TWISTING INVARIANT FACTORIZATION OF MULTIPARTICLE
DUAL AMPLITUDE

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A B S T R A C T

We exhibit a transformation of variables which allows the simultaneous factorization of a multiparticle Veneziano term and its twisted counterpart (obtained by reversing the order of the initial or final particles). The states which factorize the amplitude are linear combinations of those previously obtained which were not invariant under twisting. The behaviour of the couplings under twisting allows the definition of a signature factor. The invariance under double twist is automatically satisfied.

It is shown that the properties under the aforementioned transformation lead to relations between the couplings of the states to scalar particles (some of them are the ones called "Ward identities" in the literature).

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1. INTRODUCTION

The analysis of the intermediate unstable particles contained in the multiparticle Veneziano model ^{1),2)} has been performed by Fubini and Veneziano ³⁾ (hereafter referred to as FV) and by Bardakçi and Mandelstam ⁴⁾, from the study of the factorization properties of the multiparticle amplitude.

They were able to analyze one particular term B_{N+M} of the $N+M$ particle amplitude, corresponding to a well-defined ordering of the external particles (see Fig.1), by expanding it into an infinite sum of pole terms, each residue being a finite superposition of factorizable expressions. Every expression can be then interpreted as an intermediate state (actually reducible), and represented as in Fig. 2.

However, to the multi-Veneziano term of Fig. 1 must be added, among others, the one of Fig. 3 in which the order of the final particles, labelled by their momenta ($q_i \rightarrow \bar{q}_i = q_{M+1-i}$) has been reversed. Indeed, it is impossible to distinguish experimentally between the decay of a resonance into the states $(q_1, \dots, q_M)$ and $(q_M, \dots, q_1)$, since the pole structure of both configurations is exactly the same. This is of course related to the invariance of the $M+1$ point Veneziano function under an anticyclic permutation of the external particles ²⁾. The trouble is then that the set of states which factorizes the amplitude of Fig. 1 does not factorize that of Fig. 3. Each intermediate state in Fig. 3, which would be represented by Fig. 4, appears as a linear combination of the degenerate states (with the same mass) which appear in Fig. 2.

Because one goes from Fig. 2 to Fig. 4 by twisting the right-hand side vertex, the formal operator that realizes this linear combination has been called the twisting operator ⁵⁾. We shall find in this paper a basis of states that factorizes the amplitudes of Figs. 1 and 3 simultaneously.

The fact that the terms of Figs. 1 and 3 can be immediately summed in our factorization means that our states have a signature. Indeed, every state in our basis either gives the same contribution to the amplitudes of Figs. 1 and 3, or just the opposite. Therefore, all the states which are odd under twisting will cancel if we sum the two terms, and

this - as we shall show later - introduces factors of the form $1 \pm (-1)^J$ into the residues. In particular, the leading trajectory has positive signature, in agreement with previous results⁶⁾.

The possibility of finding different bases which factorize the amplitude of Fig. 1 comes from the existence of Ward-like identities first discovered by Fubini and Veneziano³⁾. These "Ward identities" were indeed derived by FV, by noticing that the amplitude of Fig. 1 could also be factorized in terms of states that would be represented by Fig. 5 because of the invariance of the $N+M$ point function under anticyclic permutations. In the Fubini-Veneziano basis each term of Fig. 5 is different from the corresponding one in Fig. 2, but the necessary equality of the sum over all degenerate states of both figures separately is generated by the "Ward identities". On the contrary, in our basis, the invariance under such a double twisting is automatic; we do not need the Ward identities at all.

Therefore these "Ward identities" appear merely as relations between the coupling of different states to the external particles. The multiplicity of states in our basis is the same as that of Refs. 3) and 4), except for a factor $\frac{1}{2}$ due to the signature.

The method which we use is based on a rather elementary change of variables in the multiparticle dual amplitude²⁾ B_{N+M} . In this change of variables, the subamplitudes B_{N+1} and B_{M+1} of Fig. 2 are reproduced times an extra factor. However, those factors in B_{N+M} which involve the variable of the intermediate line are transformed in such a way that the extra factors essentially cancel, while factorization in the new basis appears automatically. Another, much more laborious, way to get this factorization, or rather to check it in some simple cases, is to write the FV factorization and make the linear transformation to our basis. This leads to a complicated sum over non-factorized terms. But, if we make use of the "Ward identities", this sum reduces to the factorized expression we are looking for.

