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A PROOF OF THE EXISTENCE OF FUNCTIONS THAT SATISFY EXACTLY
BOTH CROSSING AND UNITARITY
IV. NEARLY CONSTANT ASYMPTOTIC CROSS-SECTIONS

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A B S T R A C T

A proof of the existence of functions that satisfy a Mandelstam representation with the crossing symmetry appropriate to pion-pion scattering, elastic unitarity below the inelastic threshold, and the inelastic inequalities above it, is generalized to a restricted class of amplitudes that need an arbitrary, finite number of subtractions. Although the leading angular momentum singularity is constrained not to rise above unity in the elastic region, it is possible for the total cross-section to display an almost constant asymptotic behaviour.

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1. INTRODUCTION

A previously elaborated proof ¹⁾ of the existence of functions that possess the crossing symmetry of pion-pion scattering (with Mandelstam analyticity), that satisfy the elastic unitarity condition below the four-pion threshold, and the inelastic constraints above it, is extended to encompass cross-sections that are almost constant asymptotically. Although the leading singularity in the angular momentum plane may rise indefinitely far, the exigencies of the present proof impose the restriction that this singularity should stay below unity in the elastic strip. The rho trajectory does so, but presumably the Pomeranchuk does not. Nevertheless, the total cross-section, $\sigma(s)$, can have the following asymptotic behaviour:

$$\sigma(s) \underset{s \rightarrow \infty}{\sim} (\log s)^{-2-\varepsilon}, \quad (1.1)$$

where $\varepsilon > 0$, which cannot yet be excluded experimentally. Such a behaviour would be attained, and the proof of the present paper would be applicable, if the Pomeranchuk trajectory were perfectly flat at unity, and if the singularity were not a simple pole, but a suitably complicated branch-point, or perhaps an accumulation of branch points.

The present extension follows the main lines of the earlier forms of the proof ¹⁾, but with two generalizations. First, a different metric is used in the Banach space [cf., Eq. (2.12)], which has the effect of decoupling the asymptotics from the Hölder-index, and so allowing the behaviour (1.1). Second, an extra generating function, $G(s,t)$, is employed, in addition to the inelastic double-spectral function, $v(s,t)$ (cf., Section 6). This function, which was introduced by Kupsch ²⁾, has the merit of allowing angular momentum trajectories that rise above unity, outside the elastic strip. The new generating function is to satisfy crossing symmetry and to possess Mandelstam analyticity, but with cuts starting only at the inelastic thresholds in the three channels. In addition, the partial wave projections, $G_\ell(s)$, are to satisfy the following "inelastic unitarity"

conditions:

$$\operatorname{Im} G_\ell(s) \geq 4 [1 - h(s)] |G_\ell(s)|^2, \quad (1.2)$$

for $s \geq 16$, $\ell = 0, 1, 2, \dots$, where $h(s)$ is a Hölder-continuous function that is unity at $s = 16$ and zero for $s \gg 20$ [cf., Eq. (2.10)]. In fact, it is possible in principle to use a $G(s, t)$ that has infinitely rising Regge trajectories, so long as its partial waves satisfy (1.2). The schema of the present paper then constitutes a method of unitarizing $G(s, t)$, without destroying the crossing symmetry. An outstanding problem is that one has not yet been able to construct a crossing-symmetric $G(s, t)$, with the Mandelstam cuts, but with infinitely rising trajectories, such that Eq. (1.2) holds. Kupsch²⁾ has fabricated Ansätze which have trajectories that can rise indefinitely far, but which must eventually turn over. It should be pointed out that there is as yet no theoretical reason why trajectories should not turn back.

Although the programme of the present series of papers will only be complete when a physically interesting input generating function is produced, it may be mentioned that the inelastic unitarity constraint (1.2) has a useful property of convexity³⁾. Thus, suppose that functions $G_n(s, t)$, $n = 1, 2, 3, \dots$, each separately possess the property (1.2), i.e., their partial wave projections satisfy positivity. Then, if one defines

$$G(s, t) = \sum_n a_n G_n(s, t), \quad (1.3)$$

where the constant coefficients, a_n , are non-negative and satisfy

$$\sum_n a_n = 1, \quad (1.4)$$

then one can easily show that $G(s, t)$ satisfies the positivity (1.2). The sums in Eqs. (1.3) and (1.4) can be finite or infinite, provided, in the latter case, that they converge absolutely. Clearly, these equations can be generalized to

$$G(s, t) = \int d\nu \, a(\nu) \, G(s, t; \nu), \quad (1.5)$$

and

$$\int d\nu \, a(\nu) = 1, \quad (1.6)$$

where the weight, $a(\nu)$, is non-negative, and $G(s, t; \nu)$ satisfies Eq. (1.2) for each ν . This property may well be useful in constructing a suitable series of "satellite" terms. For example, the individual terms in Eq. (1.3) or (1.5) need not be crossing-symmetric: only the sum, or integral, need be.

It has already been demonstrated that the Mandelstam equations have infinitely many solutions. This non-uniqueness corresponds, in part, to the freedom one has in selecting the generating function. Indeed, the present programme corresponds more closely to recent ideas, according to which one fixes upon a physically reasonable Ansatz, and then unitarizes it, rather than to the more old-fashioned "bootstrap" notion, where there was no generating function, and where the physical solution appears ex nihilo. It has not yet been shown whether the equations possess "bootstrap" solutions (i.e., singular solutions which do not vanish as the generating function tends to zero), in addition to the regular solutions. A very grave difficulty with regard to the possible existence of such singular solutions is precisely the satisfaction of the inelastic unitarity constraints.

