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AMPLITUDES FOR THREE-PARTICLE PRODUCTION AT HIGH ENERGIES AND
LARGE MOMENTUM TRANSFERS

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A B S T R A C T

A study is made of the asymptotic nature of the amplitude for three-particle production at high energies and large momentum transfers.

Simple assumptions are made about the nature of the discontinuities across the cuts in the complex angular momentum plane, similar to those which, in the case of elastic scattering, resulted in qualitative agreement with experiment. By means of a summation of the contribution of branch points, explicit expressions are obtained for the production amplitudes [formula (36)], similar to the expression for an elastic amplitude (an exponential drop in the amplitude with momentum transfer with superimposed oscillations).

In the last Section a potential model is developed similar to the eikonal approximation for the case of elastic scattering. With certain limitations there occurs a certain generalisation extended to the three-particle case of the usual eikonal formula [formula (54)]. A relationship is established between the potential model developed and summation of the contribution of branch points in complex angular momentum.

1. INTRODUCTION

In this article, we shall consider the behaviour of the amplitudes for three-particle production at high energies and large momentum transfers. The latter are, however, considered to be considerably smaller than the sub-energies, which enables the theory of complex angular momentum to be used. A similar problem for the case of the amplitudes of two-particle processes was examined in Refs. 1)-3). For comparison with subsequent work, it is essential to recall the situation occurring in this instance.

It is a well-known fact that at large energies the contribution to the elastic amplitude is determined by both the poles and branch points in the plane of the angular momentum. If we confine ourselves solely to the Pomeranchuk pole and the branch points linked with it, we can write

$$A(s, t) = i \sum_{n=1}^{\infty} C_n(\xi, t) s^{j_n(t)} = i \sum_{n=1}^{\infty} C_n(\xi, t) e^{j_n(t)\xi} \quad (1)$$

where

$$j_n(t) = n\alpha\left(\frac{t}{n^2}\right) - n + 1 \quad (2)$$

and

$$\xi = \ln s \quad (3)$$

s and $(-t)$ are respectively the squares of the energy and momentum transfers, $\alpha(t)$ is the trajectory of the Pomeranchuk pole.

The explicit dependence of C_n on ξ has been written, in order to stress that the coefficients C_n vary at large values of s only as functions of $\ln s$. The values of C_n depend on the discontinuities of partial wave amplitudes across the cuts in the j plane.

As follows from perturbation theory ⁴⁾, the C_n contain as multipliers the factors $(-1)^n$. The alternating nature of C_n may also be established by the continuation of multi-particle conditions of unitarity to complex values of angular momentum ⁵⁾. The physical cause of the alternating sign is that the contribution of the n^{th} branch point corresponds to an n^{th} order multiple rescattering, each of which has a diffraction character which corresponds to a purely imaginary amplitude. This question is discussed in greater detail in, for example, Ref. 3).

We may thus set

$$C_n(\xi, t) = (-1)^n B_n(\xi, t) \quad (4)$$

In a linear approximation for $\alpha(t)$

$$j_n(t) = 1 - \alpha' |t|/n \quad (5)$$

and we have

$$A(s, t) = i s \sum_{n=1}^{\infty} (-1)^n B_n(\xi, t) e^{-\alpha' \xi |t|/n} \quad (6)$$

For small values of $\alpha' \xi |t|$ (the diffraction region) it is necessary, when calculating the amplitude $A(s, t)$ to have an accurate knowledge of the coefficients $B_n(\xi, t)$. An attempt to determine B_n for small t was made in Ref. 6). For large values of $\alpha' \xi |t|$ we require, fortunately, only fairly general information about the coefficients B_n . This is because the basic contribution to the amplitude is given by the cuts with large values of n . As shown in ¹⁾, for a range of different and rather natural assumptions concerning the form of B_n the answer is not greatly different from that occurring when $B_n = \text{const}$. For a comparison with the case examined later on concerning the amplitude for three-particle production, we write only the simplest form of answer corresponding to

this assumption. For the differential scattering cross-section we have

$$\frac{d\sigma}{dt} = F(\xi, t) \exp[-2\sqrt{2\pi\alpha'\xi}|t|] \times \\ \times [1 + \lambda(\xi, t) \cos(2\sqrt{2\pi\alpha'\xi}|t| + \chi_0)] \quad (7)$$

The functions F and λ contain only a power dependence on t and ξ , χ_0 depends only on $\ln \xi$ and $\ln t$. None of these values may, at the present time, be strictly determined in the framework of the theory of complex angular momentum.

We also note that the form of the answer changes only insignificantly if we do not use a linear expression for the trajectory [see ¹⁾].

The exponential fall in cross-section, together with the oscillations predicted by formula (7), are in reasonably good qualitative agreement with the existing experimental data for pp scattering right up to 19.2 GeV/c. It is obvious that a clear confirmation of the existence of the oscillations would be of great importance to verify the theory. Although it is possible that the oscillations may be masked by spin effects and others, present experimental data justify the belief that at 30 GeV/c at least two full oscillations could be observed ³⁾. At 19.2 GeV/c only one full oscillation is visible.

The physical significance of the picture that has been described may be understood in the framework of a model of scattering on a purely imaginary potential ³⁾. We may take, as such a potential, the Fourier component of an amplitude corresponding to the exchange of one Pomeron pole ⁷⁾,

$$V(r) = \frac{-i\sqrt{5}}{32\pi^3 m} \left(\frac{\pi}{\alpha' \xi} \right)^{3/2} \sigma_{T0T} e^{-r^2/4\alpha' \xi} \quad (8)$$

(m is the nucleon mass). In order to obtain the full amplitude the potential (8) must be integrated by using the eikonal approximation (9). The significance of this integration is that, in the case of an interaction which is fairly smooth at short distances, it is impossible to transfer a large momentum with a single exchange (the amplitude is exponentially small). We have

$$A(s, t) = -2is \int e^{-i\vec{q} \cdot \vec{p}} (e^{2i\delta(\rho)} - 1) d^2\rho \quad (9)$$

where

$$\delta(\rho) = -\frac{m}{\sqrt{5}} \int_{-\infty}^{\infty} dz V(\rho^2 + z^2) \quad (10)$$

Here $\vec{\rho}$ is the impact parameter, δ is the scattering phase in the eikonal approximation, $\vec{q}$ is the two-dimensional momentum transfer ($\vec{q}^2 = -t$), (transverse to the initial direction). By expanding in powers of $\delta(\rho)$ we obtain from (8), (9) and (10)

$$A(s, t) = -is \sigma_{T0T} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! n} \left(\frac{\sigma_{T0T}}{8\pi\alpha' \xi} \right)^{n-1} e^{-\alpha' \xi q^2/n} \quad (11)$$

If we compare (11) and (6) we see that we have an amplitude similar to the Regge amplitude. The n^{th} member of the expansion corresponds to the n reggeon branch point. The factor $(-1)^n$ corresponds to the "eclipsing" effect in the Glauber picture of rescattering⁸⁾. It appears on account of the purely imaginary nature of the potential. A similar study was made in Ref. 9).

It should be stressed that calculations which are based on the explicit form of the coefficients in (11)⁹⁾ are only of a purely model nature. The real coefficients B_n which appear in the relativistic problem, may depend on q^2 and contain a more complicated dependence on ξ and, in particular, on n .