The relation between the two methods becomes clear once one realizes that our change of variables depends on one parameter, and that the corresponding transformations on the subamplitude B_{N+1} , for

instance, form a one-parameter continuous group. We shall show, in the next Section, that the properties of the infinitesimal generator of this group enable us to derive the whole set of Ward identities in closed form, while we defer to Section 3 the proof of the factorization in the new basis.

2. A CHANGE OF VARIABLES IN THE $N+1$ POINT FUNCTION AND ITS RELATIONSHIP TO THE GAUGE CONDITIONS

We start from the usual expression ²⁾ for the $N+1$ point function in the multiperipheral configuration of Fig. 6. The expression for the amplitude B_{N+1} is given by a $N-2$ -fold integral which we write symbolically

$$B_{N+1} = \int d\varphi_{N+1}(\beta_i, p_i)$$

The integrand in the usual Bardakçi-Ruegg form (we use essentially the FV notations) is written in terms of $N-2$ independent variables x_i ($i=2, \dots, N-1$) which are integrated from 0 to 1. But it is more convenient to use as independent variables products of x_i 's, which we call β_i ^{3),7)}

$$\beta_i = x_2 x_3 \dots x_i \quad i=2, \dots, N-1 \quad (2.1a)$$

and to define β_1 and β_N by :

$$\beta_1 = 1, \quad \beta_N = 0 \quad (2.1b)$$

Then

$$d\varphi_{N+1}(\beta_i, p_i) = \left(\prod_{i=2}^{N-1} d\beta_i \right) \prod_{\substack{i,j=1 \\ i < j}}^N (\beta_i - \beta_j)^{-p_i \cdot p_j} \cdot \prod_{i=1}^{N-1} \theta(\beta_i - \beta_{i+1}) (\beta_i - \beta_{i+1})^{a-1} \quad (2.2)$$

where $a = \alpha(0)$ is the intercept of the trajectory.

We now perform on the variables g_i the transformation :

$$g_i = \frac{g'_i (1+f)}{1 - (1-2g'_i)f} , \quad -1 < f < 1 \quad (2.3)$$

The transformation (2.3) is the most general projective transformation independent of the index (i) which leaves invariant the points 0 and 1. It is then also valid for $i=1$ and N : $g'_1 = 1$, $g'_N = 0$, and it leaves the integration region in the multi-Veneziano integral invariant.

The second point about (2.3) is that the result of two successive transformations with parameters f and g is a similar transformation with parameter h given by :

$$h = \frac{f+g}{1+fg} , \quad -1 < h < 1 \quad (2.4)$$

Therefore the transformations form a connected commutative group, and the inverse transformation has minus the parameter of the original. The relevance of the condition $|f| < 1$ is clear when one notices the resemblance between (2.3) and the relativistic law for the addition of velocities.

Let us now define the four-vectors $v^{(n)} \quad (3), (8)$:

$$V^{(n)} = \sum_{i=1}^N (1-2g_i)^n p_i \quad (2.5a)$$

$$V^{(0)} = \sum_{i=1}^N p_i \equiv \pi \quad (2.5b)$$

In Appendix A we show that the result of our change of variables is the identity ⁹⁾ :

$$\begin{aligned} \int d\varphi_{N+1}(g_i, p_i) F\left(\frac{g_i(1-f)}{1+(1-2g_i)f}\right) &= \\ &= \int d\varphi_{N+1}(g_i, p_i) (1-f^2)^{-\frac{\pi^2}{2}} F(p_i) \exp\left[-\sum_{n=1}^{\infty} \frac{f^n}{n} \frac{V^{(n)} \cdot \pi}{n}\right] \end{aligned} \quad (2.6)$$

where F is an arbitrary ⁹⁾ function of the internal variables $g_1, \dots, g_N$; all particles except particle 0 have to be taken on the mass shell.

The point about (2.6) is that it is not only the essential step on the way of proving the V-U factorization, as will be discussed in the next Section, but it is important in itself, being a very general statement of the gauge conditions.

