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Archives

SOME CONSEQUENCES OF THE RELATION OF $\pi\pi$ PARTIAL WAVE
AMPLITUDES TO SERIES OF STIELTJES

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A B S T R A C T

Two consequences of the Stieltjes representation for $\pi\pi$ partial wave amplitudes $a_\ell(s)$, which holds for $0 \leq s < 4$ and certain charge states, are discussed. Firstly improved bounds for the total amplitude $A(s,t)$ for these values of s for $-s < t < 0$ and $4-s < t < 4$ are obtained from the first few partial waves $a_\ell(s)$. Then from this same information upper and lower bounds to $a_\ell(s)$ are derived for all real $\ell > 2$.

In proving these results it is shown that bounds obtained previously by Baker for the series of Stieltjes may be differentiated to give bounds to the derivatives of this series.

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1. INTRODUCTION

In a recent paper ¹⁾ we have shown that the $\pi\pi$ partial wave amplitudes $a_\ell(s)$ in certain charge states are related to coefficients of series of Stieltjes when $0 \leq s < 4$. This result is a consequence of the existence of Froissart-Gribov representations for $a_\ell(s)$ when $\ell \geq 2$ and of positivity conditions following from unitarity.

For example if $a_\ell^{00}(s)$ is the amplitude for $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ scattering and $a_\ell^0(s)$ is that for $\pi\pi \rightarrow \pi\pi$ scattering in the isotopic spin $T=0$ state, then either can be written in the general form ²⁾

$$a_\ell(s) = \int_0^{1/\kappa} u^{\ell/2-1} d\phi(u) \quad \ell=2,4,\dots \quad (1.1)$$

where

$$\kappa = [\chi_0 + (\chi_0^2 - 1)^{1/2}]^2, \quad \chi_0 = \left[\frac{4+s}{4-s} \right]$$

and $\phi(u)$ is a bounded non-decreasing function of u with an s dependence which is taken to be understood ^{*}).

Two applications of the representation (1.1) for the $a_\ell(s)$ were discussed previously ¹⁾. The first was to show that given the first few partial waves for scattering in either of the above states, upper and lower bounds could be put on the total amplitude $A(s,t)$ when $0 \leq s < 4$ and $0 > t > -s$ or $4 > t > 4-s$. The second application was to show that from these partial waves one could define functions which approximated the extrapolation of $a_\ell(s)$ to complex ℓ with $\text{Re } \ell > 2$ and that sequences of these functions converged to the exact extrapolation of $a_\ell(s)$ when the number of partial waves used was increased.

^{*}) This representation holds for the general linear combination $(1 + \lambda/3)a_\ell^0(s) + (2\lambda/3)a_\ell^2(s)$ where $a_\ell^T(s)$ are the partial wave amplitudes for $\pi\pi$ scattering in the state with isotopic spin T and $\lambda \geq -1$. The results of this paper are applicable to $\pi\pi$ scattering in these linear combinations of isotopic spin states.

In this paper we will consider these two applications further. Firstly we will improve the bounds on $A(s,t)$ for the above values of s and t and secondly we will show that the functions which approximately extrapolate $a_\ell(s)$ give bounds to the exact extrapolation when ℓ is real.

The bounds for $A(s,t)$ were obtained from the representation ³⁾

$$A(s,t) = a_0(s) + \frac{1}{\pi} \int_0^\pi [x + (x^2-1)^{1/2} \cos v]^2 \left\{ -4 [x + (x^2-1)^{1/2} \cos v]^2 G'[-(x + \sqrt{x^2-1} \cos v)^2] + \right. \\ \left. + 5 G[-(x + \sqrt{x^2-1} \cos v)^2] \right\} dv \quad (1.2)$$

where

$$G(x) = \sum_{n=0}^{\infty} a_{2n+2}(s) (-x)^n = \int_0^x \frac{d\phi(u)}{1+xu} \quad (1.3)$$

and $x = 1+2t/(s-4)$. Given the first few partial waves we can derive bounds to $G(x)$ and $G'(x)$ for $-r < x < 0$ since $G(x)$ is a series of Stieltjes, and hence get bounds to $A(s,t)$.

