CM-P00057919

Ref. Th. 1005

POSITIVE DOUBLE SPECTRAL FUNCTIONS ARE INCOMPATIBLEWITH THE EXISTENCE OF THE $f(1260)$ RESONANCE

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A B S T R A C T

We show that the existence and the known properties of the $2^+, I = 0$ $f(1260)$ resonance make it very unlikely that the double spectral function $\rho(s,t)$ of the Mandelstam representation for the isospin-zero $\pi\pi$ scattering amplitude be positive. In particular, Atkinson's parametrization $\rho = \rho_{\text{elastic}} + \rho_{\text{inelastic}}$ with $\rho_{\text{inelastic}} \geq 0$ is completely ruled out.

Ref. Th. 1005

18 March 1969

Recently, Atkinson ¹⁾ has produced a constructive proof of the existence of functions satisfying the Mandelstam representation and unitarity for $\pi\pi$ scattering. In his work, the parametrization ²⁾

$$\rho(s, t) = \rho_{el}(s, t) + \rho_{inel}(s, t), \quad \rho_{inel}(s, t) \geq 0 \quad (1)$$

of the double spectral function is essential to insure unitarity. Some of the physical implications of a more general hypothesis, viz., that

$$\rho(s, t) \geq 0, \quad (2)$$

have been investigated by Martin ³⁾. We want to pursue this search further, to ascertain whether Atkinson's scheme, or, at least, the less restrictive assumption of Martin, allows for physically realistic solutions. The answer is here found to be negative; in fact, we will show that the existence and known properties of the $2^+, I=0$ $f(1260)$ resonance: (i) cast serious doubts about the possibility of having condition (2) satisfied, and (ii) rule out completely assumption (1). The proofs of these statements are as follows.

- (i) - Let us consider the isospin $I=0$ part of the $\pi\pi$ amplitude. Martin ³⁾ has shown that, under assumption (2), the Froissart-Gribov formula for the s channel partial wave amplitudes $\eta_\lambda(s)$,

$$\eta_\lambda(s) = \frac{\sqrt{s}}{k} \int d\tau Q_\lambda(\tau) A_t(s, \tau)$$

$$\text{Im } \eta_\lambda(s) = \frac{\sqrt{s}}{k} \int d\tau Q_\lambda(\tau) \rho(s, \tau); \quad \tau = 1 + t/4k^2, \quad (3)$$

where the physical values are obtained for $\lambda = 2, 4, \dots$, and where A_t is the t channel absorptive part, holds for all λ with $\text{Re } \lambda > 1$. Furthermore, and if λ is real and larger than 1,

$$\frac{\text{Im } \eta_\lambda(s)}{\text{Im } \eta_{\lambda'}(s)} \geq \frac{Q_\lambda(z_0(s))}{Q_{\lambda'}(z_0(s))} > 1, \quad \lambda' > \lambda \quad (4)$$

where

$$z_0(s) = 1 + t_0(s)/2k^2, \quad t_0(s) = \min \left\{ \frac{4s}{s-16}, \frac{16s}{s-4} \right\}.$$

The proof of (4) is based on the remark that the relation

$$\partial/\partial x \{ Q_{\lambda'}(x)/Q_\lambda(x) \} < 0 \quad \text{for } \lambda' > \lambda, x > 1,$$

proved in Ref. ³⁾ for integer λ , also holds for real λ , as is obvious from the representation ⁴⁾

$$Q_\lambda(x) = \int_0^\infty d\xi [x + (x^2-1)^{1/2} \cosh \xi]^{-\lambda-1},$$

and then proceed just as in Ref. ³⁾. By continuing analytically the elastic unitarity, $\text{Im } \eta_\ell = |\eta_\ell|^2$, one also gets

$$1 \geq \text{Im } \eta_\lambda(s) > 0, \quad \lambda \text{ real} > 1, \quad (5)$$

for s below inelastic threshold. It is then clear that Eqs. (4) and (5) together make impossible the existence of elastic resonances of spin other than zero, a fact that Mahoux and Martin had already discovered in a general context ⁵⁾.

