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REGGE FAMILIES IN THE WICK - CUTKOSKY MODEL

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A B S T R A C T

The bound solutions of the Wick-Cutkosky model are analytically continued to the whole complex l plane by a method utilizing a Volterra integral equation equivalent to the Cutkosky differential equation. General results on the behaviour of the Regge trajectories are derived. In particular it is proved that in the limit $\lambda \rightarrow 0$, where λ is the coupling strength, N Regge poles even in relative time and $N-1$ poles odd in relative time tend to the negative integer $-N$. Collision phenomena among the trajectories are proved to occur in the exact solution. Additional features relevant to the understanding of the behaviour of the trajectories are derived.

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1. INTRODUCTION

The most advanced treatment, so far, of a relativistic bound state problem is offered by the Bethe-Salpeter equation.⁽¹⁾ In particular the classical Wick-Cutkosky model^{(2) (3)} (exchange of spinless zero mass particle between spinless particles in the ladder approximation) has proved of great utility in exploring various aspects of the bound solutions of a relativistic system. From a mathematical viewpoint, the simplifying feature of the model is that its study can be reduced to that of a second order ordinary differential equation. The origin of the simplification is the validity in the model of an $O(4)$ symmetry⁽³⁾, analogous to that of the hydrogen atom⁽⁴⁾, which provides for a one-dimensional spectral representation for the wave functions.^{(2) (3)}

We shall study here the properties of the Regge trajectories of the Wick-Cutkosky model.⁽⁵⁾ As compared to the corresponding study for non relativistic potential scattering⁽⁶⁾, one encounters in the discussion of this model all the special features of a relativistic covariant problem, in particular the existence of Toller families.⁽⁷⁾

The most natural approach to the study of the Regge trajectories of a Bethe-Salpeter equation is through examination of the poles of the resolvent of the analytically continued partial wave equation.⁽⁸⁾ Unfortunately the approach fails for $\text{Re } \ell < -3/2$. A method for analytically continuing the solutions to the left of $\text{Re } \ell = -3/2$ has been devised by Tiktopoulos⁽⁹⁾; the continuation must be done step by step, by integer units in $\text{Re } \ell$. By this method one can deal with the

first daughter trajectory⁽¹⁰⁾ but the study of lower trajectories becomes increasingly cumbersome. An alternative approach to the discussion of Regge trajectories is that of using Mellin transform techniques⁽¹¹⁾; also in this case one is faced by very complicated procedures except in dealing with the higher trajectories.⁽¹²⁾

In the Wick-Cutkosky model, which we shall examine in the present paper, the analytic continuation can be given most simply through the Cutkosky differential equation. We shall prove in the text the equivalence of such a procedure of analytic continuation to the standard one.⁽⁸⁾ The advantage of the procedure lies in that it offers a compact treatment valid for all values of ℓ .

The eigensolutions of the Cutkosky equation are either even or odd, corresponding to wave functions even or odd in the relative time; moreover due to the Fock $O(4)$ degeneracy only the quantity n , related to the angular momentum ℓ by $n = \ell + n'$ ($n' = 1, 2, \dots$), appears in the equation. We shall transform the Cutkosky differential equation into a Volterra integral equation, which includes the correct boundary condition at one extreme; by imposing the symmetry requirements to the solution of the integral equation we shall obtain the conditions for a Regge pole either of even or of odd time parity.

We shall treat separately the two cases, of η^2 ($4\eta^2 = \text{square of the c.m. energy}$) below the threshold, and of η^2 above the threshold. In the first case we can easily prove that, for $\text{Re } n > -1$, the Volterra series converges and is analytic in η^2 . For $\text{Re } n < -1$ the integrals

appearing in the Volterra series diverge and we must give their analytic continuations.

In order to obtain the analytic continuations up to a given value of η , we take advantage of the fact that only a finite number of terms in the iteration series needs to be continued, while the remainder of the series is convergent. We find that the solution of the Volterra equation is analytic in the whole η -plane, except for simple poles at the negative integers; moreover we prove that, for $\text{Re } \eta < 0$ the first iteration dominates, as $\eta^2 \rightarrow -\infty$, over the rest of the series.

The result on the analytic structure in η can be used to prove that N Regge poles even in the relative time, together with $N-1$ poles odd in the relative time tend for $\lambda \rightarrow 0$ to the negative integer $-N$ in the ℓ -plane. The dominance of the first iteration can be used to prove that, as $\eta^2 \rightarrow -\infty$, no poles remain inside any assigned bounded domain of the region $\text{Re } \eta < -1$. The study of the first iteration shows a mechanism through which such a behaviour is accomplished: pairs of poles starting from the neighbourhoods of consecutive negative integers approach each other in the η -plane running along the real axis; they collide and become afterwards complex. We prove, by giving a majorization of the series, that such collision phenomena actually occur in the exact solution for small but finite λ . Similar collisions of Regge poles have been found by Cutkosky and Deo⁽¹³⁾ in the region $\text{Re } \ell > -3/2$, where no analytic continuation is needed, in their numerical treatment

of an unequal mass Bethe-Salpeter equation with exchange of a massive boson; the collisions we find here are somewhat different in character and they occur in an equal mass case and for $\text{Re} l < -2$.

For $\eta^2 > 1$ the Volterra integral equation becomes singular and thus the usual methods for proving the convergence of the iterative series fail. However we prove that it is possible to deform the integration contour in the Volterra equation such as to obtain again a regular kernel, to which the usual procedures can be applied. The treatment given for $\eta^2 < 1$ can then be extended to cover also this case, leading to similar results.

The study of the first iteration in the region $0 < \eta^2 < 1$ shows that the trajectories are real and that each solution oscillates around the corresponding negative integer in the n -plane. At $\eta^2 = 1$ each trajectory (except the leading trajectory which has a typical behaviour), passes through the negative integer exhibiting a threshold behaviour which differs for even and odd trajectories, and takes on a positive imaginary part above the threshold. The leading trajectory has instead a square-root singularity at $\eta^2 = 1$. For $\eta^2 \rightarrow \infty$ the trajectories (except the leading one which goes to $n = 0$) tend to $n = -1$. A close examination of the first iteration reveals the existence, in the region $\text{Im} n < 0$ of additional solutions, besides those which for $\eta^2 \rightarrow 0$ tend to the well-known spectrum of n -values derived by Cutkosky.⁽³⁾ Such additional solutions have the property that they tend to infinity for $\lambda \rightarrow 0$ (as opposite to the behaviour of the other solutions which tend to zero or to the negative integers for $\lambda \rightarrow 0$).

At $\eta^2 = 0$ each trajectory can be identified with a symbol $\mathcal{K}_{\pm}(n')$, where $\mathcal{K} = 0, 1, \dots$ is the nodal quantum number and n' assumes the values $1, 2, 3, \dots$; the zero energy intercepts are given by $\ell(0) = -n' - \mathcal{K} \pm \sqrt{\frac{1}{4} + \lambda} - \frac{1}{2}$.⁽³⁾ However at zero energy the integer spacing due to the Lorentz $O(4)$ symmetry is superimposed to the integer spacing due to the Fock $O(4)$ symmetry. The circumstance gives rise to mixings among the Toller families, i.e. the limits of the wave functions as $\eta^2 \rightarrow 0$ are in general superpositions of analytic continuations of different representations of the Lorentz $O(4)$. When the exchanged particle has a non-vanishing mass, the Fock $O(4)$ degeneracy is removed but for continuity one still expects the presence of two infinite sequences of Toller families.

The results we have just described provide us with an understanding of the principal features of the Regge trajectories of the Wick-Cutkosky model. Detailed quantitative behaviours of the trajectories must be studied by numerical methods, but all the qualitative features are derived analytically in the present work.

In Section 2 we treat the analytic continuation in the ℓ -plane; in Section 3 we derive the general properties of the trajectories below the threshold and in Section 4 the treatment is extended above the threshold. Section 5 is devoted to the detailed study of the first order trajectories and in Section 6 we prove the existence of collisions among Regge trajectories. The more technical results are proved in the appendices.

2. THE ANALYTIC CONTINUATION IN THE ℓ -PLANE

The homogeneous Bethe-Salpeter equation for exchange of zero mass spinless particle can be written, after Wick rotation, as⁽²⁾

$$[(p+i\eta)^2+1][(p-i\eta)^2+1]\phi(p) = \frac{\lambda}{\pi^2} \int \frac{d^4k \phi(k)}{(p-k)^2}, \quad (1)$$

where p and k are four-vectors in a Euclidean space and

$\eta = (0, 0, 0, \frac{\sqrt{t}}{2})$. The two particles bound are spinless and of same mass (set here equal to 1).

Cutkosky⁽³⁾ has shown that the general solution of the equation can be written in the form

$$\phi_n^{\ell,m}(p) = \sum_{k=0}^{n-\ell-1} \int_{-1}^{+1} g_{n,e}^k(z) \phi_{n-k}^{\ell,m}(p,z) dz, \quad (2)$$

where

$$n = \ell + n' \quad (n' = 1, 2, \dots), \quad (3)$$

and where $\phi_n^{\ell,m}(p,z)$ is given by

$$\phi_n^{\ell,m}(p,z) = \frac{y_e^m(\vec{p})}{[p^2 + 2iz p \cdot \eta + 1 - \eta^2]^{n+2}} \quad (4)$$

in terms of the solid spherical harmonics

$y_e^m(\vec{p})$. For

$n = \ell + 1$ (i.e. $n' = 1$) one has only one spectral function $g_{n,e}^k$, namely

$$g_{n,e}^0 = g_{n,n-1}^0 \equiv g_n.$$

In general, for $n = \ell + n'$ one has n' spectral functions g ; among them $g_{n,\ell}^{\circ}$ satisfies a homogeneous differential equation with λ as an eigenvalue; λ depends on n and η^2 . [The remaining spectral functions satisfy a set of inhomogeneous differential equations.] The differential equation for $g_{n,\ell}^{\circ}$ (Cutkosky equation) is given by

$$(1-z^2) g'' + 2(n-1)z g' + \left\{ \frac{\lambda}{1-\eta^2(1-z^2)} - n(n-1) \right\} g = 0 \quad (5)$$

with the boundary condition $g(\pm 1) = 0$.