It seems quite possible in fact that one of the regular solutions corresponds to the physical pion-pion amplitude. However, it is not thought that the relevant solution belongs to the CDD class zero. In particular, one expects at least one CDD pole in each of the waves $I=0=J$, $I=1=J$, and $I=0$, $J=2$. Whether these poles merely reflect important couplings to other channels, or whether they have a more elementary nature, is unknown; and the question in any case cannot be arbitrated solely on the basis of the one-channel pion equations. Because of the physical importance of CDD poles,

the proof of the present paper has been extended to include them (cf., Section 5), this extension being a straightforward generalization of the work of Ref. 4).

In this paper, solutions are obtained by an application of the Contraction Mapping Principle. No attempt has been made to estimate the largest domain in which one has a contraction: it was deemed sufficient, in view of the crudity of some of the majorizations involved, to show that this domain is non-vanishing. It may be that the physical solution lies outside the contraction domain, but this is by no means sure. Whereas McDaniel and Warnock⁵⁾ found, in their study of the Low equation, that their estimates of the contraction domain were disappointingly small, Chew, Mandelstam and Noyes⁶⁾, making a numerical study of a mathematically similar equation, had earlier found no difficulty in obtaining convergence of their iterative "S-dominant" solution, at physically interesting values of the coupling constant. This suggests that the actual contraction domain is very much greater than one would be led to expect from the crude estimates that the rigorous treatment provides. It is true that the S-dominant solution turned out to be unphysical, and that the later "P-dominant solution"⁷⁾ was not obtained by iteration. However, this latter solution was highly sensitive to the cut-off, and gave a rho resonance that was far too wide. It is quite possible that an iterative solution of the equations, with CDD poles to generate the σ , ρ and f_0 mesons, could be obtained. That physically reasonable low-energy solutions with CDD poles do at least exist, is evinced by the work of Shirkov⁸⁾ and collaborators. Should the physical solution nevertheless lie outside the contraction domain, it may be attainable by means of the Newton-Kantorovich method [cf., Section 5 of Ref. 4)]. Only numerical analysis can decide this question.

2. THE MANDELSTAM SECTOR

In this section the Mandelstam sector of the problem will be treated. The basic quantities are the elastic piece of the double

spectral function $\rho^{el}(s,t)$, and the elastic piece of the absorptive part of the S wave projection of the amplitude, $\text{Im} F_0^{el}(s)$. It is convenient to define the following functions in terms of these quantities:

$$\bar{\rho}(s,t) = \frac{s}{t} [\log(s)\log(t)]^K \beta \rho^{el}(s,t), \quad (2.1)$$

$$\bar{\sigma}(s) = [\log(s)]^{K-1} \text{Im} F_0^{el}(s). \quad (2.2)$$

Here the smallest allowed value for K will prove to be

$$K = 3 + \varepsilon, \quad (2.3)$$

$\varepsilon > 0$. The isospin crossing matrix, β , has been given in Ref. 1).

The mapping

$$\bar{\rho}'(s,t) = M(\bar{\rho}, \bar{\sigma}; s, t), \quad (2.4)$$

which will be treated in this section, is defined by

$$\rho(s,t) = \left\{ [\bar{\rho}(t,s) + v(t,s)] \frac{s}{t} + \beta [\bar{\rho}(s,t) + v(s,t)] \frac{t}{s} \right\} \cdot \{\log(s)\log(t)\}^{-K}, \quad (2.5)$$

$$\bar{d}(s,t) = \frac{1}{\pi} \int_4^\infty ds' \left[\frac{u}{s'-s} + \eta \frac{s}{s'-u} \right] \frac{\rho(t,s')}{4-s'-t}, \quad (2.6)$$

$$\rho(t) = [\bar{\sigma}(t) + u(t)] [\log(t)]^{-K+1} - \frac{1}{t-4} \int_{4-t}^0 ds \bar{d}(s,t), \quad (2.7)$$

$$d(s,t) = \bar{d}(s,t) + \rho(t), \quad (2.8)$$

$$\begin{aligned} \bar{\rho}_I'(s, t) = & \gamma_I^{M, N} h(s) \frac{s}{t} [\log(s) \log(t)]^K \\ & \cdot \iint dt_1 dt_2 K(s; t, t_1, t_2) d_M^*(s, t_1) d_N(s, t_2). \end{aligned} \quad (2.9)$$

The notations have already been explained in Refs. 1)-4), except for the term $u(t)$ in Eq. (2.7). This is the inelastic contribution to the S -wave unitarity relation, as will be seen more explicitly in the next section.

The "strip cut-off function", $h(s)$ in Eq. (2.9), is to be unity in the elastic strip, $4 \leq s \leq 16$, zero for $s \geq 20$, and is to satisfy the following Hölder condition for $16 \leq s \leq 20$:

$$|h(s_1) - h(s_2)| \leq |s_1 - s_2|^\mu. \quad (2.10)$$

The Hölder index must lie in the range

$$0 < \mu < \frac{1}{2}. \quad (2.11)$$

The proof can be made to work with a less drastic, Hölder-continuous cut-off than this, and, in a phenomenological application, a more gradual behaviour would certainly be preferable.