What about inelastic resonances? If we write

$$\eta_\ell(s) = \frac{1 - \epsilon_\ell e^{2i\alpha_\ell}}{-2i},$$

where ϵ_ℓ is the inelasticity parameter, then, at the energy of a resonance, s_R , $\alpha_{\ell R}(s_R) = \pi/2$, and thus

$$\text{Im } \eta_{\ell_R}(s_R) = [1 + \epsilon_{\ell_R}(s_R)]/2.$$

If, in particular, we consider the $f(1260)$ resonance (which is the one that, because of its quantum numbers and elasticity, is more likely to be relevant for our problem), we know that the branching ratio

$$\Gamma(f(1260) \rightarrow 2\pi) / \Gamma_{\text{total}} \gtrsim .9,$$

and hence $\text{Im } \eta_2(s_f) \gtrsim 0.9$. On the other hand, direct computation shows that

$$Q_{1+0}(z_0(s_f)) / Q_2(z_0(s_f)) \gtrsim 2, \quad (6)$$

so that

$$\text{Im } \eta_{1+0}(s_f) \gtrsim 1.9. \quad (7)$$

We have not obtained a clear-cut contradiction because Eq. (5) is only valid for $\sqrt{s} \leq 4m_\pi$. Nevertheless, if we take into account the facts that, first, the four pion forces are very weak and, second, Eq. (5) would still be valid above inelastic threshold if one could approximate inelastic effects by two-particle processes of the type $\pi + \pi \rightarrow A + B$, it is clear that the large violation of the bound $\text{Im } \eta \leq 1$ represented by Eq. (7) is very unlikely.

- (ii) - If Atkinson's condition (1) is assumed, then, since ^{1),5)} ρ_{el} is always positive, Eq. (2) is also fulfilled and, in particular, all the results of Ref. ³⁾ remain true. Moreover, if we define

$$A_t^{\ell}(s, t) = \text{Re } A_t(s, t) + i \rho_{\ell}(s, t),$$

$$\begin{aligned}\eta_{\lambda}^{el}(s) &= \frac{\sqrt{s}}{k} \int dz Q_{\lambda}(z) A_t^{el}(s, t) \\ \text{Im } \eta_{\lambda}^{el}(s) &= \frac{\sqrt{s}}{k} \int dz Q_{\lambda}(z) \rho_{el}(s, t),\end{aligned}\quad (8)$$

Then, since $\rho_{el} \leq \rho$, it follows that, for $\text{Re } \lambda > 1$, Eq. (8) is valid and defines η_{λ}^{el} analytic in λ and satisfying the conditions of Carlson's theorem ⁶⁾. In particular, for λ real > 1 ,

$$\text{Im } \eta_{\lambda}^{el}(s) \leq \text{Im } \eta_{\lambda}(s). \quad (9)$$

If we now form the combination

$$\eta_{\lambda}^{el}(s) - \eta_{\lambda}^{el}(s)^* + 2i \eta_{\lambda}(s) \eta_{\lambda}^{*}(s),$$

it vanishes for physical $\lambda = 2, 4, \dots$, and, because of Carlson's theorem ⁶⁾, everywhere to the right of $\text{Re } \lambda = 1$. Therefore, if $\lambda > 1$,

$$\text{Im } \eta_{\lambda}^{el}(s) = |\eta_{\lambda}(s)|^2. \quad (10)$$

Combining (10) and (9), it follows that, in the present situation [Eq. (1) valid] the bound (5) also holds in the inelastic region, and since the arguments leading in (i) above to (7) can still be made, the rigorous contradiction emerges : if we have (1), then the existence of the $f(1260)$ leads to a large violation of unitarity.