This equation does not contain ℓ but only $n = \ell + n'$ and thus

$g_{n,\ell}^{\circ}$ does not depend on ℓ . The circumstance is related to the $O(4)$ invariance of the problem, as discussed by Cutkosky.⁽³⁾ It is similar to the Fock $O(4)$ invariance of the Schrödinger equation for the hydrogen atom.⁽⁴⁾ The indicial equation at $z = \pm 1$ gives for the convergent behaviour

$$g_n(z) \sim (1-z^2)^n. \quad (6)$$

Clearly Eq. (5), with the condition (6), has a meaning also for n non-integer and in general complex, provided $\text{Re } n > 0$. One therefore obtains an analytic interpolation of the bound state solutions; the corresponding values of ℓ are $\ell = n - n'$, $n' = 1, 2, 3, \dots$ as stated by Eq. (3).

We shall first show that such an interpolation is exactly the Regge interpolation. We do this⁽⁵⁾ by proving its equivalence to the

analytic interpolation given by Lee and Sawyer using the partial-wave Bethe-Salpeter equation, analytically continued in ℓ ⁽¹⁴⁾

$$\psi_\ell(p, p_+) = \frac{2\lambda}{\pi} \frac{1}{[p^2 + p_+^2 + 1 - \eta^2]^2 + \eta^2 p_+^2}$$

$$\cdot \int_0^\infty \int_{-\infty}^{+\infty} Q_\ell \left(\frac{p^2 + k^2 + (p_+ - k_+)^2}{2pk} \right) \frac{k}{p} \psi_\ell(k, k_+) dk dk_+ \quad (7)$$

Due to the $O(4)$ degeneracy we can limit ourselves to $n' = 1$, i.e.

$\ell = n - 1$. To prove the equivalence of the two analytic continuations we show here that the function

$$\psi_\ell(p, p_+) = \int_{-1}^{+1} \frac{p^\ell g_{\ell+1}(z) dz}{[p^2 + p_+^2 + 2izp_+\eta + 1 - \eta^2]^{l+3}} \quad (8)$$

is a solution of Eq. (7) provided $g_n(z)$ satisfies Eq. (5) for $\text{Re } n > 0$.

We start from the relation ⁽³⁾

$$\frac{1}{\pi^2} \int \frac{d^4 k}{(\vec{p} - \vec{k})^2 + (p_+ - k_+)^2} \frac{y_\ell^m(\vec{k})}{[k^2 + k_+^2 + 2izk_+\eta + 1 - \eta^2]^{l+3}} =$$

$$= \frac{1}{(\ell+2)(\ell+1)} \frac{1}{1 - \eta^2(1 - z^2)} \frac{y_\ell^m(\vec{p})}{[p^2 + p_+^2 + 2izp_+\eta + 1 - \eta^2]^{l+1}} \quad (9)$$

derived for $\ell = 0, 1, 2, \dots$.

By integration over the spherical harmonics we obtain

$$\frac{2}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} Q_\ell \left(\frac{p^2 + k^2 + (p_+ - k_+)^2}{2pk} \right) \frac{k^{\ell+1} dk dk_+}{[k^2 + k_+^2 + 2izk_+\eta + 1 - \eta^2]^{l+3}} =$$

$$= \frac{1}{(\ell+2)(\ell+1)} \frac{1}{1 - \eta^2(1 - z^2)} \frac{p^{\ell+1}}{[p^2 + p_+^2 + 2izp_+\eta + 1 - \eta^2]^{l+1}} \quad (10)$$

Eq. (10) is valid for integer $\ell \geq 0$. In Appendix A, by means of Carlson's theorem, the validity of Eq. (10) is extended to all complex values of ℓ for which the integral converges; this region includes the half-plane $\text{Re } \ell \geq 0$. Using Eq. (10) and the representation in Eq. (8) for the wave function, through a standard integration formula⁽¹⁵⁾, the right-hand side of Eq. (7) becomes

$$\begin{aligned} & \frac{\lambda}{2} \int_{-1}^{+1} dz \int_{-1}^{+1} dt \int_0^1 \frac{dx \cdot x(1-x)^{\ell}}{1-\eta^2(1-z^2)} \frac{p^{\ell} g_{\ell+1}(z)}{[p^2 + p_+^2 + 2i\eta p_+(xt + (1-x)z) + 1 - \eta^2]^{\ell+3}} = \\ & = \frac{\lambda}{2} \int_{-1}^{+1} dz \int_{-1}^{+1} dt \int_0^1 dx \int_{-1}^{+1} d\gamma \frac{x(1-x)^{\ell}}{1-\eta^2(1-\gamma^2)} p^{\ell} g_{\ell+1}(\gamma) \cdot \\ & \cdot \frac{\delta(z - xt - (1-x)\gamma)}{[p^2 + p_+^2 + 2i\eta p_+ z + 1 - \eta^2]^{\ell+3}}. \end{aligned} \quad (11)$$

Eq. (11) has thus been shown to hold for all complex ℓ , $\text{Re } \ell \geq 0$.

One can then derive the Cutkosky integral equation for g or, equivalently, Eq. (5) with the boundary conditions $g(\pm 1) = 0$.

Summing up, we have shown that to each eigenvalue solution of Eq. (5) there corresponds a solution of the Lee and Sawyer homogeneous equation Eq. (7). For $\text{Re } n < 0$ the continuation of the boundary condition $g \sim (1-z^2)^n$ gives rise to a divergent boundary condition. This makes the direct use of Eq. (5) rather difficult. The situation here is similar to what occurs in potential scattering when one continues to the left of the line $\text{Re } \ell = -\frac{1}{2}$ (16). For this reason we shall introduce in place of g a function F which satisfies

a corresponding differential equation, but such that the boundary condition for F turns out to be independent of n ; we shall transform the obtained differential equation into a Volterra integral equation, including the boundary condition, which will be solved by a convergent iteration series. Such a series lends itself also to performing the analytic continuation for $\text{Re } n < 0$. The transition to a Volterra integral equation is similar to the treatment of potential scattering as given by Newton.⁽¹⁷⁾

We put $x = 1 - z$ and define

$$F(x) = x^{-n} g(1-x) . \quad (12)$$

$F(x)$ satisfies the differential equation⁽⁵⁾ (18)

$$x(2-x)F'' + 2(n+1-x)F' + \frac{\lambda}{1-\eta^2 x(2-x)} F = 0 . \quad (13)$$

The transition from this equation to the corresponding Volterra integral equation, which includes the boundary condition $F(0) = 1$ is performed by standard methods; we obtain

$$F(x) = 1 - \frac{\lambda}{2n} \int_0^x [1 - x^{-n}(2-x)^n y^n (2-y)^{-n}] \frac{F(y) dy}{1 - \eta^2 y(2-y)} . \quad (14)$$

Clearly the eigenfunctions of the Cutkosky equation are either even or odd in z corresponding to even or odd time parity of the wave function.⁽⁸⁾

We exploit this fact by writing the eigenvalue condition in terms of

$F(x)$ in the form

$$g'(0) = 0 , \quad \text{i.e.} \quad n F(1) + F'(1) = 0 \quad (15a)$$

for the even solutions, and in the form

$$g(0) = 0, \text{ i.e. } F(1) = 0 \quad (15b)$$

for the odd solutions.

At $\eta^2 = 0$ Eq. (13) is soluble in terms of an hypergeometric function. One finds

$$F(x) = {}_2F_1\left(\frac{1}{2} + \gamma, \frac{1}{2} - \gamma; n+1; \frac{x}{2}\right)$$

with $\gamma = \sqrt{\frac{1}{4} + \lambda}$.

For the even solutions we have thus⁽¹⁹⁾

$$\begin{aligned} n {}_2F_1\left(\frac{1}{2} + \gamma, \frac{1}{2} - \gamma; n+1; \frac{1}{2}\right) + \frac{1}{2} {}_2F_1'\left(\frac{1}{2} + \gamma, \frac{1}{2} - \gamma; n+1; \frac{1}{2}\right) = \\ = n 2^{1-n} \frac{\Gamma(n) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{\gamma}{2} + \frac{n}{2}\right) \Gamma\left(\frac{n}{2} - \frac{\gamma}{2} + \frac{1}{4}\right)} = 0, \end{aligned}$$

which gives

$$n = -\frac{1}{2} \pm \gamma - k, \quad k = 0, 2, 4, \dots \quad (16a)$$

For the odd solutions we have

$${}_2F_1\left(\frac{1}{2} + \gamma, \frac{1}{2} - \gamma; n+1; \frac{1}{2}\right) = \frac{2^{-n} \Gamma(n+1) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{\gamma}{2} + \frac{n}{2}\right) \Gamma\left(\frac{3}{4} - \frac{\gamma}{2} + \frac{n}{2}\right)} = 0,$$

which gives

$$n = -\frac{1}{2} \pm \gamma - k, \quad k = 1, 3, 5, \dots \quad (16b)$$

These are the exact intercepts at $\eta^2 = 0$ first derived for integer n by Cutkosky.⁽³⁾

3. GENERAL PROPERTIES OF REGGE TRAJECTORIES FOR $\eta^2 < 1$

In this section we shall confine ourselves to the region $\eta^2 < 1$, i.e. below the physical threshold.

A. The region $\text{Re } n > -1$

We are interested in the solution of the equation

$$F = f + K F \quad (17)$$

with $f = 1$ and

$$K_n(x, y) = - \frac{\lambda}{2n} \left[1 - x^{-n} (2-x)^n y^n (2-y)^{-n} \right] \frac{\Theta(x-y)}{1 - \eta^2 y(2-y)}. \quad (18)$$

The region of interest is $0 \leq x \leq 1$ and in particular we need the values of $F(x)$ and $F'(x)$ at $x = 1$. We shall also need the analytic continuation of $F(1)$ and $F'(1)$ for all complex values of the parameter n from which they depend.

It will be useful to consider K as an operator in the space of the continuous functions of x in $0 \leq x \leq 1$, with the norm defined as the maximum modulus.