2. PRODUCTION AMPLITUDES

When examining the asymptotic behaviour of Feynman graphs of the type shown in Fig. 1, we may develop the corresponding Regge graph technique¹⁰⁾, in exactly the same way as done in Ref. 4) for the case of the elastic amplitude. Figure 2 shows a Regge graph in which the wavy lines denote the exchange simultaneously of several reggeons (n , n_1 and n_2 , as shown in the diagram). The amplitude depends on two momentum transfers $\vec{q}_1$ and $\vec{q}_2$ (the two-dimensional vectors $\vec{q}_1$ and $\vec{q}_2$ are the components of the momenta of the particles being formed, transverse to the initial direction) and on the two pair energies $\sqrt{s_{12}}$ and $\sqrt{s_{23}}$. The square of the total energy s is expressed, as we know, asymptotically, by s_{12} and s_{23} :

$$s_{12} s_{23} = s(m_2^2 + q^2) \quad (12)$$

Here m_2 is the mass of the middle particle in the diagram, and $\vec{q} = \vec{q}_1 - \vec{q}_2$ is its transverse momentum.

With an accuracy of up to constant terms, it follows from (12) that:

$$\xi = \xi_{12} + \xi_{23}, \quad \xi = \ln s, \quad \xi_{12} = \ln s_{12}, \quad \xi_{23} = \ln s_{23} \quad (13)$$

The asymptotic behaviour of the amplitude illustrated in Fig. 2 is [see Ref. 10)*]:

*) In formula (14) we have corrected the error in the pre-exponential factor committed in Ref. 10), but this is unimportant for what follows. In comparison with the notation in (10) the sign of the momentum q_2 has also been changed.

$$A_{nn_1n_2} = \frac{S}{\xi^{n-1} \xi_{12}^{n_1-1} \xi_{23}^{n_2-1}} \left(\frac{\xi}{n} + \frac{\xi_{12}}{n_1} + \frac{\xi_{23}}{n_2} \right)^{-1} \times$$

$$\times \exp \left[-\alpha' \frac{n_2 \vec{q}_1^2 \xi_{12} \xi + n_1 \vec{q}_2^2 \xi_{23} \xi + n (\vec{q}_1 - \vec{q}_2)^2 \xi_{12} \xi_{23}}{n_1 n_2 \xi + n n_2 \xi_{12} + n n_1 \xi_{23}} \right] \quad (14)$$

For simplicity, we have omitted the factors corresponding to the vertices of the diagram in Fig. 2. The total amplitude $A(s_{12}, s_{23}, \vec{q}_1^2, \vec{q}_2^2, \vec{q}_1 \cdot \vec{q}_2)$ is determined by the sum for n, n_1, n_2 members of the form of (14) taking into account these factors. As was elucidated above, the exact form of similar coefficients C_n in formula (1) is not very important in the range $\alpha' q^2 \xi \gg 1$. Of greatest importance was simply the alternating nature of the coefficients.

We may consider that if ^{*)}

$$\alpha' \xi q^2 \gg 1 \quad (15)$$

when $n \gg 1$, $n_1 \gg 1$ and $n_2 \gg 1$ are large, the explicit form of the factors under discussion in the case of the production amplitude is also not very important. The analog of $(-1)^n$ in the coefficients C_n is now the presence, in the diagram of Fig. 2, of the factor $(-1)^{n+n_1+n_2}$, the existence of which is easily established by perturbation theory.

In view of the above remarks, we shall seek the amplitude for three-particle production in the following form (with an accuracy of up to the pre-exponential factor)

^{*)} q^2 and ξ designate any of the momentum transfers and the logarithm of any of the energies of the particles formed.

$$A(s_{12}, s_{23}, \vec{q}_1^2, \vec{q}_2^2, \vec{q}_1 \cdot \vec{q}_2) = i s \sum_{\substack{n=0 \\ n_1=1, n_2=1}}^{\infty} (-1)^{n+n_1+n_2} \times \\ \times \exp \left[-\alpha' \frac{n_2 \vec{q}_1^2 \vec{\xi}_{12} \xi + n_1 \vec{q}_2^2 \vec{\xi}_{23} \xi + n (\vec{q}_1 - \vec{q}_2)^2 \vec{\xi}_{12} \vec{\xi}_{23}}{n, n_2 \xi + n n_2 \xi_{12} + n n_1 \xi_{23}} \right] \quad (16)$$

We make a few more remarks about the model under consideration. Calculation of the sum (16) may, for example, be carried out with the aid of the steepest descent in the planes of the three complex variables n , n_1 and n_2 . The factor $(-1)^{n+n_1+n_2}$ leads to an exponential drop in the direction of the imaginary axes $\sim \exp[i\pi(n+n_1+n_2)]$ of the coefficients of the exponential in (16). This drop is fairly sharp: when $\Delta(\text{in})=1$ the coefficients differ by $e^\pi \simeq 25$ times. In the meantime, for example, the factorial dependence on n , which appears in (16) for an eikonal approximation (see below) leads to the fact that the neighbouring coefficients differ by n times. It is easy to appreciate that at present-day energies, even in the case of elastic scattering, the relevant n are comparatively small $n \sim 3-4 (|n| \sim \sqrt{q^2 \xi / \pi})$. It is obvious that it is more important to maintain in (16) the factor $(-1)^{n+n_1+n_2}$ than to take into account the factorial dependence. For this very reason the pre-exponential factors of the type $1/\xi^n$ in (14) are not very important. It is possible that all of this explains why at present energies formula (7) for an elastic cross-section gives, not only a qualitative agreement with experiment, but a more or less correct indication of the exponential drop [see Refs. 1)-3)].

The equation (16) may be written in the form

$$A = \frac{is}{(2i)^3} \iiint \frac{dn}{\sin \pi n} \frac{dn_1}{\sin \pi n_1} \frac{dn_2}{\sin \pi n_2} \exp[-\alpha' \dots] \quad (17)$$

where the contours of integration enclose the points $n=0, 1, 2$, $n_1, n_2=1, 2, 3, \dots$

As in the case of an elastic amplitude ¹⁾, the asymptotic form of the function A , when the condition (15) is fulfilled, is determined by the position of the saddle points in the planes of the variables n , n_1 and n_2 , whilst the pre-exponential factor depends on the concrete form of the factors discussed above, which were omitted in (16). The position of the saddle points for the integral (17) may be found in an explicit form and the answer for the amplitude A coincides with what will be written below. This method of determining the asymptotic form of A exactly corresponds to the method used in Ref. 1), but here it leads to very unwieldy calculations. We shall therefore use a different method which has a much clearer physical meaning ^{*)}.

