CM-P00058280

Archives

Ref.TH.1057-CERN

THE GENERALIZED VENEZIANO MODEL

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A B S T R A C T

The generalization of the Veneziano model to many-particle reactions is reviewed with a special emphasis on the earlier literature. The latest developments are also briefly indicated.

*) Review paper presented at the Royal Society Meeting on 5 June 1969 on "Duality-Reggeons and Resonances in Elementary Particle Processes".

Ref.TH.1057-CERN

21 July 1969

The successful applications by Lovelace and others ¹⁾ of the Veneziano model ²⁾ to two-body collisions have naturally led to attempts in its generalization to many-particle reactions. For certain idealized cases, it was soon found that such generalizations do indeed exist for any number of external lines. These amplitudes not only possess all the desirable properties of the original Veneziano model ²⁾, such as "duality" ³⁾, crossing symmetry, and Regge asymptotic behaviour, but have the added attractiveness of being consistent with the bootstrap hypothesis in treating all particles as bound states of others. In fact, this generalized model is formally so attractive that some authors are led to the belief that it may be more than just phenomenology for which it was originally intended, but perhaps even the beginning of a new theory of strong interactions.

This is hardly an appropriate moment for a critical review of the subject. Both phenomenological applications and attempts at building a theory are at an embryonic stage, and even our understanding of the model's internal consistency is incomplete. Moreover, the rapid development in recent weeks would make any attempt at a comprehensive summary quickly obsolete. My account, therefore, will be purely descriptive, concentrating on the earlier literature which unfortunately is rather scattered and confused. For the later developments, I shall content myself with merely indicating the directions they have taken.

1) - THE GENERALIZED BETA FUNCTIONS

We consider first an idealized system with only one family of neutral bosons, where all particles lie on the same Regge trajectory (or on its daughters), the lowest member of which has spin-parity $J^P = 0^+$. Our trajectory has thus a negative intercept : $\alpha(0) < 0$.

The scattering amplitude for the reaction :

$$0^+ + 0^+ \rightarrow 0^+ + 0^+ \quad (1)$$

in the Veneziano model ²⁾ is given by

$$T = V(1, 2, 3, 4) + V(1, 2, 4, 3) + V(1, 3, 2, 4) \quad (2)$$

where

$$V(1, 2, 3, 4) = B_4(-1 - \alpha_{12}, -1 - \alpha_{23}) \quad (3)$$

$$\alpha_{12} = \alpha_{34} = \alpha_0 + \alpha' S_{12}; \quad S_{12} = (p_1 + p_2)^2 \quad (4)$$

$$B_4(x, y) = \int_0^1 du \, u^x v^y \quad (5)$$

the variables u and v in (5) being subject to the constraint

$$u = 1 - v; \quad v = 1 - u \quad (6)$$

In what follows we shall refer to the variables u and v in (5) as being "conjugate" respectively to the variables x and y .

The properties of the amplitude (2) are well known. I shall only remind you of a few points which are particularly relevant for our later discussions :

- (i) - The beta function as defined in (5) is symmetric in x and y , which by (3) and (4) implies that $V(1, 2, 3, 4)$ is invariant under a cyclic permutation, or a reversion in ordering of the external lines. Thus,

$$V(1, 2, 3, 4) = V(2, 3, 4, 1) = V(4, 3, 2, 1) \quad (7)$$

Henceforth, we shall regard all orderings related either by cyclic permutations or reversions to be equivalent. The three terms in (2) correspond to the three possible non-equivalent orderings of the external lines. The invariant properties (7) of B_4 thus imply that the amplitude (2) is completely crossing symmetric.

- (ii) - The function B_4 is analytic apart from a sequence of poles at negative integers of x (hence also y by symmetry). The poles in x occur in the integration region near $u = 0$, and can be best studied by expanding the integrand in (5) in a Taylor series about $u = 0$. Integrating the series term by term, one sees that the residue at $x = -n$ is given by

$$\text{Res. } B_4(x, y) = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial u^{n-1}} \left[(1-u)^4 \right]_{u=0} \quad (8)$$

$x = -n$

- (iii) - There are no coincident poles in x and y . Such double poles would only occur when both u and v are zero in the integrand (5). This is impossible, however, under the constraint (6), which implies $v = 1$ when $u = 0$.

The extension of the Veneziano model to the N point 4)-11) function with spinless external lines is a straightforward generalization of the equations (2)-(6). In analogy to (2), one writes first

$$T = \sum V(1, 2, 3, \dots, N) \quad (9)$$

where the sum runs over all non-equivalent orderings of the external lines, there being altogether then $(N-1)!/2$ terms. We need consider for the moment only one such term, say that corresponding to the ordering $(1,2,3,\dots,N)$. This is expected to be invariant under any cyclic permutations or a reversal in ordering of the external lines, so that complete crossing symmetry of (9) is guaranteed.