One can prove (see Lemma 1, Appendix B) that for $\text{Re } n > -1$ the operator K_n is analytic in n and that

$$|K_n(x, y)| \leq \Theta(x-y) k_n(y), \quad (19)$$

$k_n(y)$ being an integrable function of y . This is sufficient to prove that the Volterra series

$$f + K f + K^2 f + \dots \quad (20)$$

converges uniformly in $0 \leq x \leq 1$ and that it is the solution of our equation.

The convergence of the series is also uniform in n on any closed contour lying in the region $\operatorname{Re} n > -1$; as the various terms are analytic functions of n in $\operatorname{Re} n > -1$ we conclude that the series (20) is a regular analytic function of n for $\operatorname{Re} n > -1$. Furthermore, in Appendix B it is proved that as $\eta^2 \rightarrow -\infty$ for $\operatorname{Re} n > -1$ one has

$$\int_0^1 k_n(y) dy \rightarrow 0, \quad (21)$$

and the limit is reached uniformly in any half-plane $\operatorname{Re} n \geq \alpha > -1$.

From this it follows that

$$\begin{aligned} \|F - 1\| &= \|Kf + K^2f + \dots\| \leq \\ &\leq \exp\left[\int_0^1 k_n(y) dy\right] - 1 \xrightarrow{\eta^2 \rightarrow -\infty} 0 \end{aligned} \quad (22)$$

uniformly in any half-plane $\operatorname{Re} n \geq \alpha > -1$.

The same uniform limit is clearly obtained for $\lambda \rightarrow 0$.

To deal with $F'(1)$ we observe that

$$\begin{aligned} F'(1) &= -\lambda \int_0^1 \frac{y^n (2-y)^{-n} F(y) dy}{1 - \eta^2 y (2-y)} = \\ &= 2n(1 - F(1)) - \lambda \int_0^1 \frac{F(y) dy}{1 - \eta^2 y (2-y)}. \end{aligned} \quad (23)$$

One reaches again the conclusion that, as $\eta^2 \rightarrow -\infty$ or $\lambda \rightarrow 0$,

$$F'(1) \rightarrow 0 \quad \text{uniformly in } \operatorname{Re} n \geq \alpha > -1.$$

B. Continuation of the solution to $\text{Re } n < -1$

We shall examine here the analytic continuation of the quantities $F(1)$ and $F'(1)$ for $\text{Re } n < -1$ again for $\eta^2 < 1$.

Integrals of the type Kf do not converge, in the region $\text{Re } n < -1$, and they must be obtained through analytic continuation. Before delving into the details of the argument, we sketch here the general line of reasoning. To give the analytic continuation down to $\text{Re } n > -N$ ($N =$ positive integer) we rewrite the series (20) as follows:

$$F = f + Kf + K^2f + \dots + K^N f + x^N (\tilde{K} \tilde{f} + \tilde{K}^2 \tilde{f} + \dots),$$

where (24)

$$\tilde{K}(x, y) = x^{-N} K(x, y) y^N$$
(25)

and

$$\tilde{f} = x^{-N} K^N f.$$
(26)

The terms $Kf, K^2f, \dots, K^N f$ have to be treated separately. Let us consider e.g. the first term Kf . Due to the divergence of the integral at the origin ($y = 0$) this term exhibits simple poles at $n = -1, -2, -3, \dots$. On the other hand Kf behaves at the origin like x . In the second iteration term K^2f , we can again have poles at the negative integers, both because Kf has poles there, and also because we can again meet a divergent integral. However we shall find that only simple poles can appear. At the N -th iteration, we see that $K^N f$ behaves as x^N as $x \rightarrow 0$ and that $x^{-N} K^N f$ is a vector in our space (with finite norm) except for simple poles at the negative integers. On the other hand it is proved

in Lemma 1 of Appendix B that $\tilde{K}$ for $\text{Re } n > -N-1$ is a bounded operator which is analytic and regular in n

$$|\tilde{K}_n(x, y)| \leq \Theta(x-y) \tilde{k}_n(y) \quad (27)$$

with

$$\int_0^1 \tilde{k}_n(y) dy < \infty. \quad (28)$$

In this way we find that the series

$$x^\sim (\tilde{K} \tilde{f} + \tilde{K}^2 \tilde{f} + \dots) \quad (29)$$

converges and it is analytic in $\text{Re } n > -N$ except for simple poles at the negative integers.⁽⁵⁾ A closer analysis of the operator K will enable us to derive some properties of the dependence of F on η^2 and λ also in the region $\text{Re } n < -1$. In particular we shall prove that, as $\eta^2 \rightarrow -\infty$, the first iteration dominates over the rest of the series.

We come now to the proof of our statements. In the following we shall denote by g_n a vector which is regular analytic in n $-N < \text{Re } n < 0$, by $p(n)$ a polynomial in n and by a, b, c, d non-negative integers.

In the study of the iterations we make use of the following results which are proved in Appendix B.

Lemma 3

For $0 \leq a \leq N-1$, $\text{Re } n > -N$, $K_n(x^a)$ is equal to a finite sum of terms of the type⁽²⁰⁾

$$(I) \quad \frac{p(n)}{n} (\eta^2)^b x^{a+b+c+1}$$

and

$$\frac{p(n)}{n} (\eta^2)^b \frac{x^{a+b+c+d+1}}{n+a+b+c+1},$$

and of the type

$$(II) \quad x^N g_n(x, \eta^2), \text{ with } \|g_n(x, \eta^2)\| \leq \text{const}(n) \frac{|\eta^2|^r}{\min(1-\eta^2, 1)}$$

$$0 \leq r \leq N-a-1$$

and

$$\frac{x^N g_n(x) (\eta^2)^b}{n+a+b+c+1},$$

where, for both kinds of terms (I) and (II)

$$0 \leq b \leq N-a-2$$

$$0 \leq c \leq N-a-2.$$

Lemma 4

For $-N < \text{Re } n < 0$, N positive integer, $K_n(x^{N-1} g_n(x)) = x^N g_n^{(1)}(x)$
with $g_n^{(1)}(x)$ regular analytic and

$$\|g_n^{(1)}(x)\| \leq \frac{\mathcal{H}_N(n) \|g_n(x)\|}{\min(1-\eta^2, 1)}.$$

The explicit value of $\mathcal{H}_N(n)$ is given in Appendix B.

Lemma 5

For $-N < \text{Re } n < 0$, N positive integer, $K_n(x^N g_n(x)) = x^N g_n^{(1)}(x)$
with $g_n^{(1)}(x)$ regular analytic and

$$\|g_n^{(1)}(x)\| \leq \frac{\mathcal{H}_N(n)}{1-\eta^2} \|g_n(x)\|.$$

The rules which we follow in the iteration are the following:

For a term of type (I) with exponent less than $N-1$ we apply Lemma

3. If the exponent equals $N-1$ we apply Lemma 4. Otherwise we apply Lemma 5.

We notice first that, when going on with the iteration, no double poles arise; in fact the pole is always shifted by at least one unit with respect to the exponent α of x . Terms of type (II) do not give rise to additional poles. Therefore each term of the iteration is an analytic function of n for $\operatorname{Re} n > -N$ except for simple poles at the negative integers. The exponent of the power of x is increased by at least one unit at each iteration; therefore one has

$$K^N f = x^N \frac{g_n(x, \eta^2)}{p(n)}$$

with $p(n)$ having only simple zeros at the negative integers. From the fact that $\tilde{K}_n = x^{-N} K_n y^N$ is regular analytic in $\operatorname{Re} n > -N-1$ and satisfies (27) with the condition (28), it follows that the series $x^N (\tilde{K} \tilde{f} + \tilde{K}^2 \tilde{f} + \dots)$ converges and is analytic in $\operatorname{Re} n > -N$ except for the poles of $\tilde{f} = x^{-N} K^N f$, which lie at the negative integers.

The same properties, i.e. analyticity in n except for simple poles at the negative integers, can be derived for the quantity $F'(1)$. The result can be obtained by using Eq. (23).

We shall now draw the implications of the properties we have derived for $F(1)$ and $F'(1)$, on the behaviour of the trajectories.

We saw in the preceding section that, as $\lambda \rightarrow 0$ or $\eta^2 \rightarrow -\infty$ one has $F(1) \rightarrow 1$ and $F'(1) \rightarrow 0$, uniformly in $\operatorname{Re} n \geq \alpha > -1$. From (15a) it follows that, as $\lambda \rightarrow 0$ or $\eta^2 \rightarrow -\infty$ all even trajectories retreat to the left of the line $\operatorname{Re} n = \alpha + 1$ and from (15b) that all odd trajectories retreat to the left of the line

$$\operatorname{Re} n = \alpha.$$

From the expression (24) we have derived that F is analytic in n except for simple poles at the negative integers; the poles can appear in $Kf, K^2f, \dots, K^N f$ which contains respectively $\lambda, \lambda^2, \dots, \lambda^N$ and in $\tilde{f} = x^{-N} K^N f$ which is otherwise analytic in $\operatorname{Re} n > -N$.

The series in the brackets in (24) can be majorized as follows

$$|x^N [\tilde{K}\tilde{f} + \tilde{K}^2\tilde{f} + \dots]| \leq x^N (\exp[\int_0^1 \tilde{k}_n(y) dy] - 1) \|\tilde{f}\|.$$

The expression on the right-hand side tends to zero uniformly in any bounded domain of $\operatorname{Re} n > -N$ excluding the negative integers, as $\lambda \rightarrow 0$. We then obtain that $F(1) \rightarrow 1$ as $\lambda \rightarrow 0$ uniformly in any bounded domain of n excluding the negative integers. Analogously one finds that $F'(1) \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly in the same domains. This information, when combined with (15a) and (15b), tells us that, as $\lambda \rightarrow 0$ the solutions in n either tend to infinity in the half-plane $\operatorname{Re} n \leq \alpha + 1$, $\alpha > -1$, or they tend to the non-negative integers. A more detailed result can be obtained by using Rouché theorem⁽²¹⁾ in connection with Eqs. (15).

Let us first consider the odd trajectories. Eq. (15b) can be rewritten in the form

$$(n+M)(F(1)-1) + (n+M) = 0, \quad (30)$$

where M is a positive integer.