Let $\mathcal{B}$ be the Banach space of all real functions, $F(s, t)$ with domain $4 \leq s, t < \infty$, for which there exists the norm

$$\|F\| = \sup |F(s, t)| + \sup \frac{|F(s_1, t_1) - F(s_2, t_2)|}{\left| \frac{s_1 - s_2}{s_1 s_2} \right|^\mu + \left| \frac{t_1 - t_2}{t_1 t_2} \right|^\mu}, \quad (2.12)$$

where the suprema are taken over $[4, \infty)$ for all variables.

The generating function, $v(s, t)$, is required to belong to the intersection of the ball

$$\|v\| \leq B, \quad (2.13)$$

in $\mathcal{B}$, and the cone defined by

$$v(s, t) = 0, \quad (2.14)$$

for $s \leq 16$, and for $t \leq 16$.

The function $\overline{\rho}(s, t)$ is to belong to the ball

$$\|\overline{\rho}\| \leq b, \quad (2.15)$$

and the cone

$$\overline{\rho}(s, t) = 0, \quad (2.16)$$

for $t \leq 16s/(s-4)$, $s \geq 4$.

The functions $\overline{\sigma}(t)$ and $u(t)$, that occur in Eq. (2.7), will be required to belong to certain balls in a Banach space, $\mathcal{C}$, that will be defined in the next two sections. This will guarantee the following bound:

$$|\overline{\sigma}(t)| + |u(t)| \leq c + C, \quad (2.17)$$

for all $t \geq 4$, which is enough for the needs of this section.

From the definition (2.5) one has immediately

$$|\rho(s, t)| \leq (B+b) \left[\frac{s}{t} + 3 \frac{t}{s} \right] \{ \log(s) \log(t) \}^{-K}. \quad (2.18)$$

This bound would correspond to Martin's "maximal" positive spectral function⁹⁾, if one were to take $K = 1 + \epsilon$, $\epsilon > 0$, apart from the factor of 3, which comes from the isospin crossing matrix.

Martin showed that one has positive and bounded absorptive parts of the partial wave amplitudes, with an Ansatz like (2.18). It turns out that these absorptive parts behave in general like $[\log s]^{-K+1}$, and that the real parts behave like $[\log s]^{-K+2}$, as $s \rightarrow \infty$. For the real parts to be bounded as well, one would need $K = 2 + \epsilon$; but, in fact, this is still not good enough for the proof of inelastic unitarity, which requires that the square of the real part should not be greater asymptotically than the imaginary part. This latter requirement is met if $K = 3 + \epsilon$.

It also follows from Eq. (2.5) that

$$|\rho(s_1, t) - \rho(s_2, t)| \leq (B+b) \left[\frac{s_M}{t} + 3 \frac{t}{s_m} \right] \left| \frac{s_1 - s_2}{s_1 s_2} \right|^\mu \{ \log(s_m) \log(t) \}^{-K}, \quad (2.19)$$

where $s_M = \max(s_1, s_2)$ and $s_m = \min(s_1, s_2)$. One can then prove, from Eq. (2.6) that

$$|\bar{d}(s, t)| \leq O(\psi) \left\{ \frac{s}{t} [\log(s)]^{-K+1} + \frac{t}{s} \right\} [\log(t)]^{-K}, \quad (2.20)$$

where $\psi = \max(b, B)$, and where $O(\psi)$ will be used simply to mean ψ , multiplied by a constant that depends only on μ and K . This usage obviates the need to find a different label for each new constant that appears. Hence

$$|\rho(t)| \leq O(\psi) [\log(t)]^{-K+1}, \quad (2.21)$$

if one redefines ψ as

$$\psi = \max(b, B, c, C). \quad (2.22)$$

One can also show that

$$|d(s_1, t) - d(s_2, t)| \leq O(\psi) \left[\frac{s_M}{t} + \frac{t}{s_m} \right] \cdot \left| \frac{s_1 - s_2}{s_1 s_2} \right|^\mu [\log(t)]^{-K}, \quad (2.23)$$

and Eqs. (2.20)-(2.23) suffice for the demonstration that

$$\|\bar{\rho}'\| \leq O(\psi^2). \quad (2.24)$$

In the proof of this inequality, one does not need to worry about the s -dependence, because of the cut-off, $h(s)$; only the t -dependence is crucial.

3. CDD CLASS ZERO

In this section, the Mandelstam mapping, (2.4), will be combined with an S -wave mapping

$$\bar{\sigma}' = S(\bar{\rho}, \bar{\sigma}), \quad (3.1)$$

which is defined as follows:

$$\rho(s) = [\bar{\sigma}(s) + u(s)] [\log(s)]^{-K+1} - \frac{1}{s-4} \int_{4-s}^0 dt \bar{d}(t, s), \quad (3.2)$$

$$A(t, u) = \frac{1}{\pi} \int_4^\infty ds' \frac{\rho(s')}{s'-s} + \beta \gamma \frac{s}{\pi^2} \int_4^\infty dt' \int_4^\infty du' \frac{\rho(t', u')}{(t'-t)(u'-u)(4-t'-u')}, \quad (3.3)$$

$$F(s, t) = A(t, u) + \beta A(s, u) + \gamma \beta \gamma A(t, s), \quad (3.4)$$

$$\bar{\sigma}'(s) = [\log(s)]^{K-1} \left(\frac{s-4}{s} \right)^{1/2} \left| \frac{1}{s-4} \int_{4-s}^0 dt F(s, t) \right|^2. \quad (3.5)$$

An inelastic generating function, $u(s)$, which is supposed given, has been added [in Eq. (3.2)].