Moreover, one requires that V should contain poles corresponding to the trajectory α in all possible Mandelstam channels which can be formed in the diagram of Fig. 1 without changing the order of the external lines. These channels are best enumerated by means of the so-called "dual" diagrams first introduced in perturbation theory¹²⁾. To the diagram of Fig. 1, we associate an N sided polygon, as shown in Fig. 2. Possible Mandelstam channels for Fig. 1 are then in one-one correspondence with diagonal lines joining any two vertices of the polygon. Thus, for example, the diagonal shown in Fig. 2 (henceforth denoted by the indices $1,3$) correspond to the channel shown in Fig. 3. To each diagonal $P = (i,j)$, we associate then a dynamical variable

$$x_P = x_{i,j} = -1 - \alpha_{i,j} \quad (10)$$

where

$$\begin{aligned} \alpha_{i,j} &= \alpha_0 + \alpha' S_{i,j} \\ S_{i,j} &= (p_i + p_{i+1} + \dots + p_j)^2 \end{aligned} \quad (11)$$

Furthermore, with the generalisation of the integral representation (5) in mind, we shall introduce for each $P = (i,j)$ a variable u_P conjugate to x_P .

Two Mandelstam channels are said to be dual to each other if they correspond to diagonals which intersect on the "dual" diagram. This is just a generalization of the usual concept of duality in the four-point function between, say, the s and t channels³⁾, which are clearly dual also in the sense defined above.

Clearly, dual channels cannot have coincident poles since no Feynman tree diagram exists with both trajectories as internal lines ; whereas, between non-dual channels, coincident poles are indeed possible. To ensure this, one writes as the generalization of (6) the following set of constraint equations,

$$u_P = 1 - \prod_{\bar{P}} u_{\bar{P}} \quad (12)$$

where $\bar{P}$ runs over all channels dual to P . It is clear then that two variables corresponding to dual channels cannot vanish simultaneously. Although (12) represents as many equations as there are variables, not all of them are independent. Indeed, it can be shown explicitly that the whole set can be solved in terms of $N-3$ independent variables. Now, by definition, independent variables can vanish simultaneously. They must therefore correspond to allowed coincident poles, or equivalently, to non-intersecting lines on the "dual" diagram. Although, in principle, any set of $N-3$ non-dual variables can be chosen as independent, it happens that the most convenient are those sets corresponding to poles in a multiperipheral diagram, for example, u_{1j} ($j=2,3,\dots,N-2$) corresponding to the diagram Fig. 4a, or the associated "dual" diagram Fig. 4b. In terms of u_{1j} as independent variables, the solution of (12) yields

$$u_{p,q} = \frac{(1 - \omega_{p,q-1})(1 - \omega_{p-1,q})}{(1 - \omega_{p-1,q-1})(1 - \omega_{p,q})} \quad (13)$$

where

$$\omega_{r,s} = \mu_{1,r} \mu_{1,r+1} \cdots \mu_{1,s} \quad (14)$$

and $u_{1,1} = u_{1,N-1} = 0$, by definition.

With $u_{1,j}$ ($j=2,3,\dots,N-2$) as independent variables, the generalization of (5) can then be written as :

$$B_N(1,2,3,\dots,N) = \int_0^1 \prod_P d\mu_P \prod_P \mu_P^{x_P} \prod_{P' \neq (1,j)} \delta(\mu_{P'} - 1 + \frac{\prod_P \mu_P}{\mu_{P'}}) \quad (15)$$

where P runs over all Mandelstam channels for the ordering $(1,2,\dots,N)$, P' over all channels except $(1,j)$ and $\bar{P}'$ over all channels dual to P' . Substituting the explicit solution (14) into (15) and performing the integration over the δ functions, one has

$$B_N(1,2,3,\dots,N) = \int_0^1 \prod_{j=2}^{N-2} d\mu_{1,j} \left(\frac{1}{J_1} \right) \prod_P \mu_P^{x_P} \quad (16)$$

where

$$J_1 = \prod_{i < j} (\mu_{i,j})^{j-i-1}; \quad [i = 2, \dots, N-2; j = 3, \dots, N-1] \quad (17)$$

It can readily be shown that the generalized beta-function as defined by (15) or (16) is invariant under a cyclic permutation or a reversal in order of the indices $(1,2,3,\dots,N)$, and that it is analytic apart from poles at negative integral values of the variables x_P .

Moreover, the absence of coincident poles between dual channels is already guaranteed. The residue at the $(1+1)$ th pole in $x_{1,j}$ is given, in analogy to (8), by

$$\text{Res.}_{\alpha_{1,j} = \ell} B_N(1, 2, \dots, N) = \left[\frac{1}{\ell!} \frac{\partial^\ell}{\partial \mu_{1,j}^\ell} Q \right]_{\mu_{1,j} = 0} \quad (18)$$

where

$$Q = \int_0^1 \prod_{\substack{k=2 \\ k \neq j}}^{N-2} d\mu_{1,k} \left(\frac{1}{J_1} \right) \prod_{P' \neq (1,j)} (\mu_{P'})^{x_{P'}} \quad (19)$$

Since when $\mu_{1,j} = 0$, $\mu_{\tilde{P}} = 1$ for all $\tilde{P}$ dual to $(1,j)$, the differentiation in (18) can yield at most a polynomial of total degree ℓ in the variables $x_{\tilde{P}}$, which are momentum transfers in the channel $(1,j)$. This means, of course, that the pole at $\alpha_{1,j} = \ell$ has maximum spin ℓ , giving rise to a linear trajectory. The cyclic symmetry of B_N then implies such a trajectory for all Mandelstam channels.