The equation $n+M=0$ has only one solution inside the circle C defined by $|n+M| \leq r$ with $r < 1$ while the function $(n+M)(F(1)-1)$, which is regular inside C , becomes smaller than r on the boundary of C , $|n+M|=r$, as $\lambda \rightarrow 0$. Applying the Rouché theorem we find that as $\lambda \rightarrow 0$ one and only one odd solution must lie inside the circle C , and it must eventually tend to $-M$.

Rewriting Eq. (15a) in the form

$$n(n+M) + (n+M)[n(F(1)-1) + F'(1)] = 0$$

we can similarly conclude that as $\lambda \rightarrow 0$ one and only one even solution tends to each of the points $n = 0, -1, -2, \dots$.

When these properties on the n -plane are translated into corresponding properties on the ℓ -plane, we find that, as $\lambda \rightarrow 0$ all even poles retreat to the left of $\text{Re } \ell = \alpha$ and all odd poles to the left of $\text{Re } \ell = \alpha - 1$ where α satisfies $\alpha > -1$. Furthermore N even poles and $N-1$ odd poles tend, as $\lambda \rightarrow 0$, to each negative integer $-N$, of the ℓ -plane, and there are no other finite limiting points.⁽⁵⁾

C. Dominance of the first iteration as $\eta^2 \rightarrow -\infty$

We shall now prove that, as $\eta^2 \rightarrow -\infty$ the first iteration in the solution of Eq. (14) dominates, in the region $\text{Re } n < 0$, over the rest of the iterative series. This will have as a rigorous consequence that as $\eta^2 \rightarrow -\infty$ all poles must evade from any assigned bounded domain of the region $\text{Re } n < -1$.

The proof of the dominance of the first iteration goes as follows. The highest power of η^2 appearing in the first iteration is $N-2$, for terms of type (I), while terms of type (II) are majorized by powers of η^2 with exponent less or equal to $N-1$. In the second iteration a term $(\eta^2)^h x^a$, $1 \leq a \leq N-2$, gives rise to terms of type (I) with maximum power of η^2 , $s = h + b$ with $b \leq N-a-2$ and, as $h \leq a-1$, $s \leq N-3$; and to terms of type (II) which are majorized by $(\eta^2)^s$ with $s \leq h + N-a-1 \leq N-2$. Terms with $a = N-1$ go over, as a consequence of Lemma 4, to terms of type (II) with unaltered behaviour in η^2 . The terms of type (II) have their power decreased by 1, through Lemma 5. As a result we have, after N iterations, for $\eta^2 < 0$, $-N < \text{Re } n < 0$

$$K^N f = x^N \frac{g_n(x, \eta^2)}{p(n)}$$

with

$$\|g_n(x, \eta^2)\| < \text{const}(n).$$

The series $x^N [\tilde{K} \tilde{f} + \tilde{K}^2 \tilde{f} + \dots]$ tends to zero, as $\eta^2 \rightarrow -\infty$ except for the poles at the negative integers, since $\|\tilde{K}\| \rightarrow 0$ in that limit.

The evaluation of the first iteration is given in Section 5

(see Eq. 37) and we find that

$$F(1) = 1 + \frac{\lambda \pi (-4\eta^2)^{-(n+1)}}{n \sin \pi(n+1)} + \lambda R(n, \eta^2), \quad (31)$$

where for $\eta^2 < -1$ and $n < -\varepsilon < 0$

$$|R(n, \eta^2)| < |G(n)|((- \eta^2)^{-\operatorname{Re} n - 2} + (- \eta^2)^{-1}) + \text{const } (- \eta^2)^{-1+\varepsilon}, \quad (32)$$

$G(n)$ being a function that for $\operatorname{Re} n < -\varepsilon$ has only simple poles at the negative integers. The first term on the r.h.s. of Eq. (32) majorizes what is left over of the $[\sin \pi(n+1)]^{-1}$ term in Eq. (37), and the second term in Eq. (32) majorizes the remaining terms in Eq. (37). From this equation we see that the first iteration, as $\eta^2 \rightarrow -\infty$ behaves actually as $(-\eta^2)^{-(n+1)}$ for $\operatorname{Re} n < 0$ and thus dominates over the rest of the series. In fact we have seen that the second iteration is majorized by a power $(-\eta^2)^{N-2}$, the third by $(-\eta^2)^{N-3}$, etc., while the remainder of the series in Eq. (24) is majorized by a term constant in η^2 . In order to extend this result, about the dominance of the first iteration, to the integer values of n , e.g. to $\operatorname{Re} n < -N + 3/2$, we notice that terms which can asymptotically compete with the power $(-\eta^2)^{N-2}$ appear only in the second iteration. By using the variations 3.1, 4.1 and 5.1 of Lemmas 3, 4 and 5 (see Appendix B) we see that for $-N + \frac{1}{2} < \operatorname{Re} n$ all the terms appearing in the second iteration are majorized at infinity by a term $f(n) (-\eta^2)^{N-5/2}$. We also notice that the first iteration

exhibits a pole at each negative integer with a non-vanishing residue which behaves like $(-\eta^2)^{-(n+1)}$. $F'(1)$ can be evaluated analogously through Eq. (23). One finds the following asymptotic expressions

$$F(1) + \frac{1}{n} F'(1) \simeq 1 - \frac{\lambda \pi (-4\eta^2)^{-(n+1)}}{n \sin \pi(n+1)}$$

$$F(1) \simeq 1 + \frac{\lambda \pi (-4\eta^2)^{-(n+1)}}{n \sin \pi(n+1)}.$$

A first general conclusion on the trajectories can be derived from these results when compared with Eqs. (15). Since $(\sin x)^{-1} \gg e^{-|\operatorname{Im} x|}$

we have

$$\left| F(1) + \frac{1}{n} F'(1) \right| > 0$$

and

$$|F(1)| > 0$$

in any given bounded domain of $\operatorname{Re} n < -1$ for $-\eta^2$ sufficiently large, as soon as

$$\frac{\lambda \pi (-4\eta^2)^{-(n+1)}}{n \sin \pi(n+1)}$$

dominates over all the other terms. This means that for $\eta^2 \rightarrow -\infty$ all trajectories in the n -plane go either to ∞ in $\operatorname{Re} n < -1$ or move to $\operatorname{Re} n \geq -1$ ⁽⁵⁾. In Section 5 we shall examine a concrete realization of such a behaviour.

4. EXTENSION OF THE TREATMENT TO $\eta^2 > 1$

In this section we shall extend the treatment to the values $\eta^2 > 1$, i.e. above the physical threshold. In this region the Volterra integral equation (14) becomes singular and the usual methods for proving the convergence of the iterative series break down. We shall prove however that in the calculation of the series

$$F(1) = f(1) + K f(1) + K^2 f(1) + \dots$$

one can deform the integration contour from 0 to 1 to the complex semicircle shown in fig. 1, without changing the value of $F(1)$ ⁽⁵⁾.

The integrals taken on the new contour are no longer singular and thus the convergence of the series and its behaviour as $\lambda \rightarrow 0$ or $\eta^2 \rightarrow \infty$ can be studied easily. Similarly as for $\eta^2 < 1$ we shall first consider the region $\text{Re } n > -1$.

The position of the singularities of the kernel $K(x, y) = \theta(x-y) V(x, y)$ are given by

$$y = 1 \mp \left(\sqrt{1 - \frac{1}{\eta^2}} + i\varepsilon \right). \quad (33)$$

We want to investigate the Regge trajectories on the physical sheet and thus we must give η^2 a small positive imaginary part. This determines the sign of the imaginary part in Eq. (33). We consider now the cut plane shown in fig. 1 where the position of the branch points are given by (33).

Clearly, the integral

$$\int_0^x V(x, y) dy$$

taken along a path not crossing the cuts, is an analytic function of x

in the cut x -plane.

Similarly

$$\int_0^x V(x, y) dy \int_0^y V(y, z) dz$$

is an analytic function of x in the cut x -plane and the same holds for all the terms of our iterative series.

We now fix the integration contour along the semicircle C shown in fig. (1), and define a function

$$G = f + K^{(1)} f + K^{(2)} f + \dots,$$

where

$$K_n^{(1)}(\rho, \rho') = - \frac{\lambda}{4n} \left[1 - x^{-n} (2-x)^n y^n (2-y)^{-n} \right] \cdot \frac{e^{i(\frac{\pi}{2} - \rho')}}{1 - \eta^2 y (2-y)} \theta(\rho - \rho'),$$

where

$$x = \frac{1 - e^{-i\rho}}{2} \quad \text{and} \quad y = \frac{1 - e^{-i\rho'}}{2}.$$

In Appendix C it is shown that, for $\text{Re } n > -1$,

$$|K_n^{(1)}(\rho, \rho')| < \theta(\rho - \rho') k_n^{(1)}(\rho'), \quad (34)$$

$k_n^{(1)}(\rho')$ being an integrable function of ρ' . This assures us of the convergence of the series; each term is analytic in η^2 (22) and the convergence of the series is uniform. Therefore the series is analytic in η^2 ; for $\eta^2 < 1$ we have $G(\pi) = F(1)$ since the two series coincide term by term. For small λ we can repeat the

same argument as we did in Section 3. The only difference is that, due to the weaker bound (C.3) of Appendix C one obtains a weaker restriction on the position of the poles. From (C.3) we obtain that, as $\lambda \rightarrow 0$ or $\eta^2 \rightarrow +\infty$ the poles retreat to the left of that branch of the curve

$$\frac{e^{\frac{3}{4}\pi |\operatorname{Im} n|}}{n} = \text{const} \quad (35)$$

which lies in the region $\operatorname{Re} n > 0$. (23)

In Appendix C, some Lemmas, analogous to the various Lemmas we have used to perform the analytic continuation to $\operatorname{Re} n < -1$ are proved for $\eta^2 > 1$. We obtain again that $F(1)$ and $F'(1)$ are analytic functions of n except for simple poles at the negative integers. For $\lambda \rightarrow 0$ we have $F(1) \rightarrow 1$ and $F'(1) \rightarrow 0$ except at the negative integers; one can apply the same argument based on Rouché's theorem to obtain results similar to those obtained for $\eta^2 < 1$. The dominance of the first iteration as $\eta^2 \rightarrow +\infty$, for $\operatorname{Re} n < 0$ is proved in the same way as it was proved for $\eta^2 \rightarrow -\infty$. Again we obtain that as $\eta^2 \rightarrow +\infty$ the n -plane poles must either go to infinity or to the region $\operatorname{Re} n \geq -1$.