*) Let us quote only the final expressions for the saddle point values of n , n_1 and n_2 . The greatest contribution is from the saddle point at which $\text{Im } n > 0$, $\text{Im } n_1 > 0$, $\text{Im } n_2 > 0$, and of course, the symmetrical complex conjugate point. Let ϕ_1 (ϕ_2) be the angle between $\vec{q}_1$ ($\vec{q}_2$) and $\vec{q} = \vec{q}_1 - \vec{q}_2$ and $\alpha = \arccos \sqrt{\xi_{12}/\xi}$. The steepest descent under examination exists only in the range

$$\phi_1 + \phi_2 > \frac{\pi}{2}, \quad \phi_1 < \frac{\pi}{2} + \alpha, \quad \phi_2 < \pi - \alpha$$

Here

$$\begin{aligned} n &= (1+i)(-1) \sqrt{\frac{\alpha \xi}{2\pi}} \frac{\sin \phi_1 \sin \phi_2 \cos(\phi_1 + \phi_2)}{\sin(\phi_1 + \phi_2) \sqrt{\psi}} \\ n_1 &= (1+i) \sqrt{\frac{\alpha \xi}{2\pi}} \frac{\sin \phi_2 \cos \alpha \cos(\phi_1 - \alpha)}{\sqrt{\psi}} \\ n_2 &= (1+i) \sqrt{\frac{\alpha \xi}{2\pi}} \frac{\sin \phi_1 \sin \alpha \sin(\phi_2 + \alpha)}{\sqrt{\psi}} \end{aligned} \quad (18)$$

$$\psi = \sin^2 \alpha \sin^2 \phi_1 + \cos^2 \alpha \sin^2 \phi_2 + 2 \sin \alpha \cos \alpha \sin \phi_1 \sin \phi_2 \sin(\phi_1 + \phi_2)$$

when $\phi_1 + \phi_2 \rightarrow \frac{\pi}{2}$, $n \rightarrow 0$, $\phi_1 \rightarrow \frac{\pi}{2} + \alpha$, $n_1 \rightarrow 0$ and $\phi_2 \rightarrow \pi - \alpha$, $n_2 \rightarrow 0$. Correspondingly, when $\phi_1 + \phi_2 < \frac{\pi}{2}$ (or $\phi_1 > \frac{\pi}{2} + \alpha$, or $\phi_2 > \pi - \alpha$) $n \sim 1$ (or $n_1 \sim 1$ or $n_2 \sim 1$) are already important. For a detailed discussion of this question, see below. The contribution of the saddle points to the amplitude is $\exp(2\pi i \sum n)$, which leads to an answer that coincides with (36).

Expression (16) for the amplitude A may be written with accuracy up to unimportant factors in the following form:

$$A = i s \left[G(\vec{q}_1, \xi_{12}) G(\vec{q}_2, \xi_{23}) + \int \frac{d^2 k}{(2\pi)^2} G(|\vec{k}|, \xi) \times \right. \\ \left. \times G(|\vec{k} - \vec{q}_1|, \xi_{12}) G(|\vec{k} - \vec{q}_2|, \xi_{23}) \right] \quad (19)$$

where $\vec{k}$, $\vec{k} - \vec{q}_1$ and $\vec{k} - \vec{q}_2$ are the net two-dimensional momenta of the respective beams from the n , n_1 and n_2 reggeons, whilst the G functions are two-particle amplitudes.

$$G(|\vec{k}|, \xi) = \sum_{n=1}^{\infty} (-1)^n e^{-\alpha' \xi k^2 / n} \\ G(|\vec{k} - \vec{q}_1|, \xi_{12}) = \sum_{n_1=1}^{\infty} (-1)^{n_1} e^{-\alpha' (\vec{k} - \vec{q}_1)^2 \xi_{12} / n_1} \\ G(|\vec{k} - \vec{q}_2|, \xi_{23}) = \sum_{n_2=1}^{\infty} (-1)^{n_2} e^{-\alpha' (\vec{k} - \vec{q}_2)^2 \xi_{23} / n_2} \quad (20)$$

Formula (19) follows directly from an examination of the Regge graph in Fig. 2. Substitution of (20) in (19) and integration over $d^2 k$ when n , n_1 and n_2 are fixed, does in fact lead, as shown in Ref. 10), to expression (14) [with the now added factor $(-1)^{n+n_1+n_2}$].

The two-particle amplitudes (20) were calculated in Ref. 1). With an accuracy up to a slowly varying pre-exponential factor they are equal to:

$$G(|\vec{k}|, \xi) = \text{Re} \left[e^{-\sqrt{2\pi\alpha'\xi} |\vec{k}| (1-i) - i\pi/8} \right] \quad (21)$$

and similarly for $G(|\vec{k} - \vec{q}_1|, \xi_{12})$ and $G(|\vec{k} - \vec{q}_2|, \xi_{23})$.

Let us note that the writing of the constant phase $-i\pi/8$ has not much significance since its true value (as that of the pre-exponential factor) depends on the form of the B_n coefficients which were omitted from (20) and (21). We obtain directly the above expression for the cross-section (7) if we remember that the imaginary and real parts of the amplitude, which both vary in accordance with (21), may oscillate with a certain (unknown) shift in the phase.

If (21) is substituted in (19), eight terms arise in each of which is integrated an exponential of the type:

$$\int d^2k \exp \left[-\sqrt{2\pi\alpha'} (\sqrt{\xi} |\vec{k}| (1 \pm i) + \sqrt{\xi_{12}} |\vec{k} - \vec{q}_1| (1 \pm i) + \sqrt{\xi_{23}} |\vec{k} - \vec{q}_2| (1 \pm i)) \right] \quad (22)$$

Let us first examine the integral in which all of the signs before i are identical, (for example $1+i$). The problem of calculating this integral is exactly the same as that of calculating a real integral from which the factor $(1+i)$ in the exponential is generally absent. We shall now show that in a certain range of variation of the parameters $\vec{q}_1, \vec{q}_2, \xi$ there is a real value $\vec{k} = \vec{k}_0$, at which the quantity

$$S(\vec{k}) = \sqrt{\xi} |\vec{k}| + \sqrt{\xi_{12}} |\vec{k} - \vec{q}_1| + \sqrt{\xi_{23}} |\vec{k} - \vec{q}_2| \quad (23)$$

has a minimum. The contribution from integration in the region of the saddle point $\vec{k} = \vec{k}_0$ is exponentially greater than the contribution from integration over the remaining part of the contour. This relates both to a purely real integral without a factor $(1+i)$, and to an integral with this factor. The latter can be verified simply by evaluating the modulus of the integral. In the case of a mixed

complexity in the exponential (22), it would, generally speaking, be necessary to seek complex steepest descents. We are, however, convinced below that in the region where there is a real $\vec{k}_0$ which gives a minimum to the value $S(\vec{k})$, the contribution of integrals with a mixed complexity is small so, generally, there is no need to calculate these integrals.

The possibility of calculating the integrals by the method of steepest descents is, of course, ensured by the condition (15).

In order to determine the minimum of the value $S(\vec{k})$ let us equate to zero its gradient along $\vec{k}$:

$$\nabla_{\vec{k}} S = \frac{\vec{k}}{|\vec{k}|} \sqrt{\xi} + \frac{\vec{k} - \vec{q}_1}{|\vec{k} - \vec{q}_1|} \sqrt{\xi_{12}} + \frac{\vec{k} - \vec{q}_2}{|\vec{k} - \vec{q}_2|} \sqrt{\xi_{23}} = 0 \quad (24)$$

An analytical solution of this equation results in unwieldy calculations. We shall use a certain geometrical consideration. Let us construct a triangle from two-dimensional vectors $\vec{q}_1$, $\vec{q}_2$ and $\vec{q} = \vec{q}_1 - \vec{q}_2$ (ΔOAB , in Fig. 3). Let us show, first of all, that the end of the vector $\vec{k}$ (the point K of Fig. 3) which is the solution of (24) cannot be outside ΔOAB (the beginning of the vector $\vec{k}$ is at the point O). The distances from point K to the vertices of ΔOAB are, in fact, the values $|\vec{k}|$, $|\vec{k} - \vec{q}_1|$ and $|\vec{k} - \vec{q}_2|$, which go into $S(\vec{k})$. If we reflect the point K with reference to the axis formed by the direction of any of the sides of the triangle OAB, the reflected point is obviously found at the same distance as the initial one from two of the vertices of the triangle, but the distance to the third vertex changes. Geometrically, it is clear that if the point K is outside ΔOAB , it is always possible to find a reflection which decreases this distance. Moreover, we can always, by means of the subsequent reflections in relation to the three sides, situate the reflection of the point K within ΔOAB , decreasing the value of S at each reflection. It follows from this that the point K, at which the value S has a minimum, cannot be outside ΔOAB .