In the limit when $x_{i,i+1} \rightarrow \infty$ ($i=2, \dots, N-2$) at fixed values of $x_{1,i}$ ($i=2, \dots, N-2$) and $K_j = (x_{j-1,j} x_{j,j+1}) / (x_{j-1,j+1})$, ($j=3, \dots, N-2$), namely the multi-Regge limit corresponding to Fig. 4a with 1 and N as incoming particles, it has been shown that $B_N(1, \dots, N)$ has the proper limit ¹³⁾

$$B_N \rightarrow \prod_{i=2}^{N-2} (x_{i,i+1})^{-x_{1,i}-1} G_N(x_{1,2}, \dots, x_{1,N-2}; K_3, \dots, K_{N-2}) \quad (20)$$

where the residue G_N factorizes ¹¹⁾

$$\begin{aligned}
 & G_N(x_{1,2}, \dots, x_{1,N-2}; K_3, \dots, K_{N-1}) \\
 &= \Gamma(x_{1,2}+1) V(x_{1,2}, x_{1,3}; K_3) \Gamma(x_{1,3}+1) \\
 & \quad V(x_{1,3}, x_{1,4}; K_4) \dots \Gamma(x_{1,N-3}+1) \\
 & \quad V(x_{1,N-3}, x_{1,N-2}; K_{N-2}) \Gamma(x_{1,N-2}+1).
 \end{aligned} \tag{21}$$

with

$$\begin{aligned}
 & V(x, y; K) \\
 &= \frac{1}{\Gamma(x+1)} \cdot \frac{1}{\Gamma(y+1)} \int_0^\infty d\zeta_1 d\zeta_2 \zeta_1^x \zeta_2^y \exp\left\{-\left[\zeta_1 + \zeta_2 + \frac{\zeta_1 \zeta_2}{K}\right]\right\} \tag{22}
 \end{aligned}$$

As mentioned already in the Introduction, the functions B_N have also the attractive property of being consistent with the bootstrap hypothesis. Consider, for example, the pole in $B_N(1, 2, \dots, N)$ at $\alpha_{1,j} = 0$. Since, in our present idealized system, all trajectories are the same, this pole should represent the same spinless particle as the external lines we started with. Bootstrap consistency would require that the residue at the pole factorizes into beta functions of lower orders, or explicitly,

$$\text{Res. } B_N(1, 2, \dots, N) = B_{j+1}(1, 2, \dots, j; I) B_{N-j+1}(I, j+1, \dots, N) \tag{23}$$

$\alpha_{1,j} = 0$

That this is indeed the case can be seen readily from Eq. (15) and the "dual" diagram. Since on putting $u_{1,j} = 0$ according to (18), $u_{\tilde{P}} = 1$ for all $\tilde{P}$ corresponding to lines intersecting $(1,j)$, one essentially cuts the "dual diagram" into halves corresponding to the factor of (23).

Pushing the bootstrap idea further, the same arguments as before would lead us to interpret the residue (18) at $\alpha_{1,j} = \ell$ as the amplitude for a $(j+1)$ particle process in which a particle of spin ℓ is produced, the spinning particle then decaying into $(N-j)$ spinless ones. Once this interpretation is accepted, it is clear that one would have a scheme for constructing the N point function involving any number of particles with integral spins. The scheme is entirely consistent with the bootstrap hypothesis, and will give for spinning particles also their decay correlations. By a particle with spin ℓ , however, we mean here as in the original Veneziano model, not a pure spin state with angular momentum ℓ , but a mixed state with maximum spin ℓ plus a whole sequence of degenerate daughters with angular momentum $< \ell$. As we shall see, the level structure of such a state in the present model is much more complex than expected at first sight.

2) - SATELLITES AND OTHER AMBIGUITIES

Having now found a generalization of the Veneziano formula with all the desired properties, one asks naturally the question whether the generalized formula is in some sense unique. Unfortunately, however, one knows of the answer to this question as little here as in the original Veneziano expression. It may be of some significance, though, that apart from ambiguities due to satellites¹⁾, etc., all generalizations known so far are completely equivalent¹⁶⁾.

The well-known ambiguity due to satellites in the four-point Veneziano model is best stated¹⁷⁾ in terms of an arbitrary function of u multiplying the integrand in the representation (5) of B_4 , thus

$$B_4(x, y) \rightarrow \int_0^1 du f(u) u^x (1-u)^y \quad (24)$$

As far as the physical properties, such as pole structure, duality, or asymptotic behaviour are concerned, one obtains an equally acceptable model for the amplitude so long as $f(u)$ is analytic and invariant under the transformation $u \rightarrow 1-u$. By expanding $f(u)$ in a power series, one obtains from (24) a series of beta functions with arguments shifted by integral units. These are the usual satellite terms ¹⁾.