Because the transformations (2.3) form a connected group, any of these transformations can be obtained by repeated application of the infinitesimal transformation, which therefore contains all the available information. We examine this by equating the coefficient of f in Eq. (2.6). Using Taylor's theorem, we get the general form of the gauge conditions :

$$\int d\varphi_{N+1}(g_i, p_i) \pi \cdot V^{(n)} F(g_i) = \int d\varphi_{N+1}(g_i, p_i) \tilde{F}(g_i) \quad (2.7a)$$

where

$$\tilde{F}(g_i) = \sum_{i=1}^N 2g_i(1-g_i) \frac{\partial F}{\partial g_i} \quad (2.7b)$$

One can immediately see from (2.7a) that

$$\tilde{V}^{(n)} = n(V^{(n+1)} - V^{(n-1)}) \quad (2.8)$$

Hence

$$\langle \pi \cdot V^{(n)} V^{(n)} \rangle = n(\langle V^{(n+1)} \rangle - \langle V^{(n-1)} \rangle) \quad (2.9)$$

where, as in FV, the definition

$$\langle g \rangle \equiv \int d\varphi_{N+1}(g_i, p_i) g(g_i, p_i) \quad (2.10)$$

is used.

Further, since the $\sim$ operation acts like a derivative :

$$\widetilde{F_1} F_2 = F_1 \widetilde{F_2} + \widetilde{F_1} F_2$$

we can also obtain the action of $\pi.V^{(1)}$ on any product of V's.

In Appendix B we show that the invariance under double twisting used by FV to obtain the gauge conditions can be exhibited by a change of variables of the kind just considered, and secondly that by a special choice of F in (2.6) we obtain the less general class of Ward identities of FV. Our conclusion is then that the possibility of the transformation (2.4) - (2.6) is equivalent to the gauge conditions.

3. FACTORIZATION OF THE AMPLITUDE IN TERMS OF V STATES

Let us consider the term of the multiparticle amplitude corresponding to Fig. 1. The representation which exhibits the multiperipheral structure of Fig. 7 can be written as ³⁾ :

$$B_{N+M} = \int d\varphi_{N+1}(\rho_i, p_i) \int d\varphi_{M+1}(\sigma_j, q_j) \int_0^1 dz z^{-d(\pi^2)-1} (1-z)^{a-1} \cdot \prod_{i=1}^N \prod_{j=1}^M (1 - \rho_i z \sigma_j)^{-p_i \cdot q_j} \quad (3.1)$$

where $d\varphi$ are given by (2.2), the variables σ_j being the analogues of the ρ_i 's for the subamplitude B_{M+1} , and z is the variable for the internal line which is the one upon which the factorization will be made.

Our result will be obtained by making the change of variables :

$$\rho_i = \frac{\rho_i' (1+f)}{1 - (1-2\rho_i')f}, \quad \sigma_j = \frac{\sigma_j' (1+f)}{1 - (1-2\sigma_j')f} \quad (3.2a)$$

and

$$z = \frac{4f}{(1+f)^2} \quad \text{or} \quad f = 4 \frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}} \quad (3.2b)$$

Notice that the transformations of $\int d\varphi_{N+1}$, $\int d\varphi_{M+1}$ are of the form already discussed, but have the same parameter f , which is itself related to z , the internal variable of the line to be factorized. However, the transformation $z \rightarrow f$ is not a projective one.

The only work to be done is the transformation of the z dependent factors. Inserting (3.2) we find :

$$1 - \rho_i z \sigma_j = \frac{(1-f)[1 - (1-2\rho'_i)f(1-2\sigma'_j)]}{[1 - (1-2\rho'_i)f][1 - (1-2\sigma'_j)f]} \quad (3.3)$$

The point about this last equation is that on the right-hand side the coefficients of f have the form $(1-2\rho'_i)$, namely the coefficient which comes into the definition of the V 's, whereas on the left-hand side the coefficients of z have the form ρ_i , which comes into the definition of the P 's ³⁾.

Notice that, as usual, it is understood that $\rho_1 = 1$, $\rho_N = 0$, and that since our Eq. (3.2a) is still formally valid for $i = 1, N$ (and $j = 1, M$), so is (3.3).

