Renormalization of Functional Schrödinger Equation by Background Field Method

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Abstract

Renormalization group transformations for Schrödinger equation are performed in $\varphi^4$ and in Yang-Mills theories. The dependence of the ground state wave functional on rapidly oscillating fields is found. For Yang-Mills theory, this dependence restricts a possible form of variational ansatze compatible with asymptotic freedom.
1 Introduction

It is widely accepted that many physical properties of gauge theories in the confining phase are determined by the structure of their vacua. In the literal sense of the term, the vacuum is described by a ground state wave functional – the lowest energy solution of the Schrödinger equation. To find a complete solution of the Schrödinger equation in Yang-Mills theory is by no means feasible, but the asymptotic freedom should allow to determine the dependence of the ground state wave functional on rapidly oscillating fields, since this dependence is responsible for the renormalization.

General properties of the renormalization of functional Schrödinger equation were studied within the perturbation theory. It was shown that both the Hamiltonian and the wave functional are renormalizable by usual counterterms up to multiplicative redefinition of the field variables [1]. Practically, the most convenient way to perform renormalization group transformations explicitly is based on the averaging over rapidly oscillating degrees of freedom. Background field method in the path integral framework [2] is usually associated with this procedure. The Hamiltonian counterpart of the background field method was also employed in some problems, for example, in connection with gauge fields on a torus [3, 4], in soliton quantization [5] and also in variational calculations in gauge theories [6].

The averaging procedure is common in quantum mechanics with finite number of degrees of freedom. An instructive example [7, 8] is the system with Hamiltonian

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} x^2 y^2. \quad (1.1)$$

The potential energy for this system is degenerate along the coordinate axes. Far from the origin, say, at $|x| \gg 1$, the potential valley becomes very narrow and the wave function varies along the valley much slower than in the transverse direction. So, the variables are naturally separated in the slow and the fast ones. The last two terms in (1.1) is the Hamiltonian for the fast mode $y$: $H_h = \frac{1}{2} p_y^2 + \frac{1}{2} x^2 y^2$. Its ground state energy induces an effective potential for the slow degree of freedom: $V_{\text{eff}}(x) = |x|/2$. Thus, the degeneracy of the potential is lifted and the system with Hamiltonian (1.1), in fact, has a discrete spectrum [7]. The above procedure can be generalized to a field theory [3]. We shall use it in a systematic way to renormalize Schrödinger equation in $\varphi^4$ and in the Yang-Mills theories. We consider pure gauge theory mostly for the sake of simplicity and the introduction of quarks should not cause any difficulties, since suitable methods to treat fermions in the functional Schrödinger picture are known [9].

It is worth mentioning that the averaging over fast modes is not the only way to renormalize Schrödinger equation. As usual, the renormalization group transformations consist in a proper modification of the Hamiltonian for low-energy degrees of freedom after elimination of the fast modes [10]. So, the renormalized Hamiltonian acts in the smaller Hilbert space than the bare one and the renormalization can be considered as a projection on this smaller space, or a reduction of the Hamiltonian to the block-diagonal form. General methods for partial diagonalization [11, 12], as well as for projection [13] of Hamiltonians exist. These methods were tested on simple quantum-mechanical systems [14, 13] and applied to some many-body [12, 17] and field-theoretical light-cone [15] and equal-time [16] Hamiltonians.
2 Scalar field

The Hamiltonian of $\varphi^4$ theory is

$$H = \int d^3 x \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{24} \lambda \varphi^4 \right],$$

(2.1)

where $\Pi$ is canonically conjugate to the field variable:

$$[\varphi(x), \Pi(y)] = i \delta(x - y).$$

(2.2)

Schrödinger equation for the Hamiltonian (2.1) requires regularization, thus we assume that modes with momenta larger than $\Lambda$ are somehow excluded (the concrete prescription will be given below). In order to exclude also the modes with momenta larger than $\mu$, where $\Lambda \gg \mu \gg m$, we can explicitly solve Schrödinger equation for the fields containing only these modes and then average the Hamiltonian with the wave function for the high-energy degrees of freedom. Denote by $\bar{\varphi}$, $\bar{\Pi}$ and by $\phi$, $\pi$ the slow and the fast variables – the field components which contain modes with momenta $p < \mu$ and $\mu < p < \Lambda$, respectively. According to the conventional assumption of the background field formalism [2], slowly varying fields $\bar{\varphi}$ satisfy classical equations of motion.