A similar ambiguity exists in the generalized Veneziano model formulated in the previous Section. One may choose to modify the integrand in (15) by a function $f_N(u_P)$. So long as $f_N(u_P)$ is cyclic invariant and well-behaved in the region of integration, the resultant function will have the same pole structure and asymptotic behaviour. The factorization requirement (23) imposes an additional constraint on $f_N(u_P)$. However, considerable freedom still remains. The general form for f_N which satisfies (23) has been constructed by Gross ¹⁸⁾, to whose work we shall return later while discussing the level structure of the model.

Another possible ambiguity consists of adding further trajectories at the parent level, but with the first few poles missing (the so-called sister terms). This possibility has not, as far as known, been fully investigated.

Although these extra degrees of freedom may play an important role in future developments of the model, not much is known about them, and nearly all the results reported below are based on the "leading term" represented by (15) or (16).

3) - USEFUL EQUIVALENT FORMS

Several equivalent forms of the generalized beta function (15) have been suggested which are found to be useful for different purposes. We list some of these below.

A - The series form of Hopkinson and Plahte

By expanding the integrand of (16) in a power series in various ways, one can obtain B_N in terms of an infinite series of beta functions of lower orders. Such series expansions, considered in some detail by Hopkinson and Plahte⁸⁾, yield a practical iterative method for the numerical evaluation of the beta functions. In particular, based on the expansion

$$B_5(x) = \sum_{k=0}^{\infty} (-1)^k \binom{z_{14}}{k} B_4(x_{14}, x_{45} + k) B_4(x_{12} + k, x_{13}) \quad (25)$$

where

$$z_{14} = x_{2,4} - x_{3,4} - x_{2,3}$$

Hopkinson has written a computer program for evaluating B_5 over a wide complex range of all its five arguments¹⁹⁾. This program has been used successfully in the phenomenological study of five-line processes. (See, e.g., Section 6.)

B - The Bardakçi-Ruegg formula¹¹⁾

One notices from equations (13) and (14) that factors of the form $(1 - \omega_{r,s})$ occurs in various combinations in the integrand of (16). Collecting all such factors together, one obtains the following formula, which was first given by Bardakçi and Ruegg¹¹⁾ ;

$$B_N(1, 2, 3, \dots, N)$$

$$\begin{aligned}
 &= \int_0^1 d\mu_{12} d\mu_{13} \dots d\mu_{1,N-2} \mu_{12}^{x_{12}} \mu_{13}^{x_{13}} \dots \mu_{1,N-2}^{x_{1,N-2}} \\
 &\quad \times (1 - \mu_{12})^{x_{23}} (1 - \mu_{13})^{x_{34}} \dots (1 - \mu_{1,N-2})^{x_{N-2,N-1}} \\
 &\quad \times (1 - \omega_{23})^{-2\alpha'(p_2 \cdot p_4) + \alpha_0 + \alpha'} \dots (1 - \omega_{N-3,N-2})^{-2\alpha'(p_{N-3} \cdot p_{N-1}) + \alpha_0 + \alpha'} \\
 &\quad \times (1 - \omega_{24})^{-2\alpha'(p_2 \cdot p_5)} \dots (1 - \omega_{N-4,N-2})^{-2\alpha'(p_{N-4} \cdot p_{N-1})} \\
 &\quad \times \dots (1 - \omega_{2,N-2})^{-2\alpha'(p_2 \cdot p_{N-1})}
 \end{aligned} \tag{26}$$

This has the virtue of exhibiting explicitly the simple dependence on the external momenta, and is particularly useful for studying the spin and level structures of the internal trajectories. (See, e.g., Section 4).

C - The Koba-Nielsen transformations

As already mentioned in the previous Section, any set of $(N-3)$ non-dual u_p , corresponding to non-intersecting lines on the "dual" diagram, may be chosen as independent variables. The representation (15) or (16) uses one special choice, namely $u_{1,j}$ ($j=2,3,\dots,N-2$) corresponding to Figs. 4a or 4b. It is clear that many equivalent forms of B_N can be derived by using other sets of u_p as independent variables, several of which have already been considered by Koba and Nielsen ¹⁰⁾. Although such transformations are algebraically straightforward, they may be very useful in deriving hidden symmetry properties

of the generalized beta functions. Thus, for example, Hopkinson ²⁰⁾ has found that the transformation from Fig. 4 to Fig. 5 is very useful for studying the signature of the intermediate state (1,4). Whereas, using the transformation from Fig. 4 to Fig. 6, Fubini and Veneziano ²¹⁾ were able to derive Ward-like identities relating the couplings of parent and daughter trajectories. (See, e.g., Section 4.)

Another interesting transformation considered by Koba and Nielsen ²²⁾ is based on their observation that each variable u_p of Section 1 can be expressed as an anharmonic ratio of four out of N points on the unit circle where the N points correspond to the N external lines. This enables them to rewrite (16) as an integral over the unit circle, which is particularly compact and may be useful for certain formal manipulations.