Defining the vector $V^{(n)}$ as in (2.4) and an analogous vector $U^{(n)}$ for particles with momenta q_i , i.e.,

$$U^{(n)} = \sum_{j=1}^M (1-2\sigma_j) q_j, \quad U^{(0)} = \sum_{j=1}^M q_j = -\pi$$

we then find that :

$$\begin{aligned}
 & \prod_{i=1}^N \prod_{j=1}^M (1 - \rho_i z \sigma_j)^{-\rho_i q_j} = \\
 & = (1-f)^{\pi^2} \prod_{i=1}^N [1 - (1-2\rho_i')f]^{-\rho_i \cdot \pi} \prod_{j=1}^M [1 - (1-2\sigma_j')f]^{q_j \cdot \pi} \\
 & \quad \cdot \prod_{i=1}^N \prod_{j=1}^M [1 - (1-2\rho_i')f (1-2\sigma_j')]^{-\rho_i \cdot q_j} = \\
 & = (1-f)^{\pi^2} \exp \sum_{n=1}^{\infty} \frac{f^n}{n} V^{(n)} \pi \cdot \exp \sum_{n=1}^{\infty} \frac{f^n}{n} U^{(n)} \pi \cdot \exp \sum_{n=1}^{\infty} \frac{f^n}{n} V^{(n)} U^{(n)}
 \end{aligned} \tag{3.4}$$

by exponentiating in the way of FV. Finally, since :

$$\begin{aligned}
 & \int_0^1 dz z^{-\alpha(\pi^2)-1} (1-z)^{\alpha-1} = \\
 & = \int_0^1 df 4^{-\alpha(\pi^2)} f^{-\alpha(\pi^2)-1} (1-f)^{2\alpha-1} (1+f)^{\pi^2+1}
 \end{aligned} \tag{3.5}$$

we find, inserting (2.5), (3.4) and (3.5) into (3.1), that the factors $\exp(\sum_n (f^n/n) V^{(n)} \pi)$ and $\exp(-\sum_n (f^n/n) U^{(n)} \pi)$ cancel, leaving us with

$$B_{N+M} = \int_0^1 df 4^{-\alpha(\pi^2)} f^{-\alpha(\pi^2)-1} (1-f)^{2\alpha-1} (1+f) \langle \exp \sum_{n=1}^{\infty} \frac{f^n}{n} V^{(n)} U^{(n)} \rangle \tag{3.6}$$

which shows the factorization in the new basis ¹⁰⁾.

The pole structure of B_{N+M} of (3.6) can be obtained by developing in powers of f the coefficient of $f^{-\alpha(\pi^2)-1}$ and performing the f integration. One obtains

$$B_{N+M} = \sum_{k=0}^{\infty} \frac{R_k}{k - \alpha(\pi^2)} \tag{3.7}$$

where

$$R_k = 4^{-\alpha(\pi^2)} \sum_{j=0}^k C_{k-j} S_j \tag{3.8}$$

$$C_i = (-1)^i \binom{2\alpha-1}{i} \cdot \left(1 - \frac{i}{2\alpha-i}\right) \quad (3.9)$$

$$S_0 = \langle 1 \rangle \langle 1 \rangle \quad (3.10)$$

$$S_j = \sum_{\{l_1, \dots, l_m\}} \frac{\langle V_{\mu_1}^{(1)} \dots V_{\mu_{l_1}}^{(1)} V_{\nu_1}^{(2)} \dots V_{\nu_{l_2}}^{(2)} \dots V_{\lambda_1}^{(m)} \dots V_{\lambda_{l_m}}^{(m)} \rangle \langle U_{\mu_1}^{(1)} \dots U_{\mu_{l_1}}^{(1)} U_{\nu_1}^{(2)} \dots U_{\nu_{l_2}}^{(2)} \dots U_{\lambda_1}^{(m)} \dots U_{\lambda_{l_m}}^{(m)} \rangle}{\prod_{i=1}^m i^{l_i} l_i!}$$

where the sum in (3.10) is performed over all non-negative integers l_i that satisfy