From (24) we have:

$$\begin{aligned}\xi &= \xi_{12} + \xi_{23} + 2\sqrt{\xi_{12}\xi_{23}} \cos(\vec{k} - \vec{q}_1, \vec{k} - \vec{q}_2) \\ \xi_{23} &= \xi + \xi_{12} + 2\sqrt{\xi\xi_{12}} \cos(\vec{k}, \vec{k} - \vec{q}_1)\end{aligned}\tag{25}$$

As $\xi = \xi_{12} + \xi_{23}$, this means that

$$\begin{aligned}\cos(\vec{k} - \vec{q}_1, \vec{k} - \vec{q}_2) &= 0 \\ \cos(\vec{k}, \vec{k} - \vec{q}_1) &= -\sqrt{\xi_{12}/\xi}\end{aligned}\tag{26}$$

In order to fulfil the first of these equations, let us construct a circle with a diameter AB. Then, the point K must be situated on the arc of the circle lying inside $\triangle OAB$, as is shown in Fig. 3 ($\angle AKB$ is a right angle). In order to fulfil the second equation, let us construct the angle $\angle ABC = \alpha$ with a cosine of $\sqrt{\xi_{12}/\xi}$ and join point C to point O. Then, the point of intersection of the line OC with the circle must be the point K. In fact, $\angle AKC = \angle ABC$, since both of these angles lie on the same arc. This means that the cosine of the angle $\angle OKA$ is equal to $(-\sqrt{\xi_{12}/\xi})$ a fact which is made possible by the second equation (26).

Determination of the vector $\vec{k}$ from Fig. 3 is fairly difficult. It is sufficient, however, for us to find the value $S(\vec{k})$. Let us show that

$$\frac{S(\vec{k})}{\sqrt{\xi}} = OC\tag{27}$$

As $OK = |\vec{k}|$, it is sufficient, for this, to show that

$$\begin{aligned}KC &= |\vec{k} - \vec{q}_1| \sqrt{\frac{\xi_{12}}{\xi}} + |\vec{k} - \vec{q}_2| \sqrt{\frac{\xi_{23}}{\xi}} \\ &= AK \cos \alpha + BK \sin \alpha\end{aligned}\tag{28}$$

Let us draw the axis L through the centre of the circle G , parallel to the chord KC . The values $AK \cos \alpha$ and $BK \sin \alpha$ are equal to the projections of AK and BK in the direction L : $AK \cos \alpha = A'K'$, $BK \sin \alpha = B'K'$. Furthermore

$$A'K' + B'K' = (GK' + A'G) + (GK' - B'G) \quad (29a)$$

But $A'G = B'G$, and therefore

$$A'K' + B'K' = 2GK' = K'C' = KC \quad (29b)$$

which proves Eqs. (28) and (27).

It remains to find the length of OC . This can most easily be done by introducing a co-ordinate system with the origin of the co-ordinates at the centre of the circle and with an axis X oriented along GB . Then, the co-ordinates of point O are the co-ordinates of the vector $-(\vec{q}_1 + \vec{q}_2)/2$:

$$x_0 = -\frac{q_{1x} + q_{2x}}{2}, \quad y_0 = -\frac{q_{1y} + q_{2y}}{2} \quad (30)$$

and the co-ordinates of point C are

$$\begin{aligned} x_c &= -BC \cos \alpha + GB = -g \cos^2 \alpha + g/2 = g \frac{\xi_{23} - \xi_{12}}{2\xi} \\ y_c &= BC \sin \alpha = g \sin \alpha \cos \alpha = g \sqrt{\frac{\xi_{12} \xi_{23}}{\xi^2}} \end{aligned} \quad (31)$$

From which we easily find

$$\begin{aligned} OC &= \sqrt{(x_0 - x_c)^2 + (y_0 - y_c)^2} \\ &= \xi^{-1/2} [g_1^2 \xi_{12} + g_2^2 \xi_{23} + 2g_1 g_2 \sqrt{\xi_{12} \xi_{23}} \sin \phi]^{1/2} \end{aligned} \quad (32)$$

where ϕ is the angle between the vectors $\vec{q}_1$ and $\vec{q}_2$.

And, finally:

$$S = \sqrt{q_1^2 \xi_{12} + q_2^2 \xi_{23} + 2q_1 q_2 \sqrt{\xi_{12} \xi_{23}} \sin \varphi} \quad (33)$$

Let us now find the conditions under which the saddle point does in fact exist. The easiest way of understanding this is by geometry.

1) The angle must be acute since if it is not, the point O in Fig. 3 would fall inside the circle, and the point K would be outside Δ OAB. It is clear that when $\varphi = \frac{\pi}{2}$, K and O coincide, since the steepest descent value is $\vec{k} = 0$. The steepest descent drops onto the edge of the contour of integration, and for larger values of φ the value of the integral is determined by the values of the exponential when $\vec{k} = 0$.

2) The angle φ_1 between the vectors $\vec{q}_1$ and $\vec{q}$ added to angle $\angle CAB$ equal to $\frac{\pi}{2} - \alpha$, should not exceed π , ($\varphi_1 < \frac{\pi}{2} + \alpha$). In the opposite case the point K again falls outside Δ OAB. When $\varphi_1 = \frac{\pi}{2} + \alpha$, $\vec{k} = \vec{q}_1$, and for larger values of φ_1 , the value of the integral is determined by the value of the exponential when $\vec{k} = \vec{q}_1$.

3) From similar considerations for the existence of the steepest descent the equation $\varphi_2 < \pi - \alpha$ must be fulfilled. When $\varphi_2 > \pi - \alpha$ the integral is determined by the value of the exponential when $\vec{k} = \vec{q}_2$.