The problem is to find a fixed-point of the double mapping

$$(\bar{\rho}, \bar{\sigma}) \xrightarrow{M} \bar{\rho}', \quad (3.6)$$

$$(\bar{\rho}, \bar{\sigma}) \xrightarrow{S} \bar{\sigma}', \quad (3.7)$$

at which point the total amplitude will be $F(s, t)$ given by Eq. (3.4).

Let $\mathcal{C}$ be the space of all functions, $F(s)$, defined on the domain $4 \leq s < \infty$, for which there exists the norm

$$\|F\| = \sup |F(s)| + \sup \frac{|F(s_1) - F(s_2)|}{\left| \frac{s_1 - s_2}{s_1 s_2} \right|^\mu}, \quad (3.8)$$

where the suprema are taken over the domain $[4, \infty)$. This is, of course, the one-variable analogue of the norm (2.12).

The known function, $u(s)$, is required to belong to the ball

$$\|u\| \leq C, \quad (3.9)$$

in $\mathcal{C}$, and the cone

$$u(s) = 0, \quad (3.10)$$

for $s \geq 16$. The function $\bar{\sigma}(s)$ is to belong to the ball

$$\|\bar{\sigma}\| \leq c, \quad (3.11)$$

in $\mathcal{C}$.

One can show that

$$|A(t, u)| \leq O(\psi) \left\{ [\log(s)]^{2-\kappa} + \left(1 + \frac{u}{t}\right) [\log(u)]^{1-\kappa} + \left(1 + \frac{t}{u}\right) [\log(t)]^{1-\kappa} \right\}, \quad (3.12)$$

and hence

$$|\bar{\sigma}'(s)| \leq O(\psi^2). \quad (3.13)$$

In a similar way, it is possible to prove that $\bar{\sigma}'(s)$ is Hölder-continuous, and one finds finally that

$$\|\bar{\sigma}'\| \leq O(\psi^2). \quad (3.14)$$

Because of the quadratic nature of the inequalities (2.24) and (3.14), it is clear that one can arrange, by choosing ψ small enough, that the balls (2.15) and (3.11) are mapped into themselves by M and S .

To finish the proof of the contraction mapping, one must take any two functions, $\bar{\rho}_1$ and $\bar{\rho}_2$, that belong to the ball (2.15), and any two functions, $\bar{\sigma}_1$ and $\bar{\sigma}_2$, that belong to the ball (3.11). One then proves that

$$\|\bar{\rho}_1' - \bar{\rho}_2'\| \leq O(\psi)\|\bar{\rho}_1 - \bar{\rho}_2\| + O(\psi)\|\bar{\sigma}_1 - \bar{\sigma}_2\|, \quad (3.15)$$

and

$$\|\bar{\sigma}_1' - \bar{\sigma}_2'\| \leq O(\psi)\|\bar{\rho}_1 - \bar{\rho}_2\| + O(\psi)\|\bar{\sigma}_1 - \bar{\sigma}_2\|. \quad (3.16)$$

Define the direct product $\mathcal{B} \times \mathcal{C}$, with the norm

$$\|(\bar{\rho}, \bar{\sigma})\| \equiv \max \{ \|\bar{\rho}\|, \|\bar{\sigma}\| \}, \quad (3.17)$$

as in Ref. 4). Then, Eqs. (3.15) and (3.16) mean that

$$\|(\bar{\rho}_1', \bar{\sigma}_1') - (\bar{\rho}_2', \bar{\sigma}_2')\| \leq O(\psi)\|(\bar{\rho}_1, \bar{\sigma}_1) - (\bar{\rho}_2, \bar{\sigma}_2)\|. \quad (3.18)$$

Thus one has a contraction mapping, if ψ is small enough, so that there is then a locally unique solution of the equations (2.4) and (3.1).

4. THE INELASTIC CONSTRAINTS

In general, the partial wave projections of the amplitude

$$F_\ell(s) = \frac{1}{s-4} \int_{4-s}^0 dt P_\ell\left(1 + \frac{2t}{s-4}\right) F(s,t), \quad (4.1)$$

$\ell=0,1,2,\dots$, could violate the inelastic constraints

$$\text{Im} F_\ell(s) \geq \left(\frac{s-4}{s}\right)^{1/2} |F_\ell(s)|^2, \quad (4.2)$$

for $s \gg 16$. It will now be shown that the input functions, $v(s,t)$ and $u(s)$, can be chosen in such a way that inequality (4.2) is safe for $\ell=1,2,3,\dots$. For the S wave, it will be automatic, so long as $u(s)$ is positive and ψ is small enough.