A detailed discussion of the first order iterative solution, for the whole range of η^2 is given in the following section.

5. DISCUSSION OF THE TRAJECTORIES AT FIRST ORDER

To gain some ideas on the behaviour of the trajectories we shall consider here the first order in λ .

The equation for the even time parity trajectories is

$$0 = F(1) + \frac{1}{n} F'(1) = 1 - \frac{\lambda}{2n} \int_0^1 \frac{[1 + y^n (2-y)^{-n}] dy}{1 - \eta^2 y (2-y)} \quad (36a)$$

For the trajectories of odd time parity the equation is

$$0 = F(1) = 1 - \frac{\lambda}{2n} \int_0^1 \frac{[1 - y^n (2-y)^{-n}] dy}{1 - \eta^2 y (2-y)}. \quad (36b)$$

To be definite we shall for the moment concentrate on the odd trajectories.

We shall treat later the even trajectories. We rewrite $F(1)$ in the form

$$\begin{aligned} F(1) &= 1 - \frac{\lambda}{2n} \left[\int_0^1 \frac{dy}{1 - \eta^2 y (2-y)} - \frac{1}{2\eta\sqrt{\eta^2-1}} \int_0^1 \left(\frac{1}{y-1-\sqrt{1-\eta^{-2}}} \right. \right. \\ &\quad \left. \left. - \frac{1}{y-1+\sqrt{1-\eta^{-2}}} \right) y^n (2-y)^{-n} dy \right] = \\ &= 1 - \frac{\lambda}{2n} \frac{1}{\eta\sqrt{1-\eta^2}} \operatorname{arctg} \frac{\eta}{\sqrt{1-\eta^2}} + \\ &+ \frac{\lambda}{4\eta\sqrt{\eta^2-1}} \frac{\pi}{n \sin \pi(n+1)} \left\{ [1 - 2(\eta^2 - \eta\sqrt{\eta^2-1})]^{-n} - [1 - 2(\eta^2 + \eta\sqrt{\eta^2-1})]^{-n} \right\} - \\ &- \frac{\lambda}{4\eta\sqrt{\eta^2-1}} \frac{1}{n} \left[\int_1^2 \frac{y^n (2-y)^{-n}}{y-1-\sqrt{1-\eta^{-2}}} dy - \int_1^2 \frac{y^n (2-y)^{-n}}{y-1+\sqrt{1-\eta^{-2}}} dy \right]. \end{aligned}$$

(37)

In the last step in Eq. (37) we have used the formula

$$\int_0^1 x^n (1-x)^{-n} (1-zx)^{-1} dx = \frac{\pi}{\sin \pi n} \frac{1}{2} \{ (1-z)^{-n} - 1 \}$$

which can be established starting from the standard integral representation for the hypergeometric function

$${}_2F_1(1, n+1; 2; z) = [(1-z)^{-n} - 1] / (nz).$$

A. The region $\eta^2 < 1$

For $\eta^2 < 0$ and $\text{Re } n \rightarrow -\infty$ one sees directly that on the r.h.s. of Eq. (37) the dominant terms are⁽²⁴⁾

$$1 - \frac{\lambda}{4\eta\sqrt{\eta^2-1}} \frac{\pi}{n \sin \pi(n+1)} [1 - 2(\eta^2 + \eta\sqrt{\eta^2-1})]^{-n}, \quad (38)$$

the remaining terms vanishing in the limit. For $\text{Im } n = 0$ the expression (38) diverges for $\text{Re } n \rightarrow -\infty$ showing that Eq. (37) cannot have real solutions when $\eta^2 < 0$ in a region $\text{Re } n < -K$ where K is a function of η^2 . On the other hand we know that at $\eta^2 = 0$ there are real solutions; from the dominant term (38) one sees that such solutions become complex through a mechanism of collision.

Let us consider the asymptotic form as $\eta^2 \rightarrow -\infty$ of the equation for the odd trajectories

$$1 + \frac{\lambda\pi}{n} \frac{(-4\eta^2)^{-(n+1)}}{\sin \pi(n+1)} = 0. \quad (39)$$

For $n \simeq -1$ the equation takes on the approximate form⁽⁵⁾

$$\left(\frac{\lambda}{n+1}\right)^{\frac{1}{n+1}} = -4\eta^2. \quad (40)$$

Eq. (40) has a solution with $n = \text{real} > -1$ which, as $\eta^2 \rightarrow -\infty$ tends to -1 , very slowly in η^2 . For $n' = 1$ such a solution has to be identified with the first daughter of the leading trajectory, which, by different approaches^{(10) (12)}, is found to be constant at the value $\ell = -2 + \lambda$.

Let us examine the asymptotic equation for the odd trajectories, Eq. (39) for $\text{Re } n < -1$. $(-4\eta^2)^{-(n+1)}$ is an increasing function of $-\eta^2$; this shows that there are pairs of solutions of Eq. (39) which as $-\eta^2$ increases, start from the neighbourhood of consecutive negative integers $(-2, -3)$ $(-4, -5)$..., move one against the other along the real axis, collide and turn into the complex plane orthogonally to the real axis. The preceding discussion, after Eq. (38) above, shows that the lower are the trajectories in the n -plane

the lower are the values of $-\eta^2$ for which the collision occurs [the situation is schematically illustrated in fig. (5)].

To understand the behaviour of the trajectories in the complex n -plane after their collision (clearly there are two complex conjugate branches) we note (using the inequality $|\sin x| \leq e^{|I_m x|}$) that the solutions must lie in the n -plane to the right of the line defined by

$$\ln(-4\eta^2) = \frac{\pi |I_m n| + \ln \left| \frac{n}{\lambda\pi} \right|}{- \text{Re } n - 1} \quad (41)$$

As $\eta^2 \rightarrow -\infty$ this line approaches the axis $\text{Re } n = -1$.

Let us now examine where the solutions cross the axis $\operatorname{Re} n = -1$.

We must have

$$\sinh(\pi \operatorname{Im} n) = \frac{i\pi\lambda}{-1+i \operatorname{Im} n} e^{-i(\operatorname{Im} n) \ln(-4\eta^2)}. \quad (43)$$

Equating the absolute values gives

$$|\sinh(\pi \operatorname{Im} n)| = \frac{\lambda\pi}{\sqrt{1+(\operatorname{Im} n)^2}}. \quad (42)$$

Such an equation is satisfied only for a well-defined value of

$|\operatorname{Im} n|$ which we indicate with I . One can also verify that there are infinite values of η^2 accumulating to $-\infty$ for which the phases of the two sides of Eq. (42) coincide. This means that, as $-\eta^2$ increases, an infinite number of solutions passes through the points $n = -1 \pm iI$. Inside the strip $-1 \leq \operatorname{Re} n < 0$ where the

asymptotic form of the equation is still given by Eq. (39) one easily

sees that as $-\eta^2 \rightarrow \infty$ the solutions tend to -1 . For $\varepsilon = n+1$,

$|\varepsilon| \ll 1$, we are back to Eq. (40) which we want now to consider

for complex ε . Writing $\varepsilon = \rho e^{i\alpha}$ one easily finds that along

each ray $\alpha = \text{const}$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, as $\eta^2 \rightarrow -\infty$

there are infinite intercepts of decreasing ρ which accumulate to $\rho = 0$.

On the other hand we have no accumulation of intercepts at $\rho = 0$ for

$\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$. The general picture of the trajectories that we

derive from these considerations is therefore the following: two

solution, as $-\eta^2$ increases, start from the neighbourhood of two

consecutive negative integers, approach each other along the real axis,

collide and go into complex values at a right angle with respect to the real axis. They move afterwards on the complex plane towards the right, cross the line $\text{Re } n = -1$ near the points $-1 + i I$ and $-1 - i I$, terminating into $n = -1$ for $\eta^2 \rightarrow -\infty$.

The discussion of the even time-parity trajectories goes along the same lines and one reaches similar results, with the following changes: (i) the real solution which goes to $-1 + 0$ as $\eta^2 \rightarrow -\infty$ is now absent; (ii) the pairs of solutions (which approach each other along the real axis and collide) start this time from the neighbourhoods of the negative integers $(-1, -2)$ $(-3, -4)$...

[instead of $(-2, -3)$ $(-4, -5)$...] (5)

The asymptotic formula (39) has provided us with an example of the behaviour of the trajectories for large negative η^2 . It appears that the situation is new with respect to potential scattering, where one knows that for regular potentials, as $E \rightarrow -\infty$ the trajectories go to negative integers along the real axis. (25) The origin of the different behaviour is clearly related to the singularity of the energy-dependent term in the Cutkosky equation, as $\eta^2 \rightarrow \infty$. Through the collision mechanism the trajectories comply with the general requirement derived at the end of Section 3.

We shall now discuss the odd trajectories for $0 \leq \eta^2 \leq 1$. In the region $0 \leq \eta^2 \leq 1$ the expression $1 - 2(\eta^2 - \eta \sqrt{\eta^2 - 1})$ has modulus 1 and an argument α which as η^2 increases from 0 to 1 goes from 0 to π . Therefore the third term in the final expression for $F(1)$ in Eq. (37) can be written as

$$\frac{\lambda \pi}{2 \eta \sqrt{1 - \eta^2}} \frac{\sin \alpha n}{n \sin \pi n} \quad (44)$$

It is apparent from the last expression, inserted in Eq. (37) that for small λ the trajectories [whose intercepts for $\eta^2 = 0$ are those in Eq. (16b)] oscillate on the real n -axis about the negative integers when η^2 increases from $\eta^2 = 0$ to $\eta^2 = 1$. More specifically one can deduce that the solution around $-N$ makes $N-1$ oscillation for $0 \leq \eta^2 \leq 1$ ⁽²⁶⁾. For $\eta^2 \rightarrow 1$ the trajectories go to the negative integers.

