that $Q_{\lambda}(z)/Q_{\lambda}(z_0)$ is "totally convex", i.e., to show that

$$(-1)^m \frac{\partial^m}{\partial \lambda^m} \frac{Q_{\lambda}(z)}{Q_{\lambda}(z_0)} \geq 0,$$

a fact that is again true for $n=0,1$, on the range λ or z_0 large, when $z_0 \simeq 1$, and that we have checked numerically for $0 \leq \lambda \leq 30$, $1.2 \leq z_0 \leq z \leq 20$, but we have not been able to prove in general.

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$$\mu_m = \int_0^1 du \, \psi(u) u^m, \quad m=0, 1, \dots$$
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$$\alpha_l = \lim_{f_l(s)} f_l(s) / (4-s)^l,$$

Eq. (39) should read

$$\alpha_{l+2} < \frac{1}{256 m_\pi^2} \frac{(l+2)(l+1)}{(l+5/2)(l+3/2)} \alpha_l.$$

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