D - The Fairlie-Jones formula ²³⁾

It was noticed first by these authors that the three terms in the original Veneziano formula (2) can be united in a single integral as follows :

$$T_4 = \int_{-\infty}^{\infty} |u|^{-1-\alpha_{12}} |1-u|^{-1-\alpha_{23}} du \quad (27)$$

provided the trajectories α in the three channels satisfy the constraint :

$$\alpha_{12} + \alpha_{23} + \alpha_{13} + 1 = 0 \quad (28)$$

This can be seen by writing (27) as

$$\begin{aligned} T_4 &= \int_1^{\infty} u^{-1-\alpha_{12}} (u-1)^{-1-\alpha_{23}} du \\ &+ \int_0^1 u^{-1-\alpha_{12}} (1-u)^{-1-\alpha_{23}} du \\ &+ \int_{-\infty}^0 (-u)^{-1-\alpha_{12}} (1-u)^{-1-\alpha_{23}} du \end{aligned} \quad (29)$$

The transformations $u \rightarrow \frac{1}{u}$ and $u \rightarrow \frac{u}{u-1}$ respectively in the first and last integrals then transform (29) into (2) by means of (28). The interesting thing is that they were able to obtain a similar formula for all the 12 terms in (9) for $N = 5$, which gives

$$T_5 = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt |t|^{-1-\alpha_{12}} |z|^{-1-\alpha_{45}} |1-t|^{-1-\alpha_{23}} \cdot |1-z|^{-1-\alpha_{34}} |1-tz|^{\alpha_{23} + \alpha_{34} - \alpha_{15}} \quad (30)$$

provided that the constraint

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} - \alpha_{ijk} + 1 = 0 \quad (31)$$

is satisfied for any three lines i, j, k . Moreover, it was shown that analogous representations can be written for the full amplitude T_N in (9) for general N , under conditions similar to (28) and (31).

In spite of their attractive simplicity, however, formulas such as (30) can be considered at present only as a mathematical curiosity, since the physical significance of such conditions as (28) and (31) remain nuclear²⁴⁾.

4) - LEVEL STRUCTURE

It was shown in Section 1 that B_N has poles at negative integral values of the dynamical variable x_P for every Mandelstam channel P , and that the residue at $\alpha_P \equiv -1 - x_P = \ell$ is a polynomial of total degree ℓ in the dual variables $x_{\bar{P}}$. Since in the channel P , the dual variables $x_{\bar{P}}$ appear as momentum transfers, the pole at $\alpha_P = \ell$ must represent a system with maximum spin ℓ ,

but, in general, there will also be components of lower spins. Moreover, each component with a definite spin may still contain several levels. The question then arises whether it is consistent at all to consider the pole at $\alpha_p = \ell$ as representing a finite number of single particle states, and if so, what is the spectrum or level structure. As we shall see, the spectrum is much more complex than expected and one has at present only a partial answer to the question of consistency.

If a pole represents a single level only, then unitarity requires that its residue should factorize. If it represents a finite number of single particle states, then the residue should be expressible as the sum of a finite number of factorized terms. A first requirement for consistency in our present problem, therefore, would be that the residue at $\alpha = \ell$ be expressible as a sum of factors, where the number of terms does not increase indefinitely with the number of external lines. That this requirement, at least, is satisfied has been shown by Fubini and Veneziano ²¹⁾, and by Bardakçi and Mandelstam ²⁶⁾.

Because of the cyclic symmetry of the functions B_N , it is sufficient to establish this assertion for one Mandelstam channel, say, $P = (1, j)$. Starting from the formula (26), these authors show that B_N can be written in the form

$$B_N(1, 2, \dots, N) = \int_0^1 du' \int_0^1 du'' \varphi'(u', p') \varphi''(u'', p'') \int_0^1 du_{1,j} F(u_{1,j}; u', p'; u'', p'') (u_{1,j})^{x_{1,j}} \quad (32)$$

where u' and p' refer to the variables to the left of the line $(1, j)$ in Fig. 4a, namely: $u_{1,k'}$ ($k' = 2, \dots, j-1$) and $p_{i'}$ ($i' = 1, \dots, j$), while u'' and p'' refer to the variables to the right of $(1, j)$, namely: $u_{1,k''}$ ($k'' = j+1, \dots, N-2$) and $p_{i''}$ ($i'' = j, \dots, N$). Moreover, they were able to show that the function F in (32) can be expressed in the form:

$$F = \exp \left[\sum_{n=1}^{\infty} \frac{(u_{1,j})^n}{n} G_n \right] \quad (33)$$

with

$$G_n = P_{\mu}'^{(n)} P_{\mu}''^{(n)} + C \quad (34)$$

where C is a constant and $P_{\mu}'^{(n)}$ ($P_{\mu}''^{(n)}$) are four-vectors depending only on the variables u', p' (u'', p''). Expanding F in (33) as a power series in $u_{1,j}$ thus gives us the residues automatically as a sum of factorized terms. In particular, the residue at $\alpha_{1,j} = l$ is given by the terms with $(u_{1,j})^l$, in the expansion of (33). Clearly, the number of factors of the form (34) contained in such a term depends only on the functional form of (33) and not on the number of external lines. This is what we wished to prove.