B. Threshold behaviour of the trajectories

The threshold behaviour of the trajectories can be obtained as follows. In Eq. (36b) as $\eta^2 \rightarrow 1$ the integrand becomes singular at $y = 1$ giving rise to a logarithmic singularity of the integral. Besides, the integral is singular in n at the negative integers. The compensation of the singular parts in Eq. (36b) leads to the equation

$$0 = -\frac{\lambda}{n+N} - \frac{\lambda}{2} \ln \frac{1}{1-\eta^2} \quad (45)$$

from which one obtains

$$n = -N - \frac{2}{\ln \frac{1}{1-\eta^2}}. \quad (46)$$

Above the threshold for $\eta^2 \rightarrow 1+$ Eq. (46) becomes

$$n = -N - \frac{2}{\ln \frac{1}{\eta^2-1} + i\pi}. \quad (47)$$

Eq. (47) shows that for η^2 just above the threshold, n takes on a positive imaginary part. Furthermore one can show that for $1 < \eta^2 < \infty$ Eq. (36b) [and similarly for Eq. (36a)] has no real solutions. In fact

$F(1)$ for $1 < \eta^2 < \infty$ and n real has an imaginary part (arising from the singularity due to the denominator of the integral) which never vanishes in the above interval of η^2 . Therefore for $1 < \eta^2 < \infty$ these trajectories remain in the upper half n -plane.

Let us now discuss the even trajectories. The discussion we have just made for the odd trajectories applies again apart from the following differences. The sign difference in Eq. (36a) with respect to Eq. (36b) produces a modification of the threshold behaviour which was given by Eq. (46) for the odd trajectories. Furthermore the r.h.s. of Eq. (36a) has a pole at $n = 0$ (in addition to the poles at the negative integers). For $\eta \simeq -N$ ($N = 1, 2 \dots$) Eq. (46) is now replaced by

$$0 = \frac{\lambda}{n+N} - \frac{\lambda}{2n} - \frac{\pi}{\sqrt{1-\eta^2}} \quad (48)$$

giving

$$n = -N - 2N \sqrt{1-\eta^2} \quad (49)$$

for $\eta^2 \rightarrow 1-$ and

$$n = -N + 2iN \sqrt{\eta^2 - 1} \quad (50)$$

for $\eta^2 \rightarrow 1+$. In addition one now has a solution (because of the pole at $n = 0$) which behaves as

$$n = \frac{\lambda\pi}{2} \frac{1}{\sqrt{1-\eta^2}} \quad (51)$$

for $\eta^2 \rightarrow 1^-$ and as

$$n = \frac{i \lambda \pi}{2 \sqrt{\eta^2 - 1}} \quad (52)$$

for $\eta^2 \rightarrow 1^+$. For $1 < \eta^2 < \infty$ these trajectories have a positive imaginary part.

Solving Eq. (36a) in the neighbourhood of $n = 0$ gives

$$n = \lambda \int_0^1 \frac{dy}{1 - \eta^2 y(2-y)} = \frac{\lambda}{\eta \sqrt{1-\eta^2}} \operatorname{arctg} \frac{\eta}{\sqrt{1-\eta^2}} \quad (53)$$

which for $n' = 1$, $\ell = n - n'$, reproduces exactly the well-known perturbation expression for the leading trajectory first found by Lee and Sawyer⁽⁸⁾ (for exchange of a massive boson). It is worth noting that for $\mu^2 \neq 0$ the divergence exhibited at the threshold ($\eta^2 = 1$) by Eq. (53) is a consequence of the approximation (vicinity of $n = 0$) and in fact cannot be exhibited by the actual trajectories.

In the Wick-Cutkosky model, Eq. (51), which we have derived for the threshold behaviour, shows that the singularity is actually present, and that Eq. (53) also holds in the neighbourhood of $\eta^2 = 1$.

The behaviour of the leading trajectory as given by Eq. (53) is the following: it is real for $\eta^2 < 1$, has intercept at $-1 + \lambda$ for $\eta^2 = 0$, and approaches -1 when $\eta^2 \rightarrow -\infty$ differing asymptotically from -1 by a term behaving as $\eta^{-2} \ln(-\eta^2)$.

C. The region $\eta^2 > 1$

A common character of the trajectories discussed so far, obtained from the first iteration solution, is that for $\eta^2 > 1$ they lie in

the upper half of the n -plane. A reasoning similar to that made for $\eta^2 \rightarrow -\infty$ shows that these trajectories, as η^2 increases, eventually turn to the right, cross the axis $\text{Re } n = -1$, and tend to $n = -1$ as $\eta^2 \rightarrow +\infty$.

A more detailed analysis of the first iteration, to be reported below, shows however that for $\eta^2 > 1$ additional trajectories exist which lie on the lower half of the n -plane.

Again fixing our attention on the odd-time-parity trajectories (similar considerations hold for even-time-parity) we consider the asymptotic expression for $\eta^2 \rightarrow +\infty$ in the region $\text{Re } n < 0$ of Eq. (36b). One finds

$$0 = \frac{1 - e^{-2i\pi(n+1)}}{2i} + \frac{\lambda\pi}{n} (4\eta^2)^{-(n+1)}. \quad (54)$$

We can easily see that the trajectories given by (54) cross the axis

$\text{Re } n = -1$ at either of the two points $n_1 = -1 + iI_1$ and $n_2 = -1 + iI_2$, where I_1 and I_2 are the solutions of

$$\left| \frac{1 - e^{2\pi I}}{2} \right| = \frac{\lambda\pi}{\sqrt{1 + I^2}}. \quad (55)$$

One has $I_1 > 0$ and $I_2 < 0$. As $\eta^2 \rightarrow \infty$ an infinite number of solutions of Eq. (54) pass through each of the two points n_1 and n_2 . This indicates the existence of an infinite number of trajectories on the lower half of the n -plane for $\eta^2 > 1$. A rigorous argument can be developed as follows. One rewrites Eq. (36b)

in the form

$$0 = \mathcal{F}(n, \eta^2) + \mathcal{G}(n, \eta^2), \quad (56)$$

where

$$\mathcal{F}(n, \eta^2) = \frac{1 - e^{-2i\pi(n+1)}}{2i} + \frac{\lambda\pi}{4n\eta\sqrt{\eta^2-1}} [2(\eta^2 + \eta\sqrt{\eta^2-1}) - 1]^{-n} \quad (57)$$

and $\mathcal{G}(n, \eta^2)$ vanishes as $\eta^2 \rightarrow \infty$ in $\text{Re } n < 0$. The

approximate equation $\mathcal{F}(n, \eta^2) = 0$ has solutions which as

$\eta^2 \rightarrow \infty$ cross the axis $\text{Re } n = -1$ at points which for $\text{Im } n < 0$

tend to n_1 . By application of Rouché's theorem one can show that

the presence of the term $\mathcal{G}(n, \eta^2)$ does not change this conclusion.

When $\lambda \rightarrow 0$ these solutions behave quite differently from the solutions we have treated before, which in that limit were found to

approach the negative integer values of n . For $\lambda \rightarrow 0$ one can

see that the new solutions tend to infinity in the lower half of the

n -plane. In fact we have already shown (see Section 3) that for

$\lambda \rightarrow 0$ all trajectories go either to the negative integers or to infinity. It can be established, by solving Eq. (36b) in the limit of

small λ and for n close to a negative integer, that the trajec-

tories which tend to the negative integers for $\lambda \rightarrow 0$ approach such

limits from the upper-half of the n -plane when $\eta^2 > 1$. Since for

$\eta^2 > 1$, as we have already shown, the trajectories cannot cross the

real n -axis, the only possibility for the trajectories on the lower-

half n -plane is to tend to infinity when $\lambda \rightarrow 0$. This feature

of the new trajectories we have found, for $\eta^2 > 1$, lying on the

lower-half n -plane may lead one to suspect that they may only appear in the approximate solution and could in fact not be present in the exact solution. In this case they would be "spurious" trajectories due to the mathematical approximation. However, one can show that at $\eta^2 = 0$ the first iteration Eq. (36b) only gives solutions which for $\lambda \rightarrow 0$ tend to the negative integers. Thus at least at $\eta^2 = 0$ the first iteration gives only those solutions which approximate the exact ones [see Eqs. (16)]. To verify the stated behaviour of the first iterative solutions at $\eta^2 = 0$, one notices that at such a value of η^2 Eq. (36b) can be written in the form

$$0 = F(1) = 1 - \frac{\lambda}{2n} (1 + S) + \frac{\lambda \pi}{\sin \pi n}, \quad (58)$$

where

$$S = \int_1^2 y^n (2-y)^{-n} dy. \quad (59)$$

For $\text{Re } n < 0$ one has $|S| \leq 1$. This limitation on S is sufficient to show that for sufficiently small λ all solutions must be contained inside a set of circles of radius $R(\lambda)$ centred at the negative integers, where $R(\lambda) \rightarrow 0$ for $\lambda \rightarrow 0$. As we know that for $\lambda \rightarrow 0$ only one solution tends to each negative integer, we have that at $\eta^2 = 0$ the first iteration gives only those solutions which approximate the exact ones and no solution tends to infinity as $\lambda \rightarrow 0$.

D. Behaviour near $\eta^2 = 0$ and mixing of trajectories

The exact values of the intercepts at $\eta^2 = 0$ are given by Eqs. (16). Therefore in the ℓ -plane we have at $\eta^2 = 0$

$$\ell = -n' - \kappa - \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda}. \quad (60)$$

The integer κ is the nodal quantum number ($\kappa = 0, 1, 2, \dots$)

whose parity coincides with the time parity of the solution φ .