Extracting the Hamiltonian for the high-energy degrees of freedom and expanding it to the quadratic order in $\phi$ (which is equivalent to the lowest order perturbation theory in $\lambda$), we get:

$$H_h = \int d^3 x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \bar{\varphi}^2 \phi^2 \right].$$

(2.3)

Since this Hamiltonian is quadratic, the ground state wave function has the form

$$\Psi_h = \exp \left( -\frac{1}{2} \int d^3 x d^3 y \phi(x) K(x,y) \phi(y) \right).$$

(2.4)

Substituting this expression in the Schrödinger equation

$$\int d^3 x \left[ -\frac{1}{2} \frac{\delta^2}{\delta \phi^2} + \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \bar{\varphi}^2 \phi^2 \right] \Psi_h = E_h \Psi_h,$$

(2.5)

we obtain for the operator $K$:

$$K = \left( -\partial^2 + m^2 + \frac{1}{2} \lambda \bar{\varphi}^2 \right)^{1/2},$$

(2.6)

the ground state energy being equal to

$$E_h = \frac{1}{2} \text{Tr} K.$$

(2.7)

As in the quantum-mechanical example in the introduction, the ground state energy of large-momentum modes renormalizes the low-energy Hamiltonian. When $\mu$ is much larger than any other scale in the problem, only the ultraviolet divergent contributions are essential.
UV divergences can be easily extracted by the heat kernel method, which is based on the following representation for the square root of the operator:

\[ K = \lim_{\varepsilon \to 0} \left( \frac{1}{2\varepsilon} - \frac{1}{2\pi^{1/2}} \int_0^\infty \frac{d\tau}{\tau^{3/2}} e^{-\tau K^2} \right) \]  

(2.8)

When we calculate the trace, the field-independent term merely renormalize the zero-point energy,

\[ E_0 = \int \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + m^2} \]  

Below this field-independent contribution is ignored. The remaining part of the effective potential can be calculated, for example, expanding

\[ V_{\text{eff}}(\bar{\varphi}) \equiv E_h = \text{const} - \frac{1}{4\pi^{1/2}} \int \frac{d\tau}{\tau^{3/2}} \text{Tr} \exp \left[ -\tau \left( -\partial^2 + m^2 + \frac{1}{2} \lambda \bar{\varphi}^2 \right) \right] \]  

(2.9)

in \( \lambda \). This procedure involves evaluation of the momentum integrals, where the momentum cutoff can be introduced as a lower bound of integration over \( \tau \). Therefore, the prescription to integrate over \( \tau \) from \( \Lambda^{-2} \) to \( \mu^{-2} \) naturally accounts for the presence of only large-momentum modes in \( \varphi \).

In fact, direct expansion in \( \lambda \) is not the most convenient way to extract the essential terms in the effective potential. The UV divergences are governed by the behavior of the integrand in (2.9) at small \( \tau \) and can be easily captured by DeWitt-Seeley expansion [2]. In particular [18]:

\[ \langle x | e^{-\tau(-\partial^2 + V)} | x \rangle = \frac{1}{(4\pi\tau)^{3/2}} \left[ 1 + V \tau + \left( \frac{1}{2} V^2 - \frac{1}{6} \partial^2 V \right) \tau^2 + O(\tau^3) \right] \]  

(2.10)

Substituting \( \lambda \bar{\varphi}^2/2 \) for \( V \), we get from eqs. (2.9), (2.10):

\[ V_{\text{eff}}(\bar{\varphi}) = \text{const} - \frac{1}{4\pi^{1/2}} \int \frac{d\tau}{\tau^{3/2}} \frac{e^{-m^2\tau}}{(4\pi\tau)^{3/2}} \int d^3x \left( \frac{1}{2} \lambda \bar{\varphi}^2 \tau + \frac{1}{8} \lambda^2 \bar{\varphi}^4 \tau^2 \right) + O(1/\mu^2) \]  

(2.11)

The divergent terms lead to quadratic renormalization of the mass and logarithmic renormalization of the coupling:

\[ \lambda_{\text{eff}} = \lambda - \frac{3\lambda^2}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} \]  

(2.12)

This result, of course, gives the correct expression for the \( \beta \)-function in the \( \varphi^4 \) theory.