Although from the arguments outlined above, one is able to show that any pole in the model can be consistently interpreted as a superposition of a finite number of factorized levels, the number of levels required is extremely large. It can readily be seen that the number of factors for a given l is the same as the number of ways of choosing non-negative integers l_k which satisfy the equation (partitions equation)

$$l_1 + 2l_2 + 3l_3 + \dots + kl_k + \dots = l \quad (35)$$

For l large, this number increases as $\exp a\sqrt{l}$ with $a = 2\pi/\sqrt{6}$ or in other words, the number of levels increases exponentially with the centre-of-mass energy²⁷⁾.

Another interesting point in the formula (32)-(34) is that the residues at the poles are expressed as scalar products of four-dimensional tensors. This reflects the Lorentz invariance of the present approach and explains the lack of kinematical singularities. However, because the minus sign in the Lorentz metric, this also means that the time-components of these tensors will in general have negative residues, and correspond to ghost states. Fubini and Veneziano²¹⁾ were able to show with the help of a Ward-like identity inherent in the beta function that leading ghosts at least are compensated by similar poles with positive residues. However, ghosts on lower trajectories are not so compensated.

In what follows we list some further detail on the level structure known to us at present. These have been derived from the idealized model with only one trajectory which has $\alpha(0) < 0$. Most of the statements will remain true, however, when not all the trajectories are identical.

- Parent trajectory $(J = \ell)$

A single level $^{11),21),26)}$ with normal parity $^{20),21)}$, even signature $^{20)}$, and no ghost $^{21)}$.

- First daughter trajectory $(J = \ell - 1)$

Two levels $^{21)}$ with normal parity $^{20),21)}$, mixed signature $^{20)}$ and ghosts compensated $^{21),26)}$.

- Second and lower daughters $(J < \ell - 1)$

Multiply degenerate $^{21),26)}$, mixed parity $^{20),21)}$ and signature $^{20)}$, uncompensated ghosts $^{21)}$.

The level structure reported above has been derived from the generalized beta functions (15) or (16) alone. However, the general picture remains much the same with the inclusion of satellites. The problem has been investigated by Gross¹⁸⁾ who found that the number of levels increases in general when satellites are introduced, but remains finite for a wide class of functions $f_N(u_p)$ which may be used to modify the integrand in (16), as discussed in Section 2. In other words, the

consistency condition proposed at the beginning of this Section concerning the number of daughter levels removes but little of the large ambiguity due to satellites. Moreover, from his work, it seems highly unlikely that satellites can compensate for the ghost states found by Fubini and Veneziano ²¹⁾.

5) - ATTEMPTS TOWARDS MORE REALISTIC SYSTEMS

The idealized system discussed in the previous Sections is obviously over-simplified and should be treated only as a theoretical laboratory in the same spirit as one treats, say, the φ^4 model in Lagrangian field theory. To construct a more realistic system, a formidable array of problems remain to be solved :

- (a) internal symmetry,
- (b) trajectories with positive intercepts,
- (c) meson trajectories with abnormal parity,
- (d) baryon trajectories.

A simple way of incorporating isospin, and if desired SU_3 symmetry, into the idealized model has been suggested by several authors ²⁹⁾, which at the phenomenological level is probably adequate. Consider an N particle amplitude where the external lines all have isospin 1. Let a_i ($a_i = 1, 2, 3$) denote the isospin component of the i th particle and τ_x ($x = 1, 2, 3$) the 2×2 Pauli matrices. The suggestion then is simply to multiply each term in the sum (9) by a trace of τ matrices, thus

$$V(1, 2, 3, \dots, N) \rightarrow \text{Tr}(\tau_{a_1} \tau_{a_2} \tau_{a_3} \dots \tau_{a_N}) V(1, 2, 3, \dots, N) \quad (36)$$

where the order of the τ matrices should correspond to the ordering of the external lines in V . This result can also be obtained by an application of the graphical rules of Harari and Rosner ³⁰⁾.

It is clear from the elementary properties of traces and of the τ matrices that the modification (36) will conserve isospin, retain cyclic symmetry and factorization of the original model, and exclude exotic states ($I > 1$) from all Mandelstam channels. Each term from (9), e.g., (36), will have poles in both $I = 0$ and 1 states for every channel, but on summing, some of the poles will be cancelled. Indeed, for the leading trajectories, it can be shown in general that ²⁰⁾²⁹⁾ :

- (i) trajectories with the same isospin but opposite G parities will have opposite signatures ;
- (ii) each $I = 1$ trajectory will be degenerate with an $I = 0$ trajectory with the same G parity but opposite signatures ($\rho - f_0, \omega - A_2$).