Accordingly the trajectories around $\eta^2 = 0$ can be identified by symbols $\kappa_+(n')$ and $\kappa_-(n')$ where $n' = 1, 2, 3, \dots$

The equations (14) and (15) can also be used to give power series expansions of the trajectories around the point $\eta^2 = 0$ ⁽²⁷⁾. To first order in λ and η^2 we obtain

$$\ell = -n' - \kappa + \lambda \left\{ 1 - \frac{2}{3} \eta^2 (\kappa^2 - 1) \right\} \text{ for } \kappa_+(n') \quad (61a)$$

$$\ell = -n' - \kappa - 1 - \lambda \left\{ 1 - \frac{2}{3} \eta^2 \kappa (\kappa + 2) \right\} \text{ for } \kappa_-(n'). \quad (61b)$$

The derivatives with respect to η^2 do not satisfy Domokos' formula. ⁽²⁸⁾

The reason is that the $O(4)$ -degeneracy of the Wick-Cutkosky model brings about a mixing of the various Toller families. ⁽⁵⁾ Such mixing can be

already verified in the physical bound state solutions; for example, the wave functions for $n = 3$, $\ell = 0$, $\kappa = 0$, and $n = 1$, $\ell = 0$

$\kappa = 2$ are both superpositions of the wave functions belonging to the symmetric representation of the Lorentz $O(4)$, with four-dimensional angular momentum 0 and 2. This degeneracy is typical of the point

$\eta^2 = 0$, because the integer spacing of the trajectories at η^2 due to the Lorentz $O(4)$ invariance is superimposed to the integer spacing due to the Fock $O(4)$ invariance which holds for all values of η^2 . For $\mu^2 \neq 0$ (μ = mass of the exchanged particle) this last degeneracy is removed, and one expects two infinite sequences of Toller families whose intercepts at $\eta^2 = 0$ tend for $\mu^2 \rightarrow 0$ to the values given in Eq. (60).

6. EXISTENCE OF COLLISIONS FOR THE EXACT SOLUTIONS

We want to show now that collision phenomena, such as those discussed in the preceding section, really occur in the exact solution, i.e. after summing the iteration series, at least for sufficiently small λ . To prove this we shall use again, in a more detailed form, our result of the dominance of the first iteration.

We shall consider here for definiteness the equation for the odd trajectories. It is easily seen that in the region $-N < \text{Re } n < 0$ the remainder which we call S of the iteration series, to be added to the l.h.s. of Eq. (31), is majorized for λ less than some fixed number λ_0 and for $\eta^2 < -1$ by

$$|S(n, \eta^2)| < \lambda^2 |H(n)| (-\eta^2)^{-n-5/4}, \quad (62)$$

where $H(n)$ has only simple poles at the negative integers. The proof of such a statement is as follows. We have proved in Section 3 that the second iteration is majorized, for $-N < \text{Re } n < 0$, by $(-\eta^2)^{N-2} |\varphi(n)|$ and for $-N + \frac{1}{2} < \text{Re } n < 0$, by $(-\eta^2)^{N-5/2} |T(n)|$, where $Q(n)$ and $T(n)$ are functions which in the region $-N < \text{Re } n < 0$ have only simple poles at the negative integers. Higher iterations are majorized by still lower powers of η^2 and the series in λ is convergent. Therefore, for λ less than a given λ_0 we can write the inequality (62). For $-\eta^2$ larger than a certain $-\eta_0^2$ and $\lambda < \lambda_0$ the residues of the poles of

$$\frac{\pi}{n} \lambda \frac{(-4\eta^2)^{-n-1}}{\sin \pi(n+1)}$$

are bigger than the residues of the corresponding poles of the remaining terms R and S , for $-N < n < 0$. The possibility of cancellation of the residues is thus excluded. However, for λ sufficiently small but finite, the exact equation

$$F(1) = 1 + \frac{\lambda \pi (-4\eta^2)^{-n-1}}{n \sin \pi(n+1)} + \lambda R(n, \eta^2) + S(n, \eta^2) = 0$$

has necessarily at least two real solutions inside each interval

$$-N < -2k-1 < n < -2k, \quad k = \text{integer} \gg 1.$$

Keeping now λ fixed and making $-\eta^2 \rightarrow \infty$,

the general result of Section 3 tells us that no solution can remain on the real axis for $-N < n < -2$; also the solutions are prevented from escaping along the real axis because of the poles at the integers, whose residues, as we have shown, cannot vanish for $-\eta^2 > \eta_0^2$; therefore the solutions have to become complex through a collision process. (5)

Similar considerations can be repeated for the even trajectories. These collisions of Regge poles are similar to those discovered by Cutkosky and Deo⁽¹³⁾; they examined the Regge trajectories for $\text{Re } l > -3/2$ by solving numerically the Bethe-Salpeter equation for two unequal mass spinless bosons, interacting through the exchange of a spinless boson of non-vanishing mass, in the ladder approximation.

The collisions we have found occur in the region $\text{Re } n < 0$; we shall now see that no collisions can occur in our problem in the region

$\text{Re } n > 0$ for $\eta^2 < 1$, i.e. below the physical threshold.

[By the term collision we mean the situation of two real trajectories which after colliding become complex.] To prove this we reduce the Cutkosky equation to self-adjoint form by putting $h(z) = (1-z^2)^{-\frac{n+1}{2}} g(z)$. We obtain

$$(1-z^2) \frac{d^2}{dz^2} [(1-z^2)h(z)] + \frac{\lambda(1-z^2)}{1-\eta^2(1-z^2)} h(z) = (n^2-1)h(z) \quad (63)$$

together with the condition $h(z) \sim (1-z^2)^{\frac{n-1}{2}}$ as $z \rightarrow \pm 1$.

This equation is of the form

$$Hh = (n^2-1)h,$$

where H for $\eta^2 < 1$ is a self-adjoint analytic operator. For $\text{Re } n > 0$ the eigenvalue equation coincides with our problem. Since only one of the two solutions of the indicial equation at $z = \pm 1$ corresponds to a square integrable function, the eigenvalues of H which we call $n_k^2 - 1$ are always non-degenerate. From the self-adjointness of H one derives that for $\eta^2 < 1$ the trajectories with $\text{Re } n_k > 0$ are real. From the non-degeneracy of the eigenvalues it follows that two trajectories in $\text{Re } n > 0$ can never cross for $\eta^2 < 1$ ⁽²⁹⁾. Furthermore, one can easily derive from (63) that in the region $\text{Re } n > 0$ and for $\eta^2 < 1$ the inequalities $\frac{\partial n_k}{\partial \lambda} > 0$, $\frac{\partial n_k}{\partial \eta^2} > 0$, hold. The fact that Cutkosky and Deo found collisions also for $\text{Re } \ell > -1$ does not contradict our result. In fact the problem they have examined, i.e. unequal masses bound by the exchange of a non-zero mass particle, is not reducible to an eigenvalue problem, in ℓ , for a self-adjoint operator.

7. CONCLUSIONS

We have studied in this work the Regge trajectories of the Wick-Cutkosky model. To such a purpose we have developed a procedure of analytical continuation into the complex ℓ -plane of the bound state solutions, based on the transformation of the Cutkosky differential equation into a Volterra integral equation, whose solutions are proved to be analytic on the whole complex plane except for simple poles at the negative integers. Using such a procedure we have proved the following rigorous results:

(i) For $\lambda \rightarrow 0$ where λ is the square of the coupling constant, N Regge poles even in the relative time, together with $N-1$ poles odd in the relative time, tend to the negative integer $-N$.

(ii) For $\text{Re } n < 0$ where n is the Cutkosky quantum number, the first iteration, as η^2 tends to $+\infty$ or $-\infty$, dominates over the remainder of the iterative solution of the Volterra equation. From this result we find that as $\eta^2 \rightarrow \pm \infty$ no poles remain inside any assigned bounded domain of the region $\text{Re } n < -1$. For λ smaller than a certain value, the evasion for $\eta^2 \rightarrow -\infty$ is accomplished through a collision mechanism by which two colliding real solutions become complex conjugate.

(iii) At $\eta^2 = 0$ the degeneracy implied by the Fock $O(4)$ symmetry of the problem gives rise to mixings among trajectories belonging to different Toller families.

Besides the above rigorous results, more detailed information on the behaviour of the Regge trajectories has been obtained through a study of the solutions at first order in λ . One verifies that the first iteration exhibits all the features described in (i), (ii) and (iii). In the limit $\eta^2 \rightarrow -\infty$ the even time parity trajectories (except the leading trajectory) after colliding [according to (ii)] move to the right on the ν -plane and terminate at $\nu = -1$. The odd time parity trajectories except the leading one, also collide, move to the right and terminate at $\nu = -1$; the leading one also tends to -1 as $\eta^2 \rightarrow -\infty$ but is not subject to collisions. For $0 < \eta^2 < 1$ the trajectories are real and oscillate around the negative integers; at $\eta^2 = 1$ they pass through the negative integers and assume afterwards a positive imaginary part, tending (except the even time parity leading trajectory) to $\nu = -1$ for $\eta^2 \rightarrow +\infty$. The even time parity leading trajectory for $-\infty < \eta^2 < 1$ goes from 0 to $+\infty$ along the real axis, is complex for $1 < \eta^2 < \infty$, and tends to $\nu = 0$ for $\eta^2 \rightarrow +\infty$. The solutions just described are the only ones which at $\eta^2 = 0$ assume the values given by the Cutkosky spectrum, for the first iteration zero energy intercept. For $\eta^2 > 1$ one finds additional trajectories: in particular there are trajectories which lie entirely in the lower half of the ν -plane and which for $\lambda \rightarrow 0$ tend to infinity and are thus distinct from the previous trajectories. More specific results concerning threshold behaviours and power series expansion are discussed in the text.

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APPENDIX A

We shall prove here the relation

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} Q_e \left(\frac{p^2 + k^2 + (p_+ - k_+)^2}{2pk} \right) \frac{k^{\ell+1} dk dk_+}{[k^2 + k_+^2 + 2iz\eta k_+ + 1 - \eta^2]^{\ell+3}} = \\ & = \frac{1}{(\ell+2)(\ell+1)} \frac{1}{1 - \eta^2(1-z^2)} \frac{p^{\ell+1}}{[p^2 + p_+^2 + 2iz\eta p_+ + 1 - \eta^2]^{\ell+1}} \quad (A.1) \end{aligned}$$

for all values of ℓ for which the integral on the l.h.s. converges.