The basic method, as in Refs. 1), 4), is to add certain extra "positivity" requirements to the specification of the sets to which $v(s,t)$ and $u(s)$ must belong. These conditions are as follows:

$$u(s) \geq \gamma C \left(\frac{s-16}{s}\right)^\mu, \quad (4.3)$$

for $s \gg 16$, where γ satisfies

$$0 < \gamma \leq 1. \quad (4.4)$$

In order to allow for the possibility that $v(s,t)$ oscillates, it is convenient to write

$$v(s,t) = v_1(s,t) + v_2(s,t) \quad (4.5)$$

as in Ref. 1), (third paper), and to impose different conditions on v_1 and v_2 . The function $v_1(s,t)$ is to control the positivity of the spectral function in the elastic strip, and must satisfy

$$\frac{t}{s} v_1(s, t) + \beta \frac{s}{t} v_1(t, s) \geq \gamma B \left\{ \frac{[s - \frac{16t}{t-4}][t - \frac{16s}{s-4}]}{st} \right\}^\mu \cdot \left\{ \frac{t}{s} + \frac{s}{t} \right\} \quad (4.6)$$

for $s \geq 16t/(t-4)$, $t \geq 20$, and for $t \geq 16s/(s-4)$, $s \geq 20$. Furthermore, let $v_1(s, t)$ be such that

$$g(s, t) \geq \gamma B (t-16)^\mu t^{1-\mu} [\log(t)]^{-\kappa}, \quad (4.7)$$

and

$$\beta g(s, t) \geq \gamma B (t-16)^\mu t^{1-\mu} [\log(t)]^{-\kappa}, \quad (4.8)$$

for $4 \leq s \leq 20$ and $16 < t < \infty$, where

$$g(s, t) = \frac{p}{\pi} \int_{16}^{\infty} ds' \left[\frac{u}{s'-s} + \eta \frac{s}{s'-u} - \frac{1+\eta}{t-4} \log\left(1 + \frac{t-4}{s'}\right) \right] \frac{[\log(s') \log(t)]^{-\kappa}}{4 - s' - t} \left[\frac{t}{s'} v_1(s', t) + \beta \frac{s'}{t} v_1(t, s') \right]. \quad (4.9)$$

The function $v_2(s, t)$ may oscillate, but if one defines

$$v_2^{\ell}(t) \equiv \frac{1}{s-4} \int_{16}^{\infty} ds' Q_{\ell} \left(1 + \frac{2s'}{t-4} \right) \frac{[\log(s') \log(t)]^{-\kappa}}{4 - s' - t} \cdot \left[\frac{t}{s'} v_2(s', t) + \beta \frac{s'}{t} v_2(t, s') \right], \quad (4.10)$$

for $\ell = 1, 2, 3, \dots$, then one requires

$$v_2^{\ell}(t) \geq 0, \quad (4.11)$$

and

$$\beta v_2^{\ell}(t) \geq 0, \quad (4.12)$$

for $t \geq 16$ and $\ell = 1, 2, 3, \dots$. By summing the series

$$\sum_{\ell=1}^{\infty} (2\ell+1) P_{\ell} \left(1 + \frac{2s}{t-4}\right) v_2^{\ell}(t), \quad (4.13)$$

with the help of the Heine expansion, one easily sees from Eqs. (4.10)-(4.12) that $v_2(s, t)$ also satisfies conditions of the form (4.7)-(4.9), but with v_2 replacing v_1 .

By the method of Ref. 1), (second paper, Section 3) one can now show that, if $\bar{\rho}(s, t)$ is required to belong to the cone

$$\bar{\rho}(s, t) \geq 0, \quad (4.14)$$

for $4 \leq s \leq 20$, $t \geq 20$, as well as to the set defined by Eqs. (2.15) and (2.16), then $\bar{\rho}'(s, t)$, the image of $\bar{\rho}(s, t)$, under the mapping M , can be made to belong to the same cone. Hence the fixed point also satisfies condition (4.14). By combining this result with the requirement (4.6), one sees immediately that, by choosing B small enough, one can arrange

$$\bar{\rho}(t, s) \frac{s}{t} + \frac{1}{2} \left\{ v_1(t, s) \frac{s}{t} + \beta v_1(s, t) \frac{t}{s} \right\} \geq 0 \quad (4.15)$$

everywhere. Now, one has, for $\ell=1, 2, 3, \dots$,

$$\begin{aligned} \operatorname{Im} F_{\ell}(s) - \left(\frac{s-4}{s}\right)^{1/2} |F_{\ell}(s)|^2 &= \beta v_2^{\ell}(s) - [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_{\ell}(s)|^2 + \\ &+ \frac{[\log(s)]^{-K}}{s-4} \int_{16}^{\infty} dt Q_{\ell} \left(1 + \frac{2t}{s-4}\right) (\log t)^{-K} \cdot \\ &\cdot \left\{ [\bar{\rho}(t, s) + v_1(t, s)] \frac{s}{t} + \beta v_1(s, t) \frac{t}{s} \right\}. \end{aligned} \quad (4.16)$$

Because of the positivity (4.12) and (4.15), it only remains to be shown that

$$\begin{aligned} \frac{1}{2} \frac{(\log s)^{-K}}{s-4} \int_{\frac{16s}{s-4}}^{\infty} dt Q_{\ell} \left(1 + \frac{2t}{s-4}\right) (\log t)^{-K} \left\{ v_1(t, s) \frac{s}{t} + \beta v_1(s, t) \frac{t}{s} \right\} \\ \geq [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_{\ell}(s)|^2, \end{aligned}$$

$$(4.17)$$

for $s \geq 16$. For $l=1,2,3,\dots$, one can write

$$F_\ell(s) = \frac{\beta}{s-4} \int_4^\infty dt Q_\ell\left(1 + \frac{2t}{s-4}\right) d(s,t). \quad (4.18)$$