For instance, suppose $a_\ell(s)$ for $\ell = 2, 4, \dots, 2m+2$ are given. Then, if m is even $= 2N$, we have the following bounds for $G(x)$ and its derivatives ⁴⁾

$$(-1)^p [N, N]^{(p)} \leq (-1)^p G^{(p)}(x) \leq (-1)^p (N, N-1)^{(p)} \quad (1.4)$$

for $-r < x < 0$ and $p = 0, 1, 2, \dots$. $[N, N+j]$ are the Padé approximants to $G(x)$ and $(N, N+j)$ are a set of "modified" Padé approximants to $G(x)$ which have been defined by the author ⁴⁾. Similarly, when m is odd and $= 2N-1$ then we have the bounds given by

$$(-1)^p [N, N-1]^{(p)} \leq (-1)^p G^{(p)}(x) \leq (-1)^p (N-1, N-1)^{(p)} \quad (1.5)$$

for $-r < x < 0$ and $p = 0, 1, 2, \dots$

of the reduced moment problem (1.6) will be used and in particular we will be concerned with the solution of what Shohat and Tamarkin have called ⁶⁾:

Problem M

Given $m+1$ constants $f_0, f_1, \dots, f_m$ for which the reduced moment problem

$$f_n = \int_0^{1/r} u^n d\phi(u) \quad n=0,1,2,\dots,m. \quad (1.9)$$

has a solution $\phi(u)$ which is bounded and non-decreasing.

There will in general be an infinite set of such functions $\phi(u)$ satisfying (1.9). The problem is to find for a given function $f(u)$

$$\inf_{\phi} \int_0^{1/r} f(u) d\phi(u) \quad \text{and} \quad \sup_{\phi} \int_0^{1/r} f(u) d\phi(u)$$

when ϕ ranges over this set.

This problem was first solved for certain classes of functions $f(u)$ by Markoff ⁷⁾. A set of sufficient conditions for the solution of the problem is the following. For a given $a > 1/r$, $f(u)$ is continuous with all its derivatives in the interval $[0, a)$, $f^{(n)}(u) > 0$ for all $n \leq m+1$ and all u in the interval $(0, a)$ and $f(u) \rightarrow \infty$ as $u \rightarrow a$ from below.

Two classes of functions which satisfy these conditions and which we will be considering in this paper are the following. Firstly

$$f(u) = \frac{u^p}{(1+Xu)^{p+1}} \quad p=0,1,2,\dots \quad (1.10)$$

with $-r < X < 0$. Here $a = -1/X$ and the solutions of Problem M give best upper and lower bounds to

The bounds given by $\overline{[N, N]}$ and $\overline{[N, N-1]}$ for $G(X)$ and its derivative are the best possible since they are "obtained", i.e., there exists a solution $\phi(u)$ of the incomplete moment problem

$$f_n \equiv a_{2n+2}(s) = \int_0^{1/r} u^n d\phi(u) \quad n=0,1,\dots,m. \quad (1.6)$$

for which $\overline{[N, N]}^{(p)} = G^{(p)}(X)$ and similarly for $\overline{[N, N-1]}^{(p)}$ so that the bounds to $G(X)$ and its derivatives given by these Padé approximants cannot be improved.

However, the bounds given by $(N, N-1)$ and (N, N) are not "obtained" in this sense. We will show that they can be improved upon and so will get better upper bounds for $G(X)$ and $-G'(X)$ and hence for $A(s, t)$.

Now "obtained" upper bounds for $G(X)$ have been given by Baker ⁵⁾ and are for m even and $=2N$,

$$[N, N] \leq G(X) \leq \frac{P_N^{(0)}(x) Q_N^{(-1)}(-r) + x/r P_N^{(-1)}(x) Q_N^{(0)}(-r)}{Q_N^{(0)}(x) Q_N^{(-1)}(-r) + x/r Q_N^{(-1)}(x) Q_N^{(0)}(-r)}, \quad -r < x < 0 \quad (1.7)$$

and for m odd and $=2N+1$,

$$[N+1, N] \leq G(X) \leq \frac{P_{N+1}^{(-1)}(x) Q_N^{(0)}(-r) + x/r P_N^{(0)}(x) Q_{N+1}^{(-1)}(-r)}{Q_{N+1}^{(-1)}(x) Q_N^{(0)}(-r) + x/r Q_N^{(0)}(x) Q_{N+1}^{(-1)}(-r)}, \quad -r < x < 0 \quad (1.8)$$

where $P_N^{(j)}(x)$ and $Q_N^{(j)}(x)$ are the numerator and denominator respectively of $\overline{[N, N+j]}$. For completeness we have included the "obtained" lower bounds in (1.7), (1.8).

However, we still need "obtained" upper bounds for $-G'(X)$ and we will prove that they can be got by differentiating the upper bounds in (1.7) and (1.8). To do this, known results in the theory

$$\int_0^{\infty} \frac{u^p d\phi(u)}{(1+uX)^{p+1}} = (-1)^p G^{(p)}(X) \quad (1.11)$$

which are just what are needed.