Strange mesons can be incorporated into the model by simply replacing the Pauli τ matrices with the Gell-Mann λ matrices. The couplings of the leading trajectories will then be SU_3 symmetric but the trajectories themselves need not be SU_3 degenerate. From this, the following additional results for the leading trajectories can be derived :

- (iii) each $I = 1$ trajectory will be degenerate with an $I = 0$ trajectory with the same signature but opposite G parity, ($\rho - \omega, f_0 - A_2$) ;
- (iv) there exists in addition an exchange degenerate trajectory with $I = 0$ ($\phi - f'_0$) ;
- (v) the $I = \frac{1}{2}$ trajectories are exchange degenerate ($K^* - K^{**}$).

All the results (i)-(v) have already been derived previously from various dual models, including the four-point Veneziano model ¹⁾. The interesting point here is that they can be consistently maintained in the N particle generalization.

The solution for the problem (a) discussed above suggests strongly a quark picture, as in the Harari-Rosner rule ³⁰⁾. It is tempting therefore to try a similar solution for the problems

(b) and (c), say for example, by taking traces of γ matrices instead of τ and λ . Unfortunately, all attempts in this direction so far have led to parity degenerate trajectories and ghost states, which would be highly unphysical in a realistic model for the N meson amplitude. One can take a different view point, however, as suggested by Mandelstam³¹⁾ and consider the generalized Veneziano amplitudes in their present form not as a realistic model but as the "Born term" to a future unitarized theory. If this view point is adopted, model amplitudes with degenerate parity-doublets and ghosts need not at this stage be rejected. Indeed, Mandelstam suggests that both are probably necessary even in a purely mesonic system, so long as one remains in a framework with linear trajectories and without unitarity.

With this idea in mind, Mandelstam³¹⁾ has proposed a bootstrap scheme which yields results very similar to the quark model. The problem treated by him is quite general and lies outside the scope of the present review. We only mention that in his model, degenerate parity doublets occur as well as doublets in the Toller quantum number M . The unwanted doublets are mostly ghost states with negative residues, which, he argues, will move off the real axis in a properly unitarized theory and become the trajectories which pass through the Gribov-Pomeranchuk singularities at threshold.

Based on Mandelstam's scheme³¹⁾, Bardakçi and Halpern³²⁾ have constructed N meson amplitudes which have complete factorization on all leading trajectories. They propose to use these as the Born term in a future unitarization scheme similar to those to be discussed in Section 7. It should be noted, however, that in order to split parity and remove the ghosts, unitarity corrections in such schemes must presumably be quite large. They are thus contrary to the current (though perhaps ungrounded) belief that the Veneziano model is already a reasonable approximation to nature, and will make any phenomenological applications at this stage redundant.

6) - PHENOMENOLOGICAL APPLICATIONS

Clearly, such applications are meaningful only if one believes that the Veneziano amplitudes have already something to do with reality, and are not just Born terms to a future theory. One can then no longer evade the problems listed in the preceding Section. In addition one has to face the question of the Pomeranchuk singularity which has no place in a dual model of this type without unitarity ³³⁾.

Such reasons restrict present applications effectively only to the five-point function. This has no trouble with parity-doubling, at least for spinless external particles, since all trajectories are coupled to two-body states and must therefore have normal parity : $P = (-1)$. The Pomeranchuk problem can be avoided if one chooses reactions where vacuum exchanges do not occur. One is then left with only the difficulty connected with baryon trajectories which exists already at the four-point level and cannot easily be avoided. One can indeed consider decay processes and annihilations at rest into four-body final states, similar to those treated by Lovelace ¹⁾ with the original Veneziano model. However, these reactions have practical difficulties of their own, which cannot be discussed here. For the present, it seems that phenomenological applications may have to remain on the level where one ignores the possible complications due to baryon spins.

The only known application so far of the generalized Veneziano model to actual data analysis is that by the Petersson and Törnqvist ³⁴⁾ to the reaction

$$K^- p \rightarrow \Lambda \pi^+ \pi^- \quad (37)$$

over the range of energy 3.0 GeV/c - 10 GeV/c. This particular reaction was chosen for the following reasons :

- (i) no Pomeron exchange,
- (ii) number of permissible graphs restricted by quantum numbers,
- (iii) normal parity exchange known empirically to dominate.

Of the 12 terms in the amplitude (9) corresponding to the 12 non-equivalent orderings of the external lines, six are forbidden if one allows no exotic resonances. Out of the six remaining terms, two correspond to double baryon-exchange for the reaction (37) which, for the range of energy considered, may be regarded as negligible. One is then left with the four terms corresponding to the graphs in Fig. 7. By certain arguments concerning exchange degeneracies of the K^* and Y^* trajectories, the authors further restrict themselves to the graphs of Fig. 7, a and b. Then inserting the known intercepts of the K^* , Y^* , ρ and N trajectories, a universal slope $\alpha' = 0.9$, and imaginary parts to α fitted to resonance widths from the Rosenfeld table, they are left with only one free parameter, namely the over-all normalization constant.