Thus, four regions of different asymptotic behaviour arise:

$$\begin{aligned} & \int d^3K \exp[-\sqrt{2\pi\alpha'}(1 \pm i) \{ |\vec{k}| \xi^{1/2} + |\vec{k} - \vec{q}_1| \xi_{12}^{1/2} + |\vec{k} - \vec{q}_2| \xi_{23}^{1/2} \}] \\ & \simeq \exp[-\sqrt{2\pi\alpha'}(1 \pm i) \sqrt{q_1^2 \xi_{12} + q_2^2 \xi_{23} + 2q_1 q_2 \sqrt{\xi_{12} \xi_{23}} \sin \varphi}] \quad (34a) \\ & \varphi < \pi/2, \quad \varphi_1 < \pi/2 + \alpha, \quad \varphi_2 < \pi - \alpha \end{aligned}$$

$$\simeq \exp[-\sqrt{2\pi\alpha'}(1 \pm i) [\sqrt{q_1^2 \xi_{12}} + \sqrt{q_2^2 \xi_{23}}]] \quad , \quad \varphi > \pi/2 \quad (34b)$$

$$\simeq \exp -\sqrt{2\pi\alpha'}(1\pm i)\left[\sqrt{q_1^2\xi} + \sqrt{(\vec{q}_1-\vec{q}_2)^2\xi_{23}}\right] \quad \varphi_1 > \pi/2 + \alpha \quad (34c)$$

$$\simeq \exp -\sqrt{2\pi\alpha'}(1\pm i)\left[\sqrt{q_2^2\xi} + \sqrt{(\vec{q}_1-\vec{q}_2)^2\xi_{12}}\right], \varphi_2 > \pi - \alpha \quad (34d)$$

It can clearly be seen that nowhere do any of these regions overlap, and they cover the entire possible range of variations of φ_1 , φ_2 and $\varphi = \pi - \varphi_1 - \varphi_2$ ($\varphi_1 < \pi$, $\varphi_2 < \pi$, $\varphi_1 + \varphi_2 < \pi$). The four regions of variation of the variables φ_1 and φ_2 are illustrated in Fig. 4.

Let us now turn to the discussion of the integrals of the type (22) in which the signs in front of i are not identical. Let us first note that the integrals (34) in which there is only one common factor $(1\pm i)$ in the exponential, have the same order of magnitude as a real integral in which such a factor is absent. This happens because the steepest descent point for the real and imaginary part of the exponential is one and the same, and consequently the oscillations which exist in the integral do not, in practice, decrease its size. The real part from the expressions (34) differs from the corresponding expressions for a real integral only by an additional cosine, the value of which may be considered to be of the order of unity. (This is obviously untrue in the very close vicinity of the zeros of the cosine; however, in the final expressions for the cross-section we nevertheless are concerned with several functions which oscillate with a shifted phase, so that nowhere does the cross-section become zero).

A different situation occurs for integrals in which the factors $(1+i)$ and $(1-i)$ alternate. Here, in the same real part of the exponential under the integral there are still oscillations, which do not disappear at the saddle point for the real part. It is easy to understand that these oscillations shift the saddle point into a complex plane, as a result of which the value of the integral is exponentially small in comparison with the value of the real integral. Therefore, in region I where the saddle point value of $\vec{k}_0$ exists, all the integrals except (34a) are small, and we can disregard them. In regions II-IV, where the values of the integrals are defined by the points $\vec{k}=0$, $\vec{k}=\vec{q}_1$ and $\vec{k}=\vec{q}_2$ all the integrals prove to be of the same order of magnitude ^{*)}. It is, however, not necessary here to calculate in turn all the integrals for the following reason. When $\vec{k} \rightarrow 0$ under the integral (19) it is sufficient to leave only $G(|\vec{k}|, \xi)$, since only this function will contain a term that is linear in $|\vec{k}|$. This means that in region II

$$A \cong i s G(|\vec{q}_1|, \xi_{12}) G(|\vec{q}_2|, \xi_{23}) \left[1 + \int \frac{d^2 k}{(2\pi)^2} G(|\vec{k}|, \xi) \right] \quad (35)$$

and similarly in the other regions.

We can now write the final expression for the amplitude in all four cases.

*) The difference between the cases when the size of the integral is determined by the steepest descent point and by the edge of the contour of integration, is easily seen from the following example:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 + i\beta x} dx = e^{-\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha x'^2} dx' \ll \int_{-\infty}^{\infty} dx e^{-\alpha x^2}$$

but

$$\int_A^{\infty} e^{-\alpha x + i\beta x} dx = \frac{1}{\alpha - i\beta} e^{-\alpha A + i\beta A} \sim \int_A^{\infty} e^{-\alpha x} dx = \frac{1}{\alpha} e^{-\alpha A}$$

Region I $\phi_1 + \phi_2 > \pi/2$ $\phi_1 < \pi/2 + \alpha$ $\phi_2 < \pi - \alpha$ $\phi = \pi - \phi_1 - \phi_2$

$$A = i s F_1 \exp[-\sqrt{2\pi\alpha'} \sqrt{q_1^2 \xi_{12} + q_2^2 \xi_{23} + 2q_1 q_2 \sqrt{\xi_{12} \xi_{23}} \sin \phi}] \times \\ \times \cos[\sqrt{2\pi\alpha'} (q_1^2 \xi_{12} + q_2^2 \xi_{23} + 2q_1 q_2 \sqrt{\xi_{12} \xi_{23}} \sin \phi) + \chi_1] \quad (36a)$$

Region II $\phi_1 + \phi_2 < \pi/2$

$$A = i s F_2 \exp[-\sqrt{2\pi\alpha'} (\sqrt{q_1^2 \xi_{12}} + \sqrt{q_2^2 \xi_{23}})] \times \\ \times \cos[\sqrt{2\pi\alpha'} q_1^2 \xi_{12} + \chi_2] \cos[\sqrt{2\pi\alpha'} q_2^2 \xi_{23} + \chi_2'] \quad (36b)$$

Region III $\phi_1 > \pi/2 + \alpha$

$$A = i s F_3 \exp[-\sqrt{2\pi\alpha'} (\sqrt{\vec{q}_1^2 \xi} + \sqrt{(\vec{q}_1 - \vec{q}_2)^2 \xi_{23}})] \times \\ \times \cos[\sqrt{2\pi\alpha'} q_1^2 \xi + \chi_3] \cos[\sqrt{2\pi\alpha'} (\vec{q}_1 - \vec{q}_2)^2 \xi_{23} + \chi_3'] \quad (36c)$$

Region IV $\phi_2 > \pi - \alpha$

$$A = i s F_4 \exp[-\sqrt{2\pi\alpha'} (\sqrt{q_2^2 \xi} + \sqrt{(\vec{q}_1 - \vec{q}_2)^2 \xi_{12}})] \times \\ \times \cos(\sqrt{2\pi\alpha'} q_2^2 \xi + \chi_4) \cos(\sqrt{2\pi\alpha'} (\vec{q}_1 - \vec{q}_2)^2 \xi_{12} + \chi_4') \quad (36d)$$

As in Ref. 1) we introduced in the expressions (36) for the amplitudes the arbitrary phases $\chi_1, \chi_2, \chi_2', \dots$, and the pre-exponential factors, $F_1, \dots, F_4$. $F_1, \dots, F_4$ may depend on $q_1^2, q_2^2, \xi_{12}, \xi_{23}$ only in powers, and $\chi_1, \dots, \chi_4'$ only logarithmically. These values are related to the coefficients omitted in the series (16). (Or with the form of the vertices in the diagram of Fig. 2.)