3 Yang-Mills theory

The Hamiltonian formulation of the Yang-Mills theory is most simple in the temporal gauge \( A_0 = 0 \). Then the canonical variables are gauge potentials \( A_i^B(x) \) and electric fields \( E_i^B(x) \) (we consider \( SU(N) \) gauge group, so \( B = 1, \ldots, N^2 - 1 \)):

\[ [A_i^A(x), E_j^B(y)] = i\delta^{AB} \delta_{ij} \delta(x - y). \]  

(3.1)
The Hamiltonian is
\[ H = \int d^3x \left( \frac{g^2}{2} F_{ij}^A F_{ij}^A + \frac{1}{4g^2} F_{ij}^A F_{ij}^A \right), \]  
(3.2)
where \( F_{ij}^A = \partial_i A_j^A - \partial_j A_i^A + f^{ABC} A_i^B A_j^C \). The wave functions of physical states are also subject to the Gauss’ law constraint:
\[ D_i E_i^A \Psi = 0. \]  
(3.3)

The covariant derivative \( D_i \) acts in the adjoint representation: \( D_i^{AB} = \delta^{AB} \partial_i + f^{ACB} A_i^C \).

In the “coordinate” representation, the wave function is a functional of the gauge potentials and the electric field operators act as the variational derivatives: \( E_i^A(x) = -i \delta / \delta A_i^A(x) \).

In order to find the dependence of the vacuum wave functional on the field modes with high momenta, we proceed in the same way as in the previous section. The slowly varying fields, \( \bar{A}_i \), are assumed to contain only modes with momentum \( p < \mu \) and to satisfy classical equations of motion:
\[ \bar{D}_i \bar{F}_{ij} = 0, \]  
(3.4)
where \( \bar{D}_i \) is the covariant derivative with respect to \( \bar{A}_i \) and \( \bar{F}_{ij} \) is the corresponding field strength. It is convenient to rescale the fast variables \( a_i \), which contain the modes with momenta \( \mu < p < \Lambda \), so that \( A_i = \bar{A}_i + g a_i \). To preserve the canonical commutation relations for the fast modes, the high-energy components of the electric fields should be rescaled by \( 1/g \): \( E_i = \bar{E}_i + \frac{1}{g} e_i \).

We assume that the scale \( \mu \) is sufficiently large, so that the running coupling \( g(\mu) \) is small, and we can use perturbation theory for high-energy Hamiltonian. Expanding the Hamiltonian for rapidly oscillating variables in \( g \) to the leading order, we get:
\[ H_h = \int d^3x \left[ \frac{1}{2} e_i^A e_i^A + \frac{1}{2} a_i^A (-\bar{D}^2 \delta_{ij} - 2 \bar{F}_{ij} + \bar{D}_i \bar{D}_j) A_i^B A_j^B \right]. \]  
(3.5)

Here \( \bar{F}_{ij}^{AB} = [\bar{D}_i, \bar{D}_j]^{AB} = f^{ACB} \bar{F}_{ij}^C \) acts as a matrix in the adjoint representation.