The other class is

$$f(u) = u^k \quad (1.12)$$

with $k \geq m$. Here $a = \infty$ and we get bounds to

$$a_{2k+2}(s) \equiv f_k = \int_0^{\infty} u^k d\phi(u) \quad (1.13)$$

for real $k \geq m$. But we will see later that (1.13) is just the extrapolation of $a_\ell(s)$ to all real $\ell \geq 2m+2$ and so the solutions to Problem M can be used to give the required upper and lower bounds to this extrapolation.

In Section 2 we will give without proof the complete set of solutions of Problem M for the class of functions $f(u)$ defined above. Proofs follow along the same general lines as those sketched in Ref. 6), p. 77-82, and we refer there the reader who is interested.

Then, in Section 3 we show that the bounds (1.7) and (1.8) given by Baker⁵⁾ are just the solutions of Problem M when $f(u) = 1/(1+uX)$. Using this connection we prove that the bounds in (1.7) and (1.8) can be differentiated to give "obtained" bounds to the derivatives of the series of Stieltjes $G(X)$.

The results of Section 3 are then used to get better upper bounds on the total $\pi\pi$ scattering amplitude $A(s,t)$ than those derived in our earlier work¹⁾. This is done in Section 4 and the improvement in the bound on $A(s,t)$ is illustrated by considering a simple numerical example.

In Section 5 the extrapolation of $a_\ell(s)$ to non-integer $\ell > 2$ is considered. Given the first few partial waves $a_2(s), \dots, a_{2m+2}(s)$ we construct upper and lower bounds to $a_\ell(s)$ with $\ell > 2m+2$ from the solution of Problem M with $f(u) \equiv u^k$. Finally, in Section 6 we give a brief summary of the conclusions of this paper.

2. SOLUTIONS OF PROBLEM M

The solution of Problem M is obtained by using properties of orthogonal and quasi-orthogonal polynomials with respect to the moment sequence (1.6). For the reader unfamiliar with these concepts we give the definitions and some basic properties of these polynomials ⁸⁾.

Definition The orthogonal polynomial $\omega_n(u)$ of degree n with respect to the sequence $\{f_n\}$ defined by (1.6) satisfies the conditions

$$\int_0^k \omega_n(u) u^p d\phi(u) = 0 \quad p=0,1,\dots,n-1. \quad (2.1)$$

and this defines $\omega_n(u)$ uniquely up to a constant factor. In fact the solution for $\omega_n(u)$ is ⁹⁾

$$\omega_n(u) = \begin{vmatrix} f_0 & f_1 & \dots & f_n \\ f_1 & & & \vdots \\ \vdots & & & \vdots \\ f_{n-1} & f_n & \dots & f_{2n-1} \\ 1 & u & \dots & u^n \end{vmatrix} \quad (2.2)$$

Definition The quasi-orthogonal polynomial $q(u)$ of order $n+1$ with respect to the sequence $\{f_n\}$ is a polynomial of degree at most $n+1$ satisfying (2.1) with $\omega_n(u)$ replaced by $q(u)$. When one zero $u=u_0$ of $q(u)$ is given it is uniquely defined up to a constant factor and may be written in the form ⁹⁾

$$q(u) = \begin{vmatrix} 1 & u & u^2 & \dots & u^{n+1} \\ 1 & u_0 & u_0^2 & \dots & u_0^{n+1} \\ f_0 & f_1 & \dots & \dots & f_{m+1} \\ f_1 & & & & \vdots \\ \vdots & & & & \vdots \\ f_{n-1} & \dots & \dots & \dots & f_{2n} \end{vmatrix} \quad (2.3)$$

The following properties of the zeroes of $\omega_n(u)$ and $q(u)$ are used for the solution of Problem M. Firstly, the n zeros of $\omega_n(u)$ are distinct and all lie in the open interval $(0, 1/r)$. Similarly if $q(u)$ has real coefficients its zeros are real and distinct and at most one of them lies outside the open interval $(0, 1/r)$.