In conclusion, we have shown that the condition $\rho_{inel}(s, t) \geq 0$ is, by $\sim 90\%$, incompatible with experimental evidence. Since the less restrictive hypothesis $\rho(s, t) \geq 0$ also meets difficulties, it seems that one should have $|\rho_{inel}(s, t)| > \rho_{el}(s, t)$ for

s large enough (what is not surprising) and $\rho_{inel}(s,t) < 0$ for some values of s, t (which is more unexpected). It is also to be remarked that it is only the existence of the f(1260) that gives a clear-cut contradiction. In fact, other resonances [as, e.g., the f'(1515)] have large inelasticity, $\epsilon \lesssim 0.14$, and thus fit into the scheme. Similarly, if we make the assumption of positivity for the relevant πN double spectral function, due to spin, one gets relations of the type ⁷⁾

$$\begin{aligned} \text{Im } \eta_\lambda(s) / \text{Im } \eta_{\lambda'}(s) &> Q_\lambda(z(s)) / Q_{\lambda'}(z_0(s)), \\ \eta_\ell(s) &= \frac{1}{2\ell+1} \left\{ (p_0+m)(\sqrt{s}-m) [(\ell+1)\eta_{\ell+}(s) + \ell\eta_{\ell-}(s)] + \right. \\ &\quad \left. + (p_0-m)(\sqrt{s}+m) [(\ell+1)\eta_{\ell+,-}(s) + \ell\eta_{\ell-,-}(s)] \right\}, \end{aligned} \tag{11}$$

which, due to the large number of intervening waves are always satisfied. Finally, we want to mention that it has also been shown ⁸⁾ that positive ρ 's are in contradiction with the experimental shapes of diffraction peaks. While the contradictions here appear to us less conclusive ⁹⁾ than the one we have found, in which spin (which, as exemplified by our analysis of πN scattering above may be important) is totally neglected, they substantiate the conclusion that oscillating ρ 's are necessary.

It is a pleasure to acknowledge useful discussions with Dr. A. Martin and Dr. C. Llewellyn-Smith.

REFERENCES AND FOOTNOTES

- 1) D. Atkinson - NORDITA Preprint (1968), and Rutherford Laboratory Preprint (1968), (to be published in Nuclear Phys.).
- 2) We use standard notations : ρ is the double spectral function, ρ_{el}/ρ_{inel} its elastic/inelastic part. Only the isospin zero part of the $\pi\pi$ amplitude is considered. Finally, s, t, u are the usual Mandelstam variables ; $k^2 = s-4$, and we take the pion-mass unity.
- 3) A. Martin - CERN Preprint Th. 968 (1969).
- 4) W. Magnus, F. Oberhettinger and R.P. Soni - "Formulas and Theorems for the Special Functions of Mathematical Physics", Springer Verlag, Berlin (1966).
- 5) G. Mahoux and A. Martin - Nuovo Cimento 33, 883 (1964).
- 6) See, e.g., M. Froissart - "Theoretical Physics", I.A.E.A., Vienna (1963).
- 7) These are obtained by considering Bell's analytic invariant positive amplitude, $F(s,t) = \text{Tr}(\not{k} + \not{k}')(m + \not{p})\mathcal{M}(m - \not{p})$, where $\langle p' \sigma', k' | T | p \sigma, k \rangle = \tilde{u}_{\sigma'}(\vec{p}') \mathcal{M} u_{\sigma}(\vec{p})$ is the πN scattering matrix. One then writes $F(s,t) = \sum (2\ell+1) \eta_{\ell} P_{\ell}(\cos\theta)$, where the η_{ℓ} are related to the partial waves with definite total angular momentum as in (11) and one can write $\eta_{\lambda}(s) = \int dz Q_{\lambda}(z) \rho(s,z)$ where the ρ has the best kinematical chances to be positive.
- 8) C. Goebel - Proceedings International Conference on Theoretical Aspects of High-Energy Phenomena, CERN 61-22 (1961), p. 353 ;
A.W. Martin - Rutgers University Preprint (1969).

- 9) In particular we would like to point out that the second of the arguments of A.W. Martin, Ref. 8), p. 6, is unclear unless further motivated ; in fact, the function $\rho(s, z) = \text{constant} \times \delta(s - z_0(s))$ provides a simple counterexample since it is positive, and yet the partial waves satisfy

$$\text{Im } \eta_l(s) / \text{Im } \eta_{l'}(s) \simeq 1 + \text{constant} / \log \frac{l+l'}{\sqrt{s}},$$

for $l, l' \ll \sqrt{s}$, which is A.W. Martin's argument.