$$\sum_{i=1}^{\infty} i l_i = k \quad (3.11)$$

The S_j of (3.10) have the same structure as those of FV, the difference being that here the tensors are constructed with the vectors $V^{(n)}$ and $U^{(n)}$ while in FV they were built with $P^{(n)}$ and $Q^{(n)}$ (cf., Appendix B for their definition). The discussion of the number of levels here and in FV is therefore the same. Let us remark that if we consider R_k of (3.8) as a function of the dependent $\frac{1}{2}(M+N)(M+N-3)$ trajectories which appear in the $M \rightarrow N$ amplitude we are discussing (i.e., that of Fig.1), then R_k contains π^2 or $\alpha(\pi^2)$ explicitly, besides the $4^{-\alpha(\pi^2)}$ of (3.8), while the corresponding quantity in the $P^{(n)}, Q^{(n)}$ factorization does not. This is due to the fact that $V^{(n)}$ contains all momenta p_i , while $P^{(n)}$ does not involve p_N in its definition. Then $V^{(n)} U^{(n)}$ contains scalar products as $p_N \cdot q_1$ and $p_1 \cdot q_M$ which, when developed in the dependent variables $(p_1 + p_{i-1} + \dots + p_j)^2, (q_l + q_{l-1} + \dots + q_m)^2, (p_r + p_{r-1} + \dots + p_1 + q_1 + \dots + q_s)^2$ give rise to a π^2 term.

4. DISCUSSIONS

As stated in the Introduction, our main aim is to find a basis that factorizes simultaneously a term (as that of Fig.1 which we called B_{N+M}) of the $N \rightarrow M$ multiparticle dual amplitude and that obtained from it (as that of Fig.2 we will denote by $\bar{B}_{N+M}$) by a twist of the N initial or M final particles. Let us remember that these two terms, even if they involve the initial (final) particles in the same relative order, are obtained from each other by a non-cyclic (nor anticyclic) permutation and must be, therefore, added together in the $N \rightarrow M$ scattering amplitude.

The expression for $\bar{B}_{N+M}$ is the same as that in (3.6) for B_{N+M} in which q_i are replaced by $\bar{q}_i = q_{M+1-i}$. As noted by FV, $d\varphi_M(q_i, \sigma_i)$ is unchanged if besides replacing $q_i \rightarrow q_{M+1-i}$, the replacement $\varphi_i \rightarrow \varphi_{M+1-i}$ is also performed. They also noted that under this combined operation

$$U^{(n)} \rightarrow \bar{U}^{(n)} = (-1)^n U^{(n)} \quad (4.1)$$

so, finally

$$\begin{aligned} \bar{B}_{N+M} &= \int_0^1 df \, 4^{-d(n^2)} f^{-d(n^2)-1} (1-f)^{2\alpha-1} (1+f) \langle \exp \sum_{n=1}^{\infty} \frac{f^n}{n} V^{(n)} \bar{U}^{(n)} \rangle \\ &= \int_0^1 df \, 4^{-d(n^2)} f^{-d(n^2)-1} (1-f)^{2\alpha-1} (1+f) \langle \exp \sum_{n=1}^{\infty} \frac{(-f)^n}{n} V^{(n)} U^{(n)} \rangle \end{aligned} \quad (4.2)$$

It is clear that B_{N+M} satisfies an expression like (3.7)-(3.10) in which $U^{(n)}$ are replaced by $\bar{U}^{(n)}$ as given by (4.1)

Let us define :

$$P = \sum_{i \text{ odd}} \ell_i \quad (4.3)$$

where ℓ_i are the integers appearing in (3.10). Then every factorized expression of $\bar{B}_{N+M}$ is equal to the corresponding one of B_{N+M} times $(-1)^P$. Therefore $B_{N+M} + \bar{B}_{N+M}$ will be given by an expression as that of (3.7)-(3.10) in which every tensor product in the right-hand side of (3.10) is multiplied by $1 + (-1)^P$. This implies the cancellation of the contribution with odd P .

For states with spin J lying on the leading trajectory, we have

$$l_i = 0 \text{ for } i \neq 1, \quad l_1 = J \quad (4.4)$$

and therefore, from (4.3), $P = J$. Then

$$1 + (-1)^P = 1 + (-1)^J$$

so that the leading trajectory has positive signature as is well known ⁶⁾.

Every term in (3.10) [which appears multiplied by $1 + (-1)^P$ for $B_{N+M} + \bar{B}_{N+M}$] gives rise, when reduced to O_3 , to particles with different spin j_λ . The corresponding trajectory will have positive or negative signature depending on whether j_λ and the P for the term in question have the same or opposite even-odd character.