We now give the solutions of Problem M when $f(u)$ satisfies the conditions described in the Introduction, i.e., for a given $a > 1/r$, $f(u)$ is continuous with all its derivatives in the interval $[0, a)$, $f^{(n)}(u) > 0$ for all $n \leq m+1$ and all u in the interval $(0, a)$ and $f(u) \rightarrow \infty$ as $u \rightarrow a$ from below. The solutions are separated into four categories:

i) $\sup_{\phi} \int_0^{1/r} f(u) d\phi(u)$

given $f_0, f_1, \dots, f_m$ for even $m = 2N$. The quasi-orthogonal polynomial $q(u)$ of order $N+1$ is constructed with specified zero at $u_0 = 1/r$. Then if u_i ($i = 1, 2, \dots, N$) are the other zeros in increasing magnitude,

$$0 < u_1 < u_2 < \dots < u_N < u_0 = 1/r.$$

It may be proved following the methods sketched in Ref. 6) that,

$$\sup_{\phi} \int_0^{1/r} f(u) d\phi(u) = \sum_{i=0}^N h_i f(u_i) \quad (2.4)$$

where the weights h_i are given by

$$h_i = \frac{1}{q'(u_i)} \int_0^{\frac{1}{r}} \frac{q(u) d\phi(u)}{(u-u_i)} \quad (2.5)$$

This upper bound can be evaluated from the given moments f_i since the integrand in (2.5) is just a polynomial in u of degree N .

ii)
$$I_{nf} \int_0^{\frac{1}{r}} f(u) d\phi(u)$$

given $f_0, f_1, \dots, f_m$ for even $m=2N$. The quasi-orthogonal polynomial $Q(u)$ of order $N+1$ is constructed with the specified zero $U_0=0$. Then, if U_i ($i=1, 2, \dots, N$) are the other zeros

$$I_{nf} \int_0^{\frac{1}{r}} f(u) d\phi(u) = \sum_{i=0}^N H_i f_i(U_i) \quad (2.6)$$

where

$$H_i = \frac{1}{Q'(U_i)} \int_0^{\frac{1}{r}} \frac{Q(u) d\phi(u)}{(u-U_i)} \quad (2.7)$$

iii)
$$I_{nf} \int_0^{\frac{1}{r}} f(u) d\phi(u)$$

given $f_0, f_1, \dots, f_m$ for odd $m=2N-1$. The orthogonal polynomial of degree N is constructed and if y_i ($i=1, 2, \dots, N$) are its zeros,

$$I_{nf} \int_0^{\frac{1}{r}} f(u) d\phi(u) = \sum_{i=1}^N g_i f(y_i) \quad (2.8)$$

where

$$g_i = \frac{1}{\omega_N'(y_i)} \int_0^{\frac{1}{r}} \frac{\omega_N(u) d\phi(u)}{(u-y_i)} \quad (2.9)$$

$$\text{iv)} \quad \sup_{\phi} \int_0^{\frac{1}{r}} f(u) d\phi(u)$$

given $f_0, f_1, \dots, f_m$ for odd $m = 2N+1$. This case is treated a little differently since one first has to consider the sequence $\{F_n\}$ where $F_n = f_{n+1} - f_{n+2}$. The orthogonal polynomial $W_N(u)$ of degree N with respect to this sequence $\{F_n\}$ is constructed. Then, if Y_i are the zeros of $W_N(u)$,

$$\sup_{\phi} \int_0^{\frac{1}{r}} f(u) d\phi(u) = \sum_{i=1}^N \frac{G_i f(Y_i)}{Y_i(1/r - Y_i)} + G_0 f(0) + G_{N+1} f(1/r) \quad (2.10)$$

where

$$G_i = \frac{1}{W_N'(Y_i)} \int_0^{\frac{1}{r}} \frac{W_N(u) u(1/r - u) d\phi(u)}{(u - Y_i)}; \quad G_0 = \frac{\Gamma}{W_N(0)} \int_0^{\frac{1}{r}} (1/r - u) W_N(u) d\phi(u) \quad (2.11)$$

$$G_{N+1} = \frac{\Gamma}{W_N(1/r)} \int_0^{\frac{1}{r}} u W_N(u) d\phi(u). \quad (2.12)$$

This completes the set of bounds to $\int_0^{1/r} f(u) d\phi(u)$ as $\phi(u)$ runs over all solutions of the incomplete moment problem (1.9). All these bounds are "obtained" in the sense described in the Introduction.

3. BAKER'S BOUNDS AND THEIR DERIVATIVES

The conditions on $f(u)$ for the bounds given in the previous Section to hold are satisfied when $f(u) = 1/(1+Xu)$ and $-r < X < 0$. Therefore (2.4) and (2.10) give "obtained" upper bounds to the series of Stieltjes

$$G(X) = \int_0^{\frac{1}{r}} \frac{d\phi(u)}{(1+uX)} \quad (3.1)$$

for these values of X . Baker has also derived a set of upper bounds for $G(X)$ when $-r < X < 0$ and they have been given in (1.7) and (1.8). Since they are also "obtained", they must be identical to those given by (2.4) and (2.10).