As was elucidated in Refs. 1) and 2), in the case of two-particle processes, because of the presence in the amplitude of imaginary and real parts which may be in the region $\alpha' q^2 \gg 1$ of one order and oscillate with a phase shift, but also because of the presence, for particles with spin, of several invariant functions, the cross-section is represented in the form of the sum of the squares of several amplitudes with various pre-exponential factors and phases. Although this sum may contain many terms, we nevertheless arrive at a three-parameter formula (7), since the cross-section will contain only one falling exponential and one oscillating frequency. A similar situation now occurs in region I, where the cross-section may be written in the form:

$$(\text{In region I}) \quad \frac{d\sigma}{dt} = F e^{-2\omega} (1 + \lambda \cos(2\omega + \chi^{(0)}))$$

$$\omega = \sqrt{2\pi\alpha' (q_1^2 \xi_{12}^2 + q_2^2 \xi_{23}^2 + 2q_1 q_2 \sqrt{\xi_{12}^2 \xi_{23}^2} \sin \phi)}$$

(37)

In the remaining regions we have to deal with several frequencies, for example in range II with $2\omega_1 = 2\sqrt{2\pi\alpha' q_1^2 \xi_{12}^2}$, $2\omega_2 = 2\sqrt{2\pi\alpha' q_2^2 \xi_{23}^2}$ and $2\omega^\pm = 2(\omega_1 \pm \omega_2)$. Therefore, the real expression for the cross-section in the regions II-IV, which is not unduly related to the details of the model, will contain a fair number of parameters:

$$\begin{aligned} \frac{d\sigma}{dt} = F e^{-2\omega} & [1 + \lambda_1 \cos(2\omega_1 + \chi_1^{(0)}) + \lambda_2 \cos(2\omega_2 + \chi_2^{(0)}) \\ & + \lambda_+ \cos(2\omega_+ + \chi_+^{(0)}) + \lambda_- \cos(2\omega_- + \chi_-^{(0)})] \end{aligned}$$

(38)

Here $F, \lambda_1, \dots, \lambda_-$, just like the values F and λ in formula (37), depend on $q_1^2, q_2^2, \vec{q}_1 \cdot \vec{q}_2, \xi_{12}^2, \xi_{23}^2$, only in powers whilst the phases $\chi_1^{(0)}, \dots, \chi_-^{(0)}$ do so logarithmically. For ω_1

and ω_2 we have, from (36):

(in region II)

$$\omega_1 = \sqrt{2\pi\alpha' q_1^2 \xi_{12}} \quad \omega_2 = \sqrt{2\pi\alpha' q_2^2 \xi_{23}}$$

(in Region III)

$$\omega_1 = \sqrt{2\pi\alpha' q_1^2 \xi} \quad \omega_2 = \sqrt{2\pi\alpha' (\vec{q}_1 - \vec{q}_2)^2 \xi_{23}}$$

(in Region IV)

$$\omega_1 = \sqrt{2\pi\alpha' q_2^2 \xi} \quad \omega_2 = \sqrt{2\pi\alpha' (\vec{q}_1 - \vec{q}_2)^2 \xi_{12}}$$

$$\omega_{\pm} = |\omega_1 \pm \omega_2|$$

(39)

Of course, in a concrete model in which the amplitude is represented accurately by the sum (16), all the unknown parameters may be calculated. However, as they do not, we believe, have a relation to reality we do not give the corresponding values. [Not all the unknown phases and amplitudes in Eq. (38) are independent. They are connected by complicated relations.]

As can be seen from expressions (36a), (36c) and (36d) the amplitude A possesses an interesting property. When $\vec{q}_1 = \vec{q}_2 = \vec{Q}$ (in this case, of course, $\emptyset = 0$), i.e., when the transverse momentum of the middle particle vanishes, the three-particle amplitude is reduced to the amplitude of the two-particle process

$$A = i s G(|\vec{Q}|, \xi) \simeq i s e^{-\sqrt{2\pi\alpha' Q^2 \xi}} \cos(\sqrt{2\pi\alpha' Q^2 \xi} + \tilde{\chi})$$

(40)

The possibility of the existence of a relationship of this type was pointed out in ¹⁰⁾, where it was shown that it was related to a Ward identity for reggeon graphs. In more general cases which are not related to a concrete model this theorem is true only with certain limitations, in particular, only when there exists a remaining quantum number transferred along one of the reggeon (not vacuum) lines.

In the case under examination, the relationship (40) appears, to a certain extent, to be the result of the proposed model. There are, however, grounds for believing that it is not of an unduly special nature. In fact, (40) always follows from formula (19) when, with $\vec{q}_1 - \vec{q}_2 \rightarrow \vec{Q}$, the steepest descent value $\vec{k} \rightarrow \vec{Q}$, as occurs, in particular in the model under consideration. (Here this is obvious from geometrical considerations.)

Among the results obtained in the present section the most interesting from the physical point of view is, we feel, that with specific configurations of the momenta of the particles which are formed (regions II-IV) it is of no advantage to transfer a large momentum along one of the three lines on the diagram in Fig. 2. This means that, in the regions II-IV of phase space, there already occurs between the individual particles an exchange of a small number of reggeons; in region II: $n \sim 1$; in the regions III and IV: $n_1 \sim 1$, $n_2 \sim 1$ respectively. Qualitatively, this fact is, we consider, not unduly related to the details of the model (in any case, we checked that it is maintained, provided certain reasonable alterations of the model are made) and, from this point of view, the case of the three-particle production appears to be substantially different from the case of elastic processes.

Let us note, in concluding this section, that the asymptotic behaviour of the amplitudes for three-particle production in the region of large momentum transfers was examined in Ref. 11) with the use of unitarity in the s channel. The formulas obtained therein are not the same as the formulas (36) of the present paper. This is, however, not surprising since the approximation in Ref. 11) was that in the s channel there remained only two or three-particle states, whereas the use of reggeons presupposes a large number of particles (of the order of the logarithm of the energy) in the direct channel.

3. A POTENTIAL MODEL

As was clarified in the Introductory Section, the case of elastic scattering at large momentum transfers may be understood qualitatively in the framework of scattering on a smooth, purely imaginary potential, as given, for example, by formula (8). In this case, it is necessary to bear in mind multiple rescattering, i.e., to use an eikonal approximation. We shall now show that a similar situation also occurs in the case of the production amplitude. In essence, we shall derive a formula for the production amplitude of three particles, similar to the eikonal formula.

Let us study the reaction of the production of particles 1, 2 and 3 during collisions a and b (Fig. 2). Let particles a and b be able, like particles 1, 2 and 3, to scatter on account of the potentials, V_{ab} , V_{12} , V_{13} and V_{23} . In addition, we need, obviously to introduce also the interaction W , which is responsible for the production of particles 1, 2 and 3 from a and b. The introduction of this interaction takes us, strictly speaking, outside the framework of pure quantum mechanics. In order to isolate clearly the effects due to the rescattering of the particles in the initial and final conditions we shall take account of W only in the first order of perturbation theory (for example, we shall consider W to be weak). Then the total production amplitude (taking into account rescattering) may be written in the form ^{*}):

$$A = \int \psi_f^{(-)*}(\vec{r}_3, \vec{r}_2) W(\vec{r}_3, \vec{r}_2) \psi_i^{(+)}(\vec{r}_3) d^3 r_3 d^3 r_2 \quad (41)$$

^{*}) In this normalisation

$$\frac{d\sigma}{d\tau} = 2\pi \frac{\mu_{ab}}{k_{ab}} |A|^2$$

where k_{ab} and μ_{ab} are the momentum and reduced mass of the particles in the initial configuration and (41a)

$$d\tau = \delta(E_i - E_f) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \quad (41b)$$

Here $\psi_i^{(+)}$ is the wave function of the initial state which contains in the asymptotic form a plane incident wave and an outgoing wave. $\psi_f^{(-)}$ is the function of the final state with an asymptotic form as a plane and an ingoing wave. In (41) we used the Jacobi co-ordinates: $\vec{r}_{13}$ is the radius-vector of the relative position of particles 1 and 3, and $\vec{r}_2$ is the position of the second particle relative to the centre of inertia of 1 and 3.