Let us go back to the twisting operator. Two successive twists give the unit operation $(\bar{U}^{(n)} = U^{(n)})$. As said in the Introduction, if we twist both sides of the $N \rightarrow M$ amplitude of Fig. 2 we obtain that of Fig. 5, which of course must be identical to it. This is indeed the case if we notice that $V^{(n)} U^{(n)} = \bar{V}^{(n)} \bar{U}^{(n)}$ as is clear from (4.3).

The term of Fig. 1 and its twisted counterpart (Fig. 3), which we have independently factorized, are not the only ones which have poles in the direct channel $N \rightarrow M$ particle scattering. What remains to be considered are non-cyclic (nor anticyclic) permutations among the N initial or the M final particles. These permutations lead of course to different dynamical configurations between labelled particles (i.e., different particles resonate among themselves) while the two amplitudes of Figs. 1 and 3 have exactly the same initial and final configurations.

The treatment of the other configurations would follow the one traced here. It appears, however, that the states - defined through vectors $V^{(n)}$ and $U^{(n)}$ - which factorize the amplitudes of Figs. 1 and 3, would not factorize those non-cyclic permutations we are discussing.

Of course, the whole S matrix for $N \rightarrow M$ particle scattering - which contains the sum over all non-cyclic permutations - will be factorized by states defined analogously to those discussed before but constructed with tensors given by

$$\langle V^{(2)} \dots V^{(m)} \rangle = \sum_i \langle V_i^{(2)} \dots V_i^{(m)} \rangle_i$$

where i indicates the i th permutation. The sum runs only over half of the permutations, the other half being obtained from the first by a twist.

Summarizing, our basis $V^{(n)}$ factorizes simultaneously an amplitude and its twisted counterpart (double twist being identical with the original amplitude). The behaviour under twist is simple, and represented by (4.1).

In an operator formalism $^{11), 5)}$, the twist operator Ω is diagonalized in our basis together with the propagator $\mathcal{P}$, and is simply given by

$$\Omega = (-1)^H \quad (4.5)$$

The notations being analogous to those of Refs. $^{5), 11)}$.

The double twist condition

$$\Omega^+ \mathcal{P} \Omega = \mathcal{P} \quad (4.6)$$

is now satisfied as an operator.

The vertex expression in the operator formalism in the $V^{(n)}$ basis appears to have a more complicated expression than in the $P^{(n)}$ representation; this is to be devoted to further investigation. After this paper was completed, we received some preprints (F. Gliozzi, University of Torino; C.B. Chiu, S. Matsuda and C. Rebbi, Calt.68-224; I.T. Drummond, DAMTP 69/36) in which some of the points of our paper are dealt with.

A P P E N D I X A

In this Appendix we give some details on the change of variables which leads to our main identity (2.6). We have to study the following integral :

$$\langle F \rangle = \int d\varphi_{N+1}(s_i, p_i) F(s_1, \dots, s_{N-1}) \quad (\text{A.1})$$

where $F(p_i)$ is an (almost) arbitrary ⁹⁾ function and $d\varphi_{N+1}$ is given by (2.2).

We now make the change of variables already given in Section 2 :

$$s_i = \frac{(1+f)s'_i}{1 - (1-2s'_i)f} \equiv \frac{(1+f)s'_i}{D_i} ; \quad -1 < f < 1 \quad (\text{A.2})$$

and we immediately obtain :

$$s_i - s_j = (1-f^2)(D_i D_j)^{-1}(s'_i - s'_j) \quad (\text{A.3})$$

It is important to notice that the Θ functions in (2.2) are preserved, since $s_i > s_j$ implies $s'_i > s'_j$. Moreover, the transformation (A.2) also holds in the case $i=1$ and $i=N$ (i.e., $s'_1=1$, $s'_N=0$), because the points 0 and 1 are invariant. Finally, in order to deal with the indices $i=1$ and $i=N$, it is convenient to remark that $D_0 = 1+f$ and $D_N = 1-f$.