For example if category i) is considered where $f_0, f_1, \dots, f_{2N}$ are given, then we have the identity

$$\sum_{i=0}^N \frac{h_i}{(1+u_i X)} \equiv \frac{P_N^{(0)}(X) Q_N^{(-1)}(-r) + X/r P_N^{(-1)}(X) Q_N^{(0)}(-r)}{Q_N^{(0)}(X) Q_N^{(-1)}(-r) + X/r Q_N^{(-1)}(X) Q_N^{(0)}(-r)} \quad (3.2)$$

which holds for $-r < X < 0$ and hence for all X . We will now prove the following:

Theorem 1

The "obtained" bounds to $G(X)$ for $-r < X < 0$ given by (1.7) and (1.8) may be differentiated to give "obtained" bounds to the derivatives of $G(X)$.

Proof

Let

$$f(u) = \frac{u^p}{(1+uX)^{p+1}}$$

where $p = 0, 1, 2, 3, \dots$. Then $f(u)$ again satisfies the conditions for the bounds of the previous Section to hold when $-r < X < 0$. Then, for example, in category i)

$$\begin{aligned} (-1)^p G^{(p)}(X) &= \int_0^r \frac{u^p d\phi(u)}{(1+uX)^{p+1}} \leq \sum_{i=0}^N \frac{h_i u_i^p}{(1+u_i X)^{p+1}} \\ &= (-1)^p \frac{d^p}{dx^p} \left\{ \sum_{i=0}^N \frac{h_i}{1+u_i X} \right\} \quad (3.3) \end{aligned}$$

Therefore, from (3.2) for $-r < X < 0$

$$(-1)^p G^{(p)}(X) \leq (-1)^p \frac{d^p}{dx^p} \left\{ \frac{P_N^{(0)}(x) Q_N^{(-1)}(-r) + X P_N^{(-1)}(x) Q_N^{(0)}(-r)}{Q_N^{(0)}(x) Q_N^{(-1)}(-r) + X P_N^{(-1)}(x) Q_N^{(0)}(-r)} \right\}, \quad p=0,1,2,\dots \quad (3.4)$$

and this bound is "obtained" since the bound given by (3.3) is "obtained". Similarly, we get from categories ii) to iv) for $p=0,1,2,\dots$

$$(-1)^p [N,N]^{(p)} \leq (-1)^p G^{(p)}(X) \quad (3.5)$$

$$(-1)^p [N+1,N]^{(p)} \leq (-1)^p G^{(p)}(X) \leq (-1)^p \frac{d^p}{dx^p} \left\{ \frac{P_{N+1}^{(-1)}(x) Q_N^{(0)}(-r) + X P_N^{(0)}(x) Q_{N+1}^{(-1)}(-r)}{Q_{N+1}^{(-1)}(x) Q_N^{(0)}(-r) + X P_N^{(0)}(x) Q_{N+1}^{(-1)}(-r)} \right\} \quad (3.6)$$

This completes the proof of the theorem.

We have therefore shown that the upper bounds to $G(X)$ for $-r < X < 0$ given by Baker ⁵⁾ may be differentiated any number of times to give "obtained" bounds to the derivatives of $G(X)$. At the same time we have proved the known result ⁴⁾ that the derivatives of $[N,N]$ and $[N+1,N]$ give bounds to the derivatives of $G(X)$.

The identity (3.2) gives the partial-fraction expansion of the bound to $G(X)$ and can be used to determine h_i and u_i from the Padé approximants to $G(X)$. Similarly U_i, H_i, g_i, y_i , etc., may also be determined in this way. The reader who is familiar with the handling of Padé approximants may prefer to calculate h_i , etc., in this way, than directly from (2.5), etc.

4. BOUNDS ON THE $\pi\pi$ SCATTERING AMPLITUDE $A(s,t)$

We have shown previously ¹⁾ that for the $\pi\pi$ scattering states described in the Introduction, bounds can be put on the scattering amplitude $A(s,t)$ for $0 < s < 4$ and $x_0 > x = 1+2t/(s-4) > 1$ and $-x_0 < x < -1$, when the first few partial waves are given. This result followed from the representation (1.2) for $A(s,t)$ in terms of the series of Stieltjes $G(X)$.