This region contains the half-plane $\text{Re } \ell \geq 0$. The r.h.s. gives then the analytic continuation of the l.h.s. for all ℓ -values.

We have already seen in the text that the relation is true for $\eta^2 < 1$ and $\ell = 0, 1, 2 \dots$. We shall now choose for simplicity to prove the equality of the two members for a particular range of η namely for

$$0 < \eta^2 < \frac{1}{2}$$

and for all p, p_+ .

As both members are analytic in η the integral in Eq. (A.1) coincides with the expression on the r.h.s. whenever it converges. To prove Eq. (A.1) we shall use Carlson's theorem⁽³⁰⁾. To this end we shall make sure that the two members of (A.1) as $|\ell| \rightarrow \infty$ in $\text{Re } \ell \geq 0$ are of the type $O(e^{k|\ell|})$ with $k < \pi$.

Let us study the asymptotic behaviour of

$$A = \frac{p^{\ell+1}}{(\ell+2)(\ell+1) [1 - \eta^2(1-z^2)] [p^2 + p_+^2 + 2iz\eta p_+ + 1 - \eta^2]^{\ell+1}}.$$

We have

$$\begin{aligned}
 |A| &\leq \frac{p^{\operatorname{Re} l+1}}{2 [1-\eta^2(1-z^2)] (p^2+p_+^2+1-\eta^2)^{\operatorname{Re} l+1} e^{-|\operatorname{Im} l| \operatorname{arctg} \left(\frac{2|z\eta p_+|}{p^2+p_+^2+1-\eta^2} \right)}} \\
 &\leq \frac{1}{2 [1-\eta^2(1-z^2)]} \left(\frac{1}{2\sqrt{1-\eta^2}} \right)^{\operatorname{Re} l+1} e^{|\operatorname{Im} l| \delta(\eta)} \leq \\
 &\leq \frac{1}{2 [1-\eta^2(1-z^2)]} \frac{1}{2\sqrt{1-\eta^2}} e^{[-\ln \sqrt{1-\eta^2} + \delta(\eta)] |l|}
 \end{aligned}$$

for

$$0 < \eta^2 < 1$$

with

$$\delta(\eta) = \frac{|\eta|}{\sqrt{1-\eta^2}}.$$

Thus for $0 < \eta^2 < \frac{1}{2}$ we have

$$|A| = \frac{e^{(1+\frac{\ln 2}{2})|l|}}{4 [1-\eta^2(1-z^2)] \sqrt{1-\eta^2}}.$$

We notice that

$$1 + \frac{\ln 2}{2} < \pi.$$

We examine now the l.h.s. of Eq. (A.1). From the representation of the functions Q_e

$$Q_e(z) = \frac{1}{2^{e+1}} \int_{-1}^{+1} \frac{(1-t^2)^e}{(z-t)^{e+1}} dt$$

valid for $\operatorname{Re} l > -1$, one can derive the following properties.

For $z = x = \text{real}$, $x > 1$, $\text{Re } \ell > -1$

$$(i) \quad |\varphi_\ell(x)| \leq \varphi_{\text{Re } \ell}(x)$$

$$(ii) \quad \text{for } \ell \text{ real, } \ell > 0 \quad \varphi_\ell(x) < \varphi_0(x)$$

$$(iii) \quad \frac{\partial \varphi_0(x)}{\partial x} < 0.$$

We have

$$\begin{aligned} & \left| \frac{2}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} \varphi_\ell \left(\frac{p^2 + k^2 + (p_+ - k_+)^2}{2pk} \right) \frac{k^{\ell+1} dk dk_+}{[k^2 + k_+^2 + 2iz\eta k_+ + 1 - \eta^2]^{\ell+3}} \right| \leq \\ & \leq \frac{2}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} \varphi_0 \left(\frac{p^2 + k^2}{2pk} \right) \frac{k^{\text{Re } \ell + 1} e^{|\text{Im } \ell| \delta(\eta)}}{[k^2 + k_+^2 + 1 - \eta^2]^{\text{Re } \ell + 3}} dk dk_+ \leq \\ & \leq \frac{2}{\pi} e^{|\ell| \{ \delta(\eta) - \ln \sqrt{1 - \eta^2} \}} \int_0^\infty \int_{-\infty}^{+\infty} \varphi_0 \left(\frac{p^2 + k^2}{2pk} \right) \frac{k dk dk_+}{[k^2 + k_+^2 + 1 - \eta^2]^3} \end{aligned}$$

and

$$\int_0^\infty \varphi_0 \left(\frac{p^2 + k^2}{2pk} \right) \frac{k dk}{[k^2 + 1 - \eta^2]^{5/2}} < \infty.$$

Thus both members of Eq. (A.1) are of the type $O(e^{k|\ell|})$ in

$\text{Re } \ell \geq 0$ with $k < \pi$. They coincide at the integers $\ell = 0, 1, 2, \dots$; therefore by Carlson's theorem they coincide for all ℓ .

APPENDIX B

We prove the following lemmas:

Lemma 1. The operator $X^{-N} K_n(x, y) y^N$, with N non-negative integer, for $\eta^2 < 1$, $\text{Re } n > -N-1$ is (a) bounded; (b) analytic in n ; (c) in the half-plane $\text{Re } n > \alpha > -N-1$, its norm tends uniformly to zero when $\eta^2 \rightarrow -\infty$.

Proof

Boundedness and behaviour of the norm.

We distinguish two cases: (i) $|n| > \frac{1}{2}$; (ii) $|n| \leq \frac{1}{2}$

We begin with

(i) $|n| > \frac{1}{2}$.

We have

$$\begin{aligned} & \left| -\frac{\lambda}{2n} \left(\frac{x}{y}\right)^{-N} \left[1 - x^{-n} (2-x)^n y^n (2-y)^{-n} \right] \frac{\Theta(x-y)}{1-\eta^2 y(2-y)} \right| \leq \\ & \leq \frac{\lambda}{2|n|} \left\{ 1 + \left[\begin{matrix} 1 \\ y^{\text{Re } n + N} \end{matrix} \right] \left(\begin{matrix} 1 \\ 2^{-\text{Re } n} \end{matrix} \right) \right\} \frac{\Theta(x-y)}{1-\eta^2 y(2-y)} = \\ & = \Theta(x-y) \tilde{k}_n(y), \end{aligned}$$

where the upper value in the square bracket refers to $\text{Re } n > -N$, the lower to $-N-1 < \text{Re } n < -N$; the upper value in the round bracket refers to $\text{Re } n > 0$, the lower to $-N-1 < \text{Re } n < 0$.

For

$$(ii) \quad |n| \leq \frac{1}{2}$$

we have

$$\begin{aligned} & \left| -\frac{\lambda}{2n} \left(\frac{x}{y}\right)^{-N} \left[1 - x^{-n} (2-x)^n y^n (2-y)^{-n} \right] \frac{\Theta(x-y)}{1-\eta^2 y(2-y)} \right| \leq \\ & \leq \frac{\lambda}{2} \max_{|n| \leq \frac{1}{2}} \left| \left(\frac{x}{y}\right)^{-N} \left(\frac{y(2-x)}{x(2-y)}\right)^n \ln \frac{y(2-x)}{x(2-y)} \Theta(x-y) \right| \cdot \\ & \cdot \frac{1}{1-\eta^2 y(2-y)} \leq \frac{\lambda}{\sqrt{2}} \frac{1}{\sqrt{y}} \ln\left(\frac{2}{y}\right) \frac{\Theta(x-y)}{1-\eta^2 y(2-y)} = \\ & = \tilde{k}_n(y) \Theta(x-y). \end{aligned}$$

We have in both cases

$$\int_0^1 \tilde{k}_n(y) dy < \infty.$$

One sees directly that as $\eta^2 \rightarrow -\infty$ this integral tends to zero uniformly in $\operatorname{Re} n > \alpha > -N-1$.

This proves the points a and c.

Analyticity in n .

We have to prove that for $\operatorname{Re} n > -N-1$

$$\lim_{|n'-n| \rightarrow 0} \left\| x^{-N} K'_n(x, y) y^N - \frac{x^{-N} K_{n'}(x, y) y^N - x^{-N} K_n(x, y) y^N}{n' - n} \right\| = 0.$$

Introducing Taylor's residue we have

$$\begin{aligned}
 & \left| x^{-N} K_n'(x, y) y^N - x^{-N} \frac{K_{n'}(x, y) - K_n(x, y)}{n' - n} y^N \right| \leq \\
 & \leq \frac{|n' - n|}{2} \sup_{|n'' - n| \leq |n' - n|} \left| x^{-N} K_{n''}''(x, y) y^N \right| \leq \\
 & \leq \frac{|n' - n|}{2} \frac{\lambda}{2} \sup_{|n'' - n| \leq |n' - n|} \left\{ \frac{2}{|n''|^3} + \right. \\
 & + \left[\frac{1}{y^{Re n'' + N}} \right] \left(\frac{1}{2^{-Re n''}} \right) \left(\frac{2}{|n''|^3} + \frac{2}{|n''|^2} \ln \left(\frac{2}{y} \right) + \frac{1}{|n''|} \ln^2 \left(\frac{2}{y} \right) \right) \Big\} \cdot \\
 & \cdot \frac{\theta(x-y)}{1 - \eta^2 y(2-y)}.
 \end{aligned}$$

The integral over y from 0 to 1 of this quantity is finite and tends to zero as $n \rightarrow n'$. For the point $n = 0$ it is sufficient to note that $x^{-N} K_n y^N$ is continuous at $n = 0$ as can be seen by using the following inequality:

$$\begin{aligned}
 & \left| \frac{1}{n} \left[1 - \left(\frac{x(2-y)}{y(2-x)} \right)^{-n} \right] - \ln \frac{x(2-y)}{y(2-x)} \right| \theta(x-y) \leq \\
 & \leq \frac{|n|}{2} \ln^2 \frac{x(2-y)}{y(2-x)} \sup_{|n'| < |n|} \left(\frac{x(2-y)}{y(2-x)} \right)^{-Re n'} \theta(x-y) \leq \\
 & \leq \frac{|n|}{