The two-particle wave function $\psi_i^{(+)}$ in an eikonal approximation is well-known⁸⁾.

$$\psi_i^{(+)}(\vec{r}_{13}) = e^{i\vec{k}_{ab} \cdot \vec{r}_{13}} \cdot e^{-\frac{i}{v_{ab}} \int_{-\infty}^{z_{13}} dz'_{13} V_{ab}(\vec{p}_{13}, z'_{13})} \quad (42)$$

where $\vec{r}_{13} = (\rho_{13}, z_{13})$; for the direction z is chosen the direction of the relative movement of a and b , i.e., k_{ab} ; v_{ab} is the relative velocity.

Let us find the three-particle wave function ψ_f . ψ_f satisfies the Schrödinger equation.

$$\left(-\frac{1}{2\mu_{13}} \nabla_{13}^2 - \frac{1}{2\mu_2} \nabla_2^2 + \sum V_{ij} \right) \psi_f = E \psi_f \quad (43)$$

$\mu_{13}^{-1} = m_1^{-1} + m_3^{-1}$, $\mu_2^{-1} = m_2^{-1} + (m_1 + m_3)^{-1}$ are reduced masses.

As in the two-particle case ψ_f must be sought in the form of a plane wave corresponding to free motion, multiplied by a certain slowly varying function:

$$\psi_f = e^{i\vec{k}_{13} \cdot \vec{r}_{13} + i\vec{k}_2 \cdot \vec{r}_2} \phi \quad (44)$$

$\vec{k}_{13}$ is the relative momentum of particles 1 and 3, $\vec{k}_2$ is the momentum of the second particle of the common centre of mass system. In Eq. (43), the derivatives along the transverse co-ordinates

are small, and, in exactly the same way as in the two-particle case, we obtain the equation for ϕ :

$$\left(\frac{i k_{13}}{\mu_{13}} \frac{\partial}{\partial z_{13}} + \frac{i k_2}{\mu_2} \frac{\partial}{\partial z_2} \right) \phi = \sum V_{ij} \phi \quad (45)$$

In the relativistic problem considered in the previous Section, the middle particle 2 shown in Fig. 2 is slow in comparison with 1 and 3, $k_2 \ll k_{13}$. This follows from the conditions $s_{12} \ll s$ and $s_{23} \ll s$. It is easy to see that

$k_{13} \sim \sqrt{s_{13}} \sim \sqrt{s}$ and $k_{12} \sim s^{|\lambda-1/2|}$ where the value λ determines the relative value of s_{12} and s_{23} : $s_{12} \sim s^\lambda$, $s_{23} \sim s^{1-\lambda}$, $0 < \lambda < 1$. Therefore, in the non-relativistic model too, which we are now studying, we shall consider the second particle to be comparatively slow and shall assume the condition $k_2 \ll k_{13}$. Then, in Eq. (45) the term with the derivative $\partial/\partial z_2$ drops out and we have the simple solution for ϕ :

$$\begin{aligned} \phi^{(\pm)} = \exp \left\{ \frac{-i}{U_{13}} \int_{-\infty}^{z_{13}} V_{13}(\vec{p}_{13}, z_{13}') dz_{13}' \right. \\ \left. - \frac{i}{U_{13}} \int_{-\infty}^{z_{13}} V_{12}(\vec{p}_{12}, \frac{m_3}{m_1+m_3} z_{13}' - z_2) dz_{13}' \right. \\ \left. - \frac{i}{U_{13}} \int_{-\infty}^{z_{13}} V_{23}(\vec{p}_{23}, \frac{m_1}{m_1+m_3} z_{13}' + z_2) dz_{13}' \right\} \quad (46a) \end{aligned}$$

We wrote out explicitly the arguments of the functions V_{12} and V_{23} , taking into account that integration takes place when z_2 is constant. Our integration limit is chosen as $-\infty$ or $+\infty$ for an outgoing wave $\phi^{(+)}$ or an ingoing wave $\phi^{(-)}$. By replacing the integration constants and taking into account that when $k_2 \ll k_{13}$,

$$U_{12} = \frac{U_{13} m_3}{m_1 + m_3} \quad (46b)$$

and

$$U_{23} = \frac{U_{13} m_1}{m_1 + m_3} \quad (46c)$$

we obtain from (46a):

$$\phi^{(\pm)} = \exp \left\{ \frac{-i}{U_{12}} \int_{-\infty}^{z_{12}} d\bar{z}'_{12} V_{12}(\vec{p}_{12}, \bar{z}'_{12}) - \frac{i}{U_{13}} \int_{-\infty}^{z_{13}} d\bar{z}'_{13} V_{13}(\vec{p}_{13}, \bar{z}'_{13}) - \frac{i}{U_{23}} \int_{-\infty}^{z_{23}} d\bar{z}'_{23} V_{23}(\vec{p}_{23}, \bar{z}'_{23}) \right\} \quad (47)$$

when

$$z_{13} = z_{12} + z_{23} \quad \vec{p}_{13} = \vec{p}_{12} + \vec{p}_{23} \quad (48)$$

Let us now insert (47) and (42) in (41) and introduce, as before the momentum transfers

$$\vec{q}_1 = \vec{k}_1 - \vec{k}_a \quad \vec{q}_2 = \vec{k}_b - \vec{k}_3 \quad (49)$$

In view of the non-relativistic nature of the problem we must consider the masses $m_a = m_1$ and $m_b = m_3$ as equal (this, of course is not required in relativistic conditions) since in the opposite case the momentum transfers will not be transverse. We then have:

$$A = \int d^2 p_{12} d^2 p_{23} e^{-i \vec{q}_1 \cdot \vec{p}_{12} - i \vec{q}_2 \cdot \vec{p}_{23}} F(\vec{p}_{12}, \vec{p}_{23}) \quad (50)$$

where

$$F(\vec{p}_{12}, \vec{p}_{23}) = \iint d\bar{z}_{12} d\bar{z}_{23} \exp \left\{ \frac{-i}{U_{ab}} \int_{-\infty}^{z_{13}} d\bar{z}'_{13} V_{ab} - \frac{i}{U_{12}} \int_{-\infty}^{z_{12}} d\bar{z}'_{12} V_{12} - \frac{i}{U_{13}} \int_{-\infty}^{z_{13}} d\bar{z}'_{13} V_{13} - \frac{i}{U_{23}} \int_{-\infty}^{z_{23}} d\bar{z}'_{23} V_{23} \right\} W(\vec{p}_{12}, \bar{z}_{12}, \vec{p}_{23}, \bar{z}_{23})$$

A difference from the case of elastic scattering in formula (50) is that it is not possible in the general form to perform the integration over dz_{12} and dz_{23} . We shall, however, examine two examples when this can be done.