Inequalities (3.4) and (3.6) give better upper bounds to $G(X)$ and $-G'(X)$ than those used in our previous work since they are "obtained" and hence give improved upper bounds to $A(s,t)$. Suppose $a_0(s), a_2(s), \dots, a_{2m+2}(s)$ are given with $m = 2N$. Then, using the identity (3.2)

$$G(X) \leq \sum_{i=0}^N \frac{h_i}{1+u_i X} \quad (4.1)$$

and

$$X G'(X) \leq -X \sum_{i=0}^N \frac{h_i u_i}{(1+u_i X)^2} \quad (4.2)$$

for $-r < X < 0$. Substituting in (1.2) we get after some manipulation

$$A(s,t) \leq a_0(s) + \sum_{i=0}^N \frac{h_i}{u_i} \left\{ \frac{1}{2} (1-u_i) \left[\frac{1}{(1+2\sqrt{u_i}x+u_i)^{3/2}} + \frac{1}{(1-2\sqrt{u_i}x+u_i)^{3/2}} \right] - 1 \right\} \quad (4.3)$$

for the above values of s and t .

Similarly, if $a_0(s), \dots, a_{2m+2}(s)$ with $m = 2N+1$ are given, then from (2.10)

$$\begin{aligned} A(s,t) \leq & a_0(s) + \sum_{i=1}^N \frac{G_i}{\gamma_i^2 (1-\gamma_i)} \left\{ \frac{1}{2} (1-\gamma_i) \left[\frac{1}{(1+2\sqrt{\gamma_i}x+\gamma_i)^{3/2}} + \frac{1}{(1-2\sqrt{\gamma_i}x+\gamma_i)^{3/2}} \right] - 1 \right\} + \\ & + r G_{N+1} \left\{ \frac{1}{2} (1-1/r) \left[\frac{1}{(1+2x/\sqrt{r}+1/r)^{3/2}} + \frac{1}{(1-2x/\sqrt{r}+1/r)^{3/2}} \right] - 1 \right\} + \\ & + 5 G_0 P_2(x) \end{aligned} \quad (4.4)$$

The upper bounds on $A(s,t)$ given by (4.3) and (4.4) are to be compared with those given by (5.26) of Ref. 1) which they replace.

We consider now a simple numerical example to illustrate how these new upper bounds on $A(s,t)$ improve on the old ones. This example is the one considered in Ref. 1), i.e., we assume

$$a_0(s) = 0, \quad a_2(s) = 1 \quad \text{and} \quad a_4(s) = \frac{1}{2} \left[\frac{Q_4\left(\frac{4+s}{4-s}\right)}{Q_2\left(\frac{4+s}{4-s}\right)} \right] \quad (4.5)$$

The value for $a_4(s)$ is half its maximum value allowed by unitarity when $a_2(s)$ is unity¹⁰⁾.

Given these two partial waves, an upper bound to $A(s,t)$ may be obtained from (4.4) with $N=0$. The coefficients G_0 and G_1 may be evaluated by constructing the $[0,0]$ and $[1,0]$ Padé approximants to $G(X)$ and then using the identity between (2.10) when $f(u) = (1+uX)^{-1}$ and the right-hand side of (1.8). The bound for $A(s,t)$ is then

$$A(s,t) \leq a_0(s) + a_4(s) r^2 \left\{ \frac{1}{2} (1 - \frac{1}{r}) \left[\frac{1}{(1+2x/\sqrt{r} + \frac{1}{r})^{3/2}} + \frac{1}{(1-2x/\sqrt{r} + \frac{1}{r})^{3/2}} \right] - 1 \right\} +$$

$$+ 5 [a_2(s) - a_4(s) r] P_2(x) \quad (4.6)$$

In Table 1 we give the values of this upper bound which we denote by $C_0(s,t)$ for $s=2$ and $2 < t < 4$ along with the upper bound $B_{0,0}(s,t)$ and the lower bound $A_{1,0}(s,t)$ given in Ref. 1). It will be seen that the new upper bound is an appreciable improvement on $B_{0,0}(s,t)$.

5. BOUNDS ON THE EXTRAPOLATION OF $a_\ell(s)$ TO NON-INTEGERS ℓ

For the $\pi\pi$ scattering states considered in this paper and for $0 \leq s \leq 4$ the Froissart-Gribov formula for the partial wave amplitudes may be written ¹⁾

$$a_\ell(s) = \frac{1}{\pi} \int_{x_0}^{\infty} \Phi_\ell(x) d\psi(x) \quad \ell=2,4,6,\dots \quad (5.1)$$

with $\psi(x)$ a non-decreasing function. This formula gives the unique extrapolation $a(\ell,s)$ of $a_\ell(s)$ to complex ℓ ¹⁾ with $\text{Re } \ell > 2$ if it is required that $a(\ell,s)$ be analytic for these values of ℓ and to satisfy

$$|a(\ell,s)| < C \exp[k|\ell|] \quad , \quad k < \pi \quad (5.2)$$

where k and C are constants.