The Jacobian J turns out to be

$$J = (1-f^2)^N \prod_{i=1}^N D_i^{-2} \quad (\text{A.4})$$

while straightforward algebra, making use of the mass shell condition $a + \frac{1}{2}\mu^2 = 0$ for particles $1, \dots, N$, shows that the two terms in $d\varphi_{N+1}$ are transformed as follows :

$$\prod_{\substack{i,j=1 \\ i < j}}^N (g_i - g_j)^{-p_i \cdot p_j} = (1-f^2)^{-\frac{\pi^2}{2} - Na} \prod_{i=1}^N D_i^{\pi \cdot p_i + 2a} \prod_{\substack{i,j=1 \\ i < j}}^N (g'_i - g'_j)^{-p_i \cdot p_j} \quad (\text{A.5})$$

$$\prod_{i=1}^{N-1} \Theta(g_i - g_{i+1}) (g_i - g_{i+1})^{a-1} = (1-f^2)^{N(a-1)} \prod_{i=1}^N D_i^{2-2a} \prod_{i=1}^{N-1} \Theta(g'_i - g'_{i+1}) (g'_i - g'_{i+1})^{a-1} \quad (\text{A.6})$$

Collecting together the three terms (A.4), (A.5) and (A.6), we obtain the transformation law for the integrand of the $N+1$ point function :

$$d\varphi_{N+1}(g_i, p_i) = d\varphi_{N+1}(g'_i, p_i) (1-f^2)^{-\frac{\pi^2}{2}} \prod_{i=1}^N D_i^{\pi \cdot p_i} \quad (\text{A.7})$$

or

$$d\varphi_{N+1}(g_i, p_i) = d\varphi_{N+1}(g'_i, p_i) (1-f^2)^{-\frac{\pi^2}{2}} \exp - \sum_{n=1}^{\infty} \frac{f^n}{n} \pi \cdot V^{(n)} \quad (\text{A.8})$$

It may be worth noticing that

$$(1-f^2)^{-\frac{\pi^2}{2}} \exp - \sum_{n=1}^{\infty} \frac{f^n}{n} \pi \cdot V^{(n)} = \exp - \sum_{n=1}^{\infty} \frac{f^n}{n} \pi \cdot v^{(n)} \quad (\text{A.9})$$

with $U^{(n)}$ defined by 3) :

$$v^{(n)} = V^{(n)} \text{ for } n \text{ odd} ; \quad v^{(n)} = V^{(n)} - \pi \text{ for } n \text{ even}$$

Inserting (A.8) into (A.1) and using the fact that the parameter of the inverse transformation of (A.2) is $-f$, we finally obtain Eq. (2.6).

A P P E N D I X B

First we shall show that a modified form of the change of variables (2.3) transforms the $\overline{P}Q$ factorization of FV into the $P\overline{Q}$ one. Therefore all the gauge conditions following from the equivalence of the two factorizations must follow more directly from the change of variables in the way that we have already investigated in Section 2.

Secondly, we shall derive the special class of gauge conditions given by equation (4.7) of FV ³⁾, and see that it corresponds to a special choice of F in Eq. (2.6).

We define

$$\begin{aligned} P^{(n)} &= \sum_{i=1}^N g_i^n p_i & \overline{P}^{(n)} &= \sum_{i=1}^N (1-g_i)^n p_i \\ \overline{Q}^{(n)} &= \sum_{i=1}^M \sigma_i^n q_i & Q^{(n)} &= \sum_{i=1}^M (1-\sigma_i)^n q_i \end{aligned}$$

The apparent difference in our definition of the Q 's compared with that of FV is due to the fact that we found it convenient to label the lines in a different way.

Let us write the change of variables in the form :

$$g_i = \frac{r_i}{1-z(1-r_i)} \equiv \frac{r_i}{E_i} \quad -1 < z < 1 \quad (B.1)$$

To show that (B.1) is indeed of the class (2.3), one has simply to set

$$f = \frac{z}{2} \left(1 - \frac{z}{2}\right)^{-1} \quad (B.2)$$

We could then use (B.2), (A.8) and equations (4.8) and (4.10) of FV to obtain the transformation law of the integrand, but it is as simple to work out again the change of variables. The Jacobian is given by :

$$J = (1-z)^N \prod_{i=1}^N E_i^{-2} \quad (B.3)$$

while

$$g_i - g_j = (1-z)(E_i E_j)^{-1} (r_i - r_j) \quad (\text{B.4})$$

where we have used $E_1 = 1$, $E_N = 1-z$. Comparing with (A.3) and (A.4) we see that we need not perform any new calculation, since we have simply to make in (A.8) the substitutions :