The first example is of a fairly non-physical nature but clarifies the situation well. Let W be a function with a sharp maximum at $z_{12}=0$ and $z_{23}=0$, so that the residual factors can be extracted at this point from under the integral. Then,

$$F(\vec{p}_{12}, \vec{p}_{23}) = \tilde{W} \exp \{ i \delta_{ab}(p_{13}) + i \delta_{12}(p_{12}) + i \delta_{13}(p_{13}) + i \delta_{23}(p_{23}) \} \quad (51)$$

where

$$\tilde{W} = \int_{-\infty}^{\infty} dz_{12} dz_{23} W$$

$$\delta_{ab} = -\frac{1}{v_{ab}} \int_{-\infty}^0 V_{ab} dz_{13}', \quad \delta_{12} = -\frac{1}{v_{12}} \int_0^{\infty} V_{12} dz_{12}' \quad \text{e.t.c.} \quad (52)$$

The expressions (52) represent the scattering phase in the eikonal approximation. In the case of elastic scattering we have an integral for z from $-\infty$ to $+\infty$, which gives a factor $e^{2i\delta(p)}$. Here, in the interval $-\infty < z < 0$ only the particles (a, b) are scattered but after this, when $z > 0$ the particles 1, 2 and 3 are scattered. The condition that W is a sharp function of z clearly isolates the scattering of the particles in the initial and final states. (51) is thus a simple consequence of unitarity.

Another example consists of the special choice of W , which is that particle production occurs on account of an interaction of the same type as that of scattering

$$W = \lambda V_{12}(\vec{r}_{12}) V_{23}(\vec{r}_{23}) \quad (53)$$

λ is a certain constant.

In addition, we shall consider the particles a and b identical to 1 and 3. It is this very situation that occurs, obviously, when summing the reggeon graphs due to the exchange of vacuum poles. By inserting (53) in (50) and noting that:

$$V_{12} \exp \frac{-i}{v_{12}} \int_{-\infty}^{\bar{z}_{12}} V_{12} d\bar{z}'_{12} = -i v_{12} \frac{d}{d\bar{z}_{12}} \exp \frac{-i}{v_{12}} \int_{-\infty}^{\bar{z}_{12}} V_{12} d\bar{z}'_{12}$$

and, similarly for the integral with V_{23} , we easily obtain

$$A = -\lambda v_{12} v_{23} \int e^{-i\vec{q}_1 \cdot \vec{p}_{12} - i\vec{q}_2 \cdot \vec{p}_{23}} e^{2i\delta_{13}(p_{13})} \cdot (e^{2i\delta_{12}(p_{12})} - 1)(e^{2i\delta_{23}(p_{23})} - 1) d^2 p_{12} d^2 p_{23} \\ \vec{p}_{13} = \vec{p}_{12} + \vec{p}_{23} \quad (54)$$

where

$$\delta_{ij} = \frac{-1}{v_{ij}} \int_{-\infty}^{\infty} V_{ij} d\bar{z}' \quad (55a)$$

(54) and (55a) form the natural generalisation of formulas (9) and (10) for an elastic amplitude. Let us note that by the interaction of V_{13} , A has a phase $e^{2i\delta_{13}}$, which is due to the scattering both in the final and in the initial state. The interactions V_{12} (V_{23}) give, however, only

$$e^{i\delta_{12}} \cdot e^{i\delta_{23}} \quad (55b)$$

since

$$(e^{2i\delta_{12}} - 1)(e^{2i\delta_{23}} - 1) = -4 e^{i(\delta_{12} + \delta_{23})} \sin \delta_{12} \sin \delta_{23} \quad (55c)$$

If we turn from the representation in terms of impact parameters to the usual momentum representation by introducing:

$$A_{ij}(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{p}} (e^{2i\delta_{ij}(\vec{p})} - 1) d^2p \quad (56)$$

(54) takes the form

$$A = -\lambda v_{12} v_{23} \left[A_{12}(\vec{q}_1) A_{23}(\vec{q}_2) + \int \frac{d^2k}{(2\pi)^2} A_{13}(\vec{k}) A_{12}(\vec{k}-\vec{q}_1) \cdot A_{23}(\vec{k}-\vec{q}_2) \right] \quad (57)$$

The last formula coincides with expression (19).

Formulas (19) and (57) are obviously somewhat more general than the model considered in the previous Section or the eikonal model. In both of these models, apart from formulas (19) and (57), there is also a formula for calculating the two-particle amplitudes which go under the integral. Formulas (57) or (54), in fact, have such a graphical and physical sense that one is tempted to use them outside the dependence on the models used for the derivation. Let us assume, as an example, that all two-particle amplitudes are equal to:

$$A_{12}(\vec{k}) = A_{13}(\vec{k}) = A_{23}(\vec{k}) = e^{-bk} \quad (58)$$

$$b = \text{const}$$

which, more or less, corresponds to the experimental situation. (The oscillations observed at the present time are fairly insignificant, as is the dependence of the exponential drop on the energy.) Then, by calculating the integral (57) by the steepest descent method, we easily obtain the following answer for the three-particle amplitude:

$$\begin{aligned}
 A &\sim \exp - b \sqrt{q_1^2 + q_2^2 + 2q_1 q_2 \sin(\varphi - \pi/6)}, \quad \varphi_1 < 2\pi/3, \\
 &\quad \varphi_2 < 2\pi/3, \quad \varphi < 2\pi/3 \\
 A &\sim e^{-b(q_1 + q_2)} \quad \varphi > 2\pi/3 \\
 A &\sim e^{-b(q_1 + |\vec{q}_1 - \vec{q}_2|)} \quad \varphi_1 > 2\pi/3 \\
 A &\sim e^{-b(q_2 + |\vec{q}_1 - \vec{q}_2|)} \quad \varphi_2 > 2\pi/3
 \end{aligned}
 \tag{59}$$

All the qualitative properties of the asymptotic form of (36) (apart, of course, from the oscillations) remain true.

If we use the expression (8) for the potentials V_{ij} , considering that they are defined by exchanging the Pommeranchuk pole, it is easy to find, by means of (54):

$$\begin{aligned}
 A &\sim \sum_{n, n_1, n_2} \frac{(-1)^{n+n_1+n_2}}{n! n_1! n_2!} \left(\frac{\sigma_{13}}{8\pi\alpha' \xi} \right)^{n-1} \left(\frac{\sigma_{12}}{8\pi\alpha' \xi_{12}} \right)^{n_1-1} \left(\frac{\sigma_{23}}{8\pi\alpha' \xi_{23}} \right)^{n_2-1} \\
 &\cdot \frac{1}{\frac{\xi}{n} + \frac{\xi_{12}}{n_1} + \frac{\xi_{23}}{n_2}} \cdot \exp \left[-\alpha' \cdot \frac{n_2 \vec{q}_1^2 \xi_{12} + n_1 \vec{q}_2^2 \xi_{23} + n(\vec{q}_1 - \vec{q}_2)^2 \xi_{12} \xi_{23}}{n_1 n_2 \xi + n n_2 \xi_{12} + n n_1 \xi_{23}} \right]
 \end{aligned}
 \tag{60}$$

This series represents a sum of terms of the same type as the contribution of the branch points (14) with fully defined coefficients. As for the case of elastic scattering, the explicit form of these coefficients has scarcely any relationship with the full relativistic problem. As was clarified above, we may consider that with present-day energies only the alternating nature of the coefficients is really relevant. This is why we have confined ourselves to a summation of the series (16).

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FIGURE CAPTIONS

- Figure 1 :
Feynman diagram for three-particle production.
- Figure 2 :
General Pomeron exchange contribution to three-particle production.
- Figure 3 :
Geometrical construction for saddle point calculation.
- Figure 4 :
Regions of different asymptotic behaviour.

FIG. 1

FIG. 2

FIG. 3

FIG. 4