But we also have the representation

$$a_\ell(s) = \int_0^{1/4} u^{\ell/2-1} d\phi(u) \quad , \quad 0 \leq s \leq 4; \ell=2,4,6,\dots \quad (5.3)$$

and this can be used to extrapolate $a_\ell(s)$ to complex ℓ with $\text{Re } \ell > 2$. This extrapolation satisfies the above conditions of analyticity and asymptotic behaviour and is therefore identical with $a(\ell,s)$. Hence

$$a(\ell,s) = \int_0^{1/4} u^{\ell/2-1} d\phi(u) \quad \text{Re } \ell > 2 \quad (5.4)$$

Consider the situation when the first few partial wave amplitudes $a_2(s), a_4(s), \dots, a_{2m+2}(s)$ are known for some value of s between 0 and 4. We have seen ¹⁾ that an approximant to $a(\ell,s)$ may be constructed from these amplitudes and that sequences of such approximants can be defined, which converge to $a(\ell,s)$ for all $\text{Re } \ell > 2$. From the above information we can also get bounds to the extrapolation of the partial wave amplitude $a_\ell(s)$ as follows from the following result.

Theorem 2

When $a_2(s), \dots, a_{2m+2}(s)$ are given, upper and lower bounds to the extrapolation $a(\ell, s)$ of the partial wave amplitude may be constructed for real $\ell > 2$ and they are given by (2.4) to (2.12) with $f(u) = u^{\ell/2-1}$ and $f_n = a_{2n+2}(s)$, $n = 0, 1, \dots, m$.

The proof follows immediately from the fact that when $f(u)$ has the above form it satisfies the requirements for the solution of Problem M when $\ell > 2m+2$, and that the integrals $\int_0^1 f(u) d\phi(u)$ are immediately identifiable from (5.4) as $a(\ell, s)$ the unique extrapolation of $a_\ell(s)$.

Consider the case when $m = 2N-1$ with N an integer. A lower bound for $a(\ell, s)$ is then given by (2.8). Therefore when $\ell > 4N$ and $0 \leq s < 4$

$$a(\ell, s) \geq \sum_{i=1}^N g_i y_i^{\ell/2-1} \quad (5.5)$$

where g_i and y_i are given by the identity

$$\sum_{i=1}^N \frac{g_i}{(1+u y_i)} \equiv [N, N-1]. \quad (5.6)$$

But, comparing with Eq. (6.4) and (6.7) of Ref. 1), we see that the right-hand side of (5.5) is the approximant $a_N(\ell, s)$ to the extrapolation $a(\ell, s)$ of the partial wave amplitude. Therefore, the $a_N(\ell, s)$ not only converge to $a(\ell, s)$ for all $\text{Re } \ell > 2$ but also give lower bounds to $a(\ell, s)$ when $\ell > 4N$.

If just $a_2(s)$ and $a_4(s)$ are given then the bounds to $a(\ell, s)$ may be obtained from (2.8) and (2.10) and are

$$a_2(s) \left[\frac{a_4(s)}{a_2(s)} \right]^{\ell/2-1} \leq a(\ell, s) \leq a_4(s) \tau^{2-\ell/2}, \quad \ell > 4 \quad (5.7)$$

If $a_6(s)$ is also known then the bounds are obtained from (2.4) and (2.6) and are for $\ell > 6$

$$a_6(s) \left[\frac{a_6(s)}{a_4(s)} \right]^{\frac{l}{2}-3} \leq a(l,s) \leq \left\{ a_6(s)r^2 - 2a_4(s)r + a_2(s) \right\}^{-1} \times$$

$$\times \left\{ r^2 [a_6(s)a_2(s) - (a_4(s))^2] \left(\frac{1}{r} \right)^{\frac{l}{2}-1} + \left[\frac{a_6(s)-a_4(s)}{a_4(s)-a_2(s)r} \right]^{\frac{l}{2}-1} \left[(a_4(s))^2 + r^2(a_2(s))^2 - 2a_2(s)a_4(s)r \right] \right\} \quad (5.8)$$

We illustrate these bounds by considering the simple numerical example used previously ¹⁾, i.e., we take

$$a(l,s) = a_2(s) Q_l\left(\frac{4+s}{4-s}\right) / Q_2\left(\frac{4+s}{4-s}\right) \quad (5.9)$$

and normalize $a_2(s)$ to unity. Let $b_1(l,s)$ and $b_2(l,s)$ be the lower bounds to $a(l,s)$ given in (5.7) and (5.8) respectively and $b_3(l,s)$ and $b_4(l,s)$ be the upper bounds given respectively by (5.8) and (5.7). In Table 2 we give values of these functions for $s=2$ and $l \geq 2$ along with the values of $a(l,s)$.