Samples of their calculations are compared with experimental data in Figs. 8-12. The agreement is impressive for the little freedom they have available. At present, the calculation must be regarded as crude, since

- (i) they have ignored the spin of the baryons, treating them essentially just as spinless mesons ;
- (ii) they calculated graphs a and b of Fig. 7, whereas the duality graphic rules of Harari-Rosner³⁰⁾ would prefer instead c and d of the same figure, which is somewhat disturbing.

Nevertheless, the results are very encouraging and represent a great step forward from previous calculations with the multi-Regge model³⁵⁾.

Besides the analysis reported above, one should mention a paper by Bia~~l~~as and Pokorski¹⁵⁾ which studies in detail the phenomenological predictions of the five-point Veneziano model in both the single and double Regge limits. Such predictions would be interesting to test when data are available.

7) -

THE UNITARITY PROBLEM

The problem of unitarization of the Veneziano model is receiving a lot of attention and is being rapidly developed. I shall give here only a brief sketch of the preliminary work so far attempted.

Besides practical unitarization schemes on the phenomenological level, such as complex trajectories, K matrix approach, etc., which are similar to those in the four-point Veneziano model ³⁶⁾, two different aspects of the unitarity problem have been considered.

In the first approach, the generalized Veneziano model is accepted as a reasonable approximation to nature. It can then be used for example to study the shadow effect of inelastic channels on elastic scattering. A crude numerical calculation in this direction has been attempted by Roberts ³⁷⁾ giving quite encouraging results. One can go further, and try to derive for the Veneziano amplitudes, equations similar to that of Amati, Fubini and Stanghellini ³⁸⁾ for the multiperipheral model. The result would be a scheme analogous to the multiperipheral bootstrap proposed by Chew, Pignotti and collaborators ³⁹⁾. This has been attempted by Bardakçi and Ruegg ¹¹⁾ and is being pursued by others ⁴⁰⁾.

The second approach is much more ambitious and aims at a complete theory of strong interactions. The Veneziano model is considered merely as the Born term in a new perturbation series in which "duality" is consistently maintained. General rules for constructing loop diagrams consistent with "duality", which correspond to the higher terms in the perturbation series, have been given by Kikkawa, Sakita and Virasoro ⁴¹⁾. In order to guarantee unitarity in the sense of perturbation theory, the internal propagators of loop diagrams should represent a complete sum over all possible intermediate states, while the coupling constants of internal lines should be identical to those occurring in the tree diagrams. For pursuing this program therefore, a careful study of the level structure and the factorization properties of tree diagrams is imperative. The initial study of such problems for the

generalized beta functions reported in Section 4 has now been extended by Fubini, Gordon and Veneziano ⁴²⁾, using an elegant operator formalism. This allows one, at least in principle, to construct all diagrams with any number of loops for this idealized system. However, such diagrams are found ⁴³⁾ to be badly divergent and cannot at present be given a meaning. The prospects in this direction are extremely hazy and do not look at all bright. However, the mere fact that attempts are being made to construct a complete theory for strong interactions serves to indicate what exciting new possibilities are being opened up by the generalized Veneziano model.

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by several authors, in particular for the general N
point problem by Koba and Nielson ²⁵⁾. If all the tra-
jectories are identical, as in the cases so far consi-
dered, then the conditions (28) and (31) together with
their N point generalizations determine α_0 and α'
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FIGURE CAPTIONS

- Figure__1 Diagram representing the term in the N point amplitude which corresponds to the ordering $(1,2,\dots,N)$ of the external lines.
- Figure__2 The "dual" diagram associated with Fig. 1. The diagonal shown is denoted by the indices $(1,3)$.
- Figure__3 The Mandelstam channel corresponding to the diagonal $(1,3)$ of Fig. 2.
- Figure__4 Diagrams representing the set of independent variables u_{1j} ($j = 2,3,\dots,N-2$).
- Figure__5 Diagrams representing a change of variables used in Ref. 20).
- Figure__6 Diagrams representing a change of variables used in Ref. 21).
- Figure__7 Dominant diagrams for the reaction : $K^-p \rightarrow \pi^- \pi^+ \Lambda$ considered in Ref. 34).
- Figure__8 The energy dependence for the total reaction cross-section and the partial cross-sections for $Y^{*\pm}$ production ³⁴⁾.
- Figure__9 Percentage effective mass distributions at 3 GeV/c. The curves are the predicted results ³⁴⁾.
- Figure__10 Percentage effective mass distributions at 5.5 GeV/c. The curves are the predicted results ³⁴⁾.
- Figure__11 The distributions in t of ρ and $Y^{*}(1385)$ and their decay angular distributions at 5.5. GeV/c. The curves are the predicted results ³⁴⁾.
- Figure__12 The distribution in the radial angle ω on the hexagonal plot ³⁴⁾.

FIG 2

FIG 1

FIG 3

(a)

(b)

FIG 4

(a)

(b)

FIG 5

(a)

(b)

FIG 6

a

b

c

d

FIG 7

FIG. 8

FIG. 9

FIG. 10

FIG. 11

FIG. 12