Denote by \( L \) the quadratic form of the potential in eq. (3.5):
\[ L_{ij} = -\bar{D}^2 \delta_{ij} - 2 \bar{F}_{ij} + \bar{D}_i \bar{D}_j, \]  
(3.6)

Then the ground state wave functional for the fast variables is
\[ \Psi_h = \exp \left( -\frac{1}{2} a L^{1/2} a \right), \]  
(3.7)
where summation over color and spatial indices and integration over spatial coordinates is implied. The wave functional (3.7) satisfies Gauss’ law constraint up to the two first orders in \( g \):
\[ \left( \frac{1}{g} \bar{D}_i e_i^A + \bar{D}_i \bar{E}_i^A + f^{ABC} \bar{A}_i^B e_i^C \right) \Psi_h = 0. \]  
(3.8)

The \( O(1/g) \) part of the Gauss’ law generates transformations
\[ a_i \to a_i + \bar{D}_i \omega. \]  
(3.9)
The invariance of the wave functional under these transformations follows from the identity $L_{ij} \bar{D}_j \omega = 0$ [6]. Terms of order $g^0$ in the Gauss’ law generate usual gauge transformations of the background fields and homogeneous transformations of the fast variables:

$$\bar{A}_i \to \Omega^I (\partial_i + \bar{A}_i) \Omega, \quad a_i \to \Omega^I a_i \Omega.$$  (3.10)

The invariance of (3.7) under these transformations is evident, since the kernel of the operator $L^{1/2}$ transforms homogeneously: $L^{1/2}(x, y) \to \Omega^I (x) L^{1/2}(x, y) \Omega(y)$. To recover the invariance of the wave functional under the transformations generated by higher order terms in the Gauss’ law, it is necessary to include higher orders in the Schrödinger equation.

The effective potential generated by averaging of the Hamiltonian with the wave functional (3.7) is

$$V_{\text{eff}}(\bar{A}) = \frac{1}{2} \text{Tr} L^{1/2}. \quad (3.11)$$

Although the operator $L$ has a large number of gauge zero modes, they do not cause any problems with the calculation of the trace of its square root, hence, there is no reason to fix the gauge and to introduce ghosts as in the conventional background field method, where one usually deals with logarithms of the operators. The field-independent part of the effective potential corresponds to the zero-point energy and is not traced below. If the scale $\mu$ is sufficiently large, only UV divergent contributions are essential. The only contribution of this kind, the gauge coupling renormalization, can be easily extracted by the heat kernel method, as in Ref. [6], where a similar expression was considered in the context of the variational approach to the QCD vacuum.

DeWitt-Seeley coefficients for the operator $L$ were calculated in Ref. [19], where a more general operator,

$$L_{ij}(\alpha) = -\bar{D}^2 \delta_{ij} - 2 \bar{F}_{ij} + \left(1 - \frac{1}{\alpha}\right) \bar{D}_i \bar{D}_j,$$  (3.12)

was considered. Taking the limit $\alpha \to \infty$ of the small $\tau$ expansion of the heat kernel given in Ref. [19]*, we find:

$$V_{\text{eff}}(\bar{A}) = \text{const} - \frac{1}{4 \pi^{1/2}} \int \frac{d\tau}{\tau^{3/2}} \int d^3 x \text{ tr} \langle x | e^{-\tau L} | x \rangle = -\frac{1}{4 \pi^{1/2}} \int \frac{d\tau}{\tau^{3/2}} \frac{1}{(4 \pi \tau)^{3/2}} \left(\text{const} + \tau^2 \frac{11N}{6} \int d^3 x \bar{F}_{ij} \bar{F}_{ij} + O(\tau^3)\right) \quad = \text{const} - \frac{11N}{48 \pi^2} \ln \frac{\Lambda^2}{\mu^2} \frac{1}{4} \int d^3 x \bar{F}_{ij} \bar{F}_{ij} + O(1/\mu^2).$$  (3.13)

Here $\text{tr}$ denotes the trace with respect to color and spatial indices. The averaging over the fast modes reproduces the usual coupling constant renormalization:

$$\frac{1}{g^2_{\text{eff}}} = \frac{1}{g^2} - \frac{11N}{24 \pi^2} \ln \frac{\Lambda}{\mu}, \quad (3.14)$$

as expected.