It is easily confirmed that the $b_i(l,s)$ give the bounds to $a(l,s)$ for the values of l indicated in (5.7) and (5.8). Another point to note is that for $l \geq 6$ the bounds to $a(l,s)$ given by (5.8) are better than those given by (5.7). This is what one would expect since in (5.7) we have the bounds for $\int_0^{1/r} f(u) d\phi(u)$ for a larger set of functions $\phi(u)$ than in (5.8). The former set is all solutions of the incomplete moment problem (1.6) with $m=1$ which is larger than the set of solutions of (1.6) with $m=2$. It is a general and expected result that better bounds on $a(l,s)$ are obtained by using more partial waves $a_2(s), a_4(s), \dots$

6. CONCLUSIONS

The main objective of this paper was to extend and improve some results on the consequences of representation (1.1) for the

partial wave scattering amplitudes in certain $\pi\pi$ states. To do this we had to consider the reduced moment problem (1.6) and the solution of the associated Problem M. Although the method for solving this problem is well known ⁶⁾ the complete set of solutions is not usually given. We therefore thought it worth while to give the complete solution in Section 2.

In Section 3 we have shown that there is a connection between these solutions and bounds obtained by Baker ⁵⁾ for a series of Stieltjes with non-zero radius of convergence r . Using this connection we were able to prove the interesting result that Baker's bound could be differentiated to give bounds to the derivatives of the series of Stieltjes for points on the real axis between the origin and $-r$.

In the following Sections we used these mathematical results to improve the upper bounds to $A(s,t)$, the total scattering amplitude, and to obtain bounds to the extrapolation $a(\ell,s)$ of the partial wave amplitudes to non-integer ℓ . From the latter application we have the important result that given the first few partial waves $a_2(s), a_4(s), \dots, a_{2m+2}(s)$ bounds can be on all higher waves $a_{2m+4}(s)$, etc.

The bounds to $A(s,t)$ and $a(\ell,s)$ are "obtained" in the sense that there exists a solution $\varnothing(u)$ of the incomplete moment problem (1.1) with $\varnothing(u)$ non-decreasing for which each of these quantities equals its bound. However, this $\varnothing(u)$ may not correspond to a $\psi(x)$ in (5.1) which is also non-decreasing ^{*)}, which has to be the case for the physical partial wave amplitudes for the states under consideration. Therefore, we have not derived attained bounds for $A(s,t)$ and $a(\ell,s)$ since there may not exist a $\psi(x)$ for which either of these quantities equals its bound.

*) An explicit counter-example is constructed in Appendix 2 of Ref. 1).

Finally we think that our results have helped to stress the importance of the representation (1.1) for the partial wave amplitudes $a_\ell(s)$.

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TABLE 1

Bound functions for the scattering amplitude $A(s,t)$
when $s = 2$

t	$A_{1,0}(s,t)$	$C_0(s,t)$	$B_{0,0}(s,t)$
2.2	8.728	8.764	8.854
2.5	15.94	16.24	16.95
2.8	25.83	27.30	30.62
3.1	39.34	45.30	57.91
3.4	58.00	82.00	128.70
3.7	84.38	155.10	437.50

TABLE 2

$a(l,s)$ and its bounds

l	$b_1(l,s)$	$b_2(l,s)$	$a(l,s)$	$b_3(l,s)$	$b_4(l,s)$
2.0	1.00E+0	0.908E+0	1.00E+0	1.00E+0	0.762E+0
2.5	3.87E-1	3.60E-1	3.81E-1	3.83E-1	3.16E-1
3.0	1.50E-1	1.43E-1	1.47E-1	1.48E-1	1.31E-1
3.5	5.80E-2	5.66E-2	5.73E-2	5.74E-2	5.42E-2
4.0	2.24E-2	2.24E-2	2.24E-2	2.24E-2	2.24E-2
4.5	8.69E-3	8.90E-3	8.85E-3	8.83E-3	9.30E-3
5.0	3.36E-3	3.53E-3	3.50E-3	3.49E-3	3.85E-3
5.5	1.30E-3	1.40E-3	1.39E-3	1.39E-3	1.60E-3
6.0	5.04E-4	5.55E-4	5.55E-4	5.55E-4	6.61E-4
7.0	7.55E-5	8.72E-5	8.88E-5	8.97E-5	1.13E-4
8.0	1.13E-5	1.37E-5	1.43E-5	1.47E-5	1.95E-5
9.0	1.69E-6	2.15E-6	2.33E-6	2.44E-6	3.34E-6
10.0	2.54E-7	3.39E-7	3.81E-7	4.09E-7	5.73E-7

R E F E R E N C E S

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