Following Kupsch, one can divide this integral into two pieces, respectively over the ranges $4 \leq t \leq 16s/(s-4)$ and $16s/(s-4) < t < \infty$. The latter piece can be bounded, with the help of inequalities (2.20) and (2.21). The result is a term of the form

$$O(\psi) \frac{1}{s} \int_{\frac{16s}{s-4}}^\infty dt Q_\ell\left(1 + \frac{2t}{s-4}\right) \left\{ \frac{s}{t} (\log s)^{-K+1} + \frac{t}{s} + \log t \right\} (\log t)^{-K}, \quad (4.19)$$

for $s \geq 16$. In the piece of the integral from 4 to $16s/(s-4)$, one makes the following use of the Schwarz inequality:

$$\begin{aligned} \left| \frac{1}{s-4} \int_4^{16s/(s-4)} dt Q_\ell\left(1 + \frac{2t}{s-4}\right) d(s,t) \right| &\leq \left\{ \frac{1}{s-4} \int_4^{16s/(s-4)} dt |d(s,t)| \right\}^{1/2} \\ &\cdot \left\{ \frac{1}{s-4} \int_4^{16s/(s-4)} dt [Q_\ell\left(1 + \frac{2t}{s-4}\right)]^2 |d(s,t)| \right\}^{1/2}. \end{aligned} \quad (4.20)$$

In the second factor on the right of Eq. (4.20), one inserts the inequality

$$[Q_\ell\left(1 + \frac{2t}{s-4}\right)]^2 \leq \frac{3}{2} \left(1 + \frac{2t}{s-4}\right) \log\left(1 + \frac{s-4}{t}\right) Q_\ell\left(1 + \frac{8t}{s-4} + \frac{8t^2}{(s-4)^2}\right), \quad (4.21)$$

for $l \geq 1$, $t \geq 4$, $s \geq 4$, which was proved by Kupsch in Ref. 1), (first paper, Appendix B), and also changes the integration variable to

$$t' = 4t + \frac{4t^2}{s-4}. \quad (4.22)$$

The interval $4 \leq t \leq 16s/(s-4)$, for $s \geq 16$, maps into a subset of $16s/(s-4) \leq t' \leq 6400/27$, and one finds finally that (4.20) is bounded by the square-root of the expression (4.19), multiplied by $\psi^{\frac{1}{2}}$. From

$$|\int_4^\infty|^2 \leq |\int_4^{16s/(s-4)}|^2 + 2|\int_4^{16s/(s-4)}| |\int_{16s/(s-4)}^\infty| + |\int_{16s/(s-4)}^\infty|^2, \quad (4.23)$$

and the easily provable fact that the expression (4.19) is bounded by

$$O(\psi) (\log s)^{-K+2} \quad (4.24)$$

for all $\ell=1,2,3,\dots$, one finds finally that

$$|F_\ell(s)|^2 \leq O(\psi^2) \frac{(\log s)^{-K+2}}{s} \int_{16s/(s-4)}^\infty dt Q_\ell(1 + \frac{2t}{s-4}) (\log t)^{-K} \cdot \left\{ \frac{s}{t} (\log s)^{-K+1} + \frac{t}{s} + \log t \right\}. \quad (4.25)$$

This is still not in a suitable form for the proof of inequality (4.17), because of the two terms t/s and $\log t$ within the parentheses $\{\}$. However, it is not hard to show that

$$\int_{s(\log s)^{-K}}^\infty dt \xi_\ell(s, t) \leq C \int_{16s/(s-4)}^{s(\log s)^{-K}} dt \xi_\ell(s, t), \quad (4.26)$$

for $s \gg 16$, $\ell=1,2,3,\dots$, where C is a constant, and where

$$\xi_\ell(s, t) = Q_\ell(1 + \frac{2t}{s-4}) \left\{ \frac{t}{s} + \log t \right\} (\log t)^{-K}. \quad (4.27)$$

Moreover, for $t \leq s(\log s)^{-K}$, one certainly has

$$\frac{t}{s} + \log t \leq \frac{2s}{t} (\log s)^{-K+1}. \quad (4.28)$$

Hence, inequality (4.25) can be coarsened to

$$|F_\ell(s)|^2 \leq O(\psi^2) \frac{(\log s)^{-2K+3}}{s} \int_{16s/(s-4)}^\infty dt Q_\ell(1 + \frac{2t}{s-4}) \frac{s}{t}, \quad (4.29)$$

and this suffices for the proof of Eq. (4.17), since $-2K+3 < -K$, and $v(s,t)$ satisfies (4.6). Note that the factor $[1-h(s)]$ in Eq. (4.17) ensures that this inequality can be maintained right down to $s=16$.