*In fact, the first, field-independent, DeWitt-Seeley coefficient of the operator $L(\alpha)$ diverges as $\alpha \to \infty$. Although we do not trace the field-independent contribution, the origin of this divergence is clarified in Appendix. The correct prescription consists in simply dropping the divergent term.
4 Discussion

The most common application of the background field techniques, that to QCD sum rules [20], is based on the parametrization of the slowly varying fields by vacuum condensates [21]. In the Hamiltonian picture, this would correspond to treating rapidly oscillating fields perturbatively, as above, and parameterizing the wave functional for the low-energy degrees of freedom by a finite number of parameters. Probably, this approach may provide some useful information.

One may try to parametrize the vacuum in Yang-Mills theory by a finite number of parameters using some reasonable ansatz for the ground state wave function and then to find an approximate solution of the Schrödinger equation from the variational principle (for a recent discussion of the variational approach to the QCD vacuum, see [22, 6]). Apparently, an accuracy of such kind of approximations is not under control. In particular, the correct renormalization of physical quantities is not automatically guaranteed in the variational calculations. Our results show that any approximate wave functional compatible with the asymptotic freedom must depend on the large-momentum modes of the fields in a very specific way – as given by eq. (3.7). This property substantially restricts valid functional form of variational ansatze for the ground state in Yang-Mills theory.

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Appendix A  Heat kernel of the operator $L$

In order to relate the operators $L$ and $L(\alpha)$, it is convenient to introduce the non-Abelian transverse projector [6]:

$$P_{ij} = \delta_{ij} - \bar{D}_i \frac{1}{\bar{D}^2} \bar{D}_j. \tag{A.1}$$

Denote $L(1)$ by $G$:

$$G_{ij} = -\bar{D}^2 \delta_{ij} - 2\bar{F}_{ij}. \tag{A.2}$$

Then, with the use of the equations of motion (3.4) and the commutation relations $[\bar{D}_i, \bar{D}_j] = \bar{F}_{ij}$, the following equality can be proved:

$$L_{ij} = G_{ik} P_{kj} = P_{ik} G_{kj}, \tag{A.3}$$

So, the operator $L$ is a transverse projection of the operator $G$.

As a consequence of eq. (A.3), and since $P$ is the projection operator,

$$e^{-\tau L} = e^{-\tau G} P. \tag{A.4}$$
One the other hand, eqs. (3.6), (A.2) and (A.3) imply that $\bar{D}_i \bar{D}_j = -G_{ik}(\delta_{kj} - P_{kj})$ and, consequently,

$$L_{ij}(\alpha) = G_{ik}P_{kj} + \frac{1}{\alpha} G_{ik}(\delta_{kj} - P_{kj}).$$  \hspace{1cm} (A.5)

Since $P$ and $1 - P$ are the orthogonal projectors,

$$e^{-\tau L(\alpha)} = e^{-\tau G} P + e^{-\frac{\tau}{\alpha} G} (1 - P).$$  \hspace{1cm} (A.6)

The diagonal matrix elements of the second term on the right hand side of eq. (A.6) can be expanded in local operators. By dimensional reasons the parameter of this expansion is $\tau/\alpha$:

$$\text{tr} \langle x | e^{-\tau \frac{\tau}{\alpha} G} (1 - P) | x \rangle = \sum_{n=0}^{\infty} \left( \frac{\tau}{\alpha} \right)^{n-\frac{3}{2}} O_n(x),$$  \hspace{1cm} (A.7)

where $\text{tr}$ denotes the trace with respect to spatial and color indices and $O_n(x)$ are gauge-invariant operators of dimension $2n$. As there are no such operators of dimension 2, only the first term proportional to $\alpha^{3/2}$ survives the limit $\alpha \to \infty$. It is field-independent and can be calculated in the momentum representation. Finally we get:

$$\text{tr} \langle x | e^{-\tau L} | x \rangle = \lim_{\alpha \to \infty} \left[ \text{tr} \langle x | e^{-\tau L(\alpha)} | x \rangle - \frac{(N^2 - 1)\alpha^{3/2}}{(4\pi \tau)^{3/2}} \right].$$  \hspace{1cm} (A.8)

The small $\tau$ expansion of the right hand side is given in Ref. [19] up to the two first terms and, indeed, has a finite $\alpha \to \infty$ limit.

References