$$g'_i \rightarrow r_i, \quad D_i \rightarrow E_i, \quad 1+f \rightarrow 1, \quad 1-f \rightarrow 1-z$$

so that we get :

$$d\varphi_{N+1}(g_i, p_i) = d\varphi_{N+1}(r_i, p_i) (1-z)^{-\frac{\pi^2}{2}} \prod_{i=1}^N E_i^{\pi \cdot p_i} \quad (\text{B.5})$$

or

$$d\varphi_{N+1}(g_i, p_i) = d\varphi_{N+1}(r_i, p_i) (1-z)^{-\frac{\pi^2}{2}} \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \pi \cdot \bar{P}^{(n)} \quad (\text{B.6})$$

By repeating the calculations of Section 3, but without changing the variable z , we display the equivalence of the $P\bar{Q}$ and $\bar{P}Q$ factorizations since :

$$\prod_{i=1}^N \prod_{j=1}^M (1 - g_i z \sigma_j)^{-p_i \cdot q_j} = \prod_{i=1}^N \prod_{j=1}^M \left[\frac{(1-z)(1 - (1-r_i)z(1-s_j))}{[1-z(1-r_i)][1-z(1-s_j)]} \right]^{-p_i \cdot q_j}$$

and

$$\prod_{i=1}^N \prod_{j=1}^M [1 - (1-r_i)z(1-s_j)]^{-p_i \cdot q_j} = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \bar{P}^{(n)} \varphi^{(n)}$$

In these equations s_j is obtained from σ_j by the transformation (B.1).

For the second point, let us choose as function $F(g_i)$:

$$F(\beta_i) = (1-z)^{-\pi \cdot q_0/2} \prod_{i=1}^N (1-\beta_i z)^{\beta_i \cdot q_0} = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathcal{P}^{(n)} \cdot q_0 \quad (\text{B.7})$$

where q_0 is an arbitrary four-vector and $\mathcal{P}^{(n)}$ is defined by ³⁾ :

$$\mathcal{P}^{(n)} = P^{(n)} - \frac{\pi}{2}$$

From (B.5) we have

$$\langle F \rangle = \int d\varphi_{N+1}(\tau_i, p_i) (1-z)^{\frac{1}{2} \pi \cdot (\pi - q_0)} \prod_{i=1}^N [1-z(1-\tau_i)]^{(\pi - q_0) \cdot p_i}$$

or

$$\langle \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathcal{P}^{(n)} \cdot q_0 \rangle = \langle \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \overline{\mathcal{P}}^{(n)} \cdot (\pi - q_0) \rangle \quad (\text{B.8})$$

We recover equation (4.7) of FV if we set :

$$q'_0 = \frac{1}{2}(\pi - \Delta)$$

We see that we need not make any "off-mass shell extrapolation" for the particles q_0 and $q_1 = \pi - q_0$.

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- 8) Notice that our $V^{(n)}$'s differ from those of Fubini and Veneziano by a normalization factor 2^n . Furthermore, we use the metric $g_{00} = -g_{ii} = +1$, and take all momenta ingoing.
- 9) There may be some regularity conditions on F at the points $\varphi_i = 0$ and 1; (2.5) is certainly true if F is regular at these points, and this is all what we need in what follows. It is easily seen that we can also allow poles of finite order. Furthermore, we can relax the bootstrap condition ³⁾ $\alpha(\mu^2) = 0$ and give arbitrary masses to the $N+1$ external particles. Equation (2.5) is then modified, but only the masses μ_1 and μ_N of the particles (1) and (N) and the variables φ_2, φ_{N-1} appear

in the modified formula which differs from (2.5) only in the extra factor contained in the right-hand side :

$$\left[\frac{1 - (1 - 2\rho_2)f}{1 + f} \right]^{\alpha(\mu_2^2)} \left[\frac{1 - (1 - 2\rho_{N-1})f}{1 - f} \right]^{\alpha(\mu_N^2)}$$

- 10) It appears that $V-\bar{U}$ factorization can also be obtained by a direct manipulation of the Koba-Nielsen formula for B_{N+M} ; see H. Stapp, Trieste preprint (1969), Eq. (24).
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FIG.1

FIG.2

FIG.