5. CDD SECTOR

The purpose in this section is to generalize the work of Section 3, in order to allow the possibility of CDD poles. This will be done in such a way that the proof of the last section is still applicable. For simplicity, only one CDD pole will be inserted in the S wave. The Mandelstam mapping, M , of Section 2, is now to be combined with a mapping, N , that involves an N/D decomposition of the S wave, as in Ref. 4),

$$\bar{\sigma}' = N(\bar{\sigma}, \bar{\sigma}). \quad (5.1)$$

The form of N is a little different from that given in Ref. 4), since it will be written for $\bar{\sigma}$, the elastic part of the S-wave absorptive part, rather than for the whole absorptive part. Since the inelastic N/D equations will be written in the Frye-Warnock form¹⁰⁾, it is more convenient to work with the S-wave elasticity function, $\eta(s)$, as input function, rather than the $u(s)$ of Sections 2 and 3. The connection is

$$4q(s)u(s) = 1 - [\eta(s)]^2. \quad (5.2)$$

One first defines the left-hand cut discontinuity by

$$\Delta(s) = \frac{2\beta}{4-s} \int_4^{4-s} dt \operatorname{Re} d(s,t). \quad (5.3)$$

Then, the function, $n(s)$, is defined as the solution (which will be proved to exist) of the linear integral equation

$$\begin{aligned} \eta(s) n(s) = & B(s) + c_1 \frac{B(s) - B(s_1)}{s - s_1} + \\ & + \frac{1}{\pi} \int_4^\infty ds' \frac{B(s) - B(s')}{s - s'} q(s') n(s'), \end{aligned} \quad (5.4)$$

where

$$B(s) = B_R(s) + B_L(s), \quad (5.5)$$

$$B_R(s) = \frac{P}{\pi} \int_{16}^\infty \frac{ds'}{s' - s} \frac{1 - \eta(s')}{2 q(s')}, \quad (5.6)$$

$$B_L(s) = \frac{1}{\pi} \int_{-\infty}^0 \frac{ds'}{s' - s} \Delta(s'). \quad (5.7)$$

In Eqs. (5.4) and (5.6)

$$q(s) = \left(\frac{s-4}{s} \right)^{\frac{1}{2}}. \quad (5.8)$$

In Eq. (5.4), c_1 and s_1 are respectively the residue and position of the CDD pole. It will be convenient to take s_1 real and greater than 4, and it will prove necessary for c_1 to have a pre-determined sign, and not to be too large [see below, and Ref. 4)].

One next defines

$$Re D(s) = 1 + \frac{c_1}{s - s_1} - \frac{P}{\pi} \int_4^\infty \frac{ds'}{s' - s} q(s') n(s'), \quad (5.9)$$

and finally

$$\bar{\sigma}'(s) = \frac{1}{4q(s)} \left[\frac{\{[1-\eta(s)] \operatorname{Re} D(s)\}^2 + \{[1+\eta(s)] q(s) n(s)\}^2}{[\operatorname{Re} D(s)]^2 + [q(s) n(s)]^2} - \right. \\ \left. - 1 + [\eta(s)]^2 \right] (\log s)^{K-1}. \quad (5.10)$$

The analysis of these equations follows that of Ref. 4), Section 3, very closely, and so it will suffice to summarize the method, and to point out the differences.

The first step is to analyze the integral equation (5.4) in the Banach space, $\mathcal{D}$, of all real, continuous functions, $x(s)$, on $[4, \infty)$, with the norm

$$\|x\|_{\mathcal{D}} = \sup |(\log s)^{K-2} x(s)|. \quad (5.11)$$

One shows that Eq. (5.4), considered as a mapping of $\mathcal{D}$ into itself, is completely continuous in the norm topology, and that the norm of the kernel function is proportional to ψ . To do this, one needs to impose the restriction that $1-\eta(s)$ belongs to $\mathcal{C}^1$ [see Eq. (3.8)], that

$$\|1-\eta\|_{\mathcal{C}^1} = O(\psi), \quad (5.12)$$

and, more than this, that $\eta(s)$ be twice differentiable, with

$$\eta(\infty) = 1, \\ \eta'(\infty) = 0 = \eta''(\infty), \quad (5.13)$$

and

$$|\eta''(s_1) - \eta''(s_2)| \leq O(\psi) \left| \frac{s_1 - s_2}{s_1 s_2} \right|^m s_m^{-2}. \quad (5.14)$$

One can then show that, for small enough ψ , Eq. (5.4) has a unique solution, $n(s)$, in $\mathcal{D}$, with a norm of order ψ .

As in Ref. 4), one now shows that $\text{Re } D(s)$ is well defined by Eq. (5.9), and that $|D(s)|$ can have no zeros on the whole physical sheet, if c_1 has the correct sign, and is small enough. One then demonstrates that $\bar{\sigma}'(s)$, defined by Eq. (5.10), belongs to a ball of radius $O(\psi^2)$ in $\mathcal{C}$, so that N maps the balls (2.15) and (3.11) into (3.11), for ψ small enough.

Finally, one shows that, for any $\bar{\rho}_1, \bar{\rho}_2$ belonging to the ball (2.15), and for any $\bar{\sigma}_1, \bar{\sigma}_2$ belonging to the ball (3.11), one has

$$\begin{aligned} \|N(\bar{\rho}_1, \bar{\sigma}_1) - N(\bar{\rho}_2, \bar{\sigma}_2)\|_{\mathcal{C}} \leq O(\psi) \{ \|\bar{\rho}_1 - \bar{\rho}_2\|_{\mathcal{B}} + \\ + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{\mathcal{C}} \}, \end{aligned} \quad (5.15)$$

which suffices for the proof of the contraction condition, for small enough ψ , just as in Section 3.

6. GENERATING FUNCTIONS

In this section, a programme will be outlined for the generalization of the work of Sections 2.5, to allow the possibility of an amplitude that would need an n -times subtracted Mandelstam representation. The case $n = \infty$, corresponding to an indefinitely rising Regge trajectory, is not excluded.

The method derives directly from that of Kupsch²⁾. One adds to the once-subtracted Mandelstam representation, Eq. (3.4), the extra generating function, $G(s, t)$ in addition to the quantities $v(s, t)$ and $u(s)$. Call the new amplitude $\hat{F}$, with

$$\hat{F}(s,t) = F(s,t) + G(s,t). \quad (6.1)$$

The function $G(s,t)$ is to be holomorphic in s, t , cut along $[16, \infty)$ in s, t and $u=4-s-t$; that is, it has Mandelstam analyticity with only the inelastic cuts. It is to satisfy crossing symmetry:

$$G(s,t) = \beta G(t,s) = \eta G(s,u), \quad (6.2)$$

and is to be polynomially bounded in t at fixed s , although this bound need not be uniform with respect to s . However, it is necessary that

$$|G(s,t)| \leq g t (\log t)^{-\kappa+1}, \quad (6.3)$$

for $4 \leq s \leq 20$, and all $t \geq 4$, where g is constant, and that the t discontinuity of G be Hölder-continuous with respect to s , and satisfy the bound (for the same range of s)

$$|[G(s,t)]_t| \leq g t (\log t)^{-\kappa}. \quad (6.4)$$

This means that the leading angular-momentum singularity must stay below unity, in the elastic strip.

One adds the t discontinuity of βG to Eq. (2.6):

$$\hat{d}(s,t) = \bar{d}(s,t) + \beta [G(s,t)]_t. \quad (6.5)$$

It is convenient to generalize the definition (2.22) to

$$\psi = \max(b, B, c, C, g), \quad (6.6)$$

and then there is no difficulty to show that the double contraction condition holds as before, for ψ small enough.

Positivity conditions have to be imposed on $G(s,t)$ to ensure that the inelastic inequalities still hold. These are firstly, that

$$[G(s,t)]_t \geq 0, \quad (6.7)$$

and

$$\beta [G(s,t)]_t \geq 0, \quad (6.8)$$

for $4 \leq s \leq 20$, $t > 16$. Secondly, the partial wave projections of G , namely

$$G_\ell(s) = \frac{1}{s-4} \int_{4-s}^0 dt P_\ell\left(1 + \frac{2t}{s-4}\right) G(s,t), \quad (6.9)$$

should satisfy

$$\text{Im } G_\ell(s) \geq 4 [1 - h(s)] |G_\ell(s)|^2, \quad (6.10)$$

for $s \gg 16$, $\ell = 0, 1, 2, \dots$. One then finds, for $\ell = 1, 2, 3, \dots$

$$\begin{aligned} \text{Im } \hat{F}_\ell(s) - \left(\frac{s-4}{s}\right)^{1/2} |\hat{F}_\ell(s)|^2 &= \beta v_2^\ell(s) + \text{Im } G_\ell(s) - \\ &\quad - [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_\ell(s) + G_\ell(s)|^2 + \\ &\quad + \frac{(\log s)^{-k}}{s-4} \int_{\frac{16s}{s-4}}^{\infty} dt Q_\ell\left(1 + \frac{2t}{s-4}\right) \cdot \\ &\quad \cdot \left\{ [\bar{\rho}(t,s) + v_1(t,s)] \frac{s}{t} + \beta v_1(s,t) \frac{t}{s} \right\}. \end{aligned} \quad (6.11)$$

One can still prove inequality (4.15), but one has now to deal with the term involving $|F_\ell(s) + G_\ell(s)|^2$. Consider each of two possibilities in turn:

a) those values of ℓ and s for which

$$|F_\ell(s)| \leq |G_\ell(s)|. \quad (6.12)$$

Here

$$\begin{aligned} \operatorname{Im} G_\ell(s) - [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_\ell(s) + G_\ell(s)|^2 \\ \geq \operatorname{Im} G_\ell(s) - 4 [1 - h(s)] |G_\ell(s)|^2, \end{aligned} \quad (6.13)$$

and this is non-negative by condition (6.10). The rest of Eq. (6.11) is also non-negative;

b) those values of ℓ and s for which, on the contrary,

$$|F_\ell(s)| > |G_\ell(s)|. \quad (6.14)$$

Here, the term on the left-hand side of (6.13) is minimized instead by

$$\operatorname{Im} G_\ell(s) - 4 [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_\ell(s)|^2. \quad (6.15)$$

Now $\operatorname{Im} G_\ell(s)$ is non-negative, and one can show that

$$\begin{aligned} \frac{1}{2} \frac{(\log s)^{-K}}{s-4} \int_{\frac{16s}{s-4}}^{\infty} dt Q_\ell \left(1 + \frac{2t}{s-4}\right) (\log t)^{-K} \left\{ v_1(t, s) \frac{s}{t} + \beta v_1(s, t) \frac{t}{s} \right\} \\ \geq 4 [1 - h(s)] \left(\frac{s-4}{s}\right)^{1/2} |F_\ell(s)|^2, \end{aligned} \quad (6.16)$$

exactly as in Section 4, following Eq. (4.17), since the extra

factor of 4 makes no difference to the argument. This completes the proof of the positivity of the left-hand side of (6.11) for $l=1,2,3,\dots$. The positivity condition for the S wave is immediate, given the positivity of $\text{Im } G_0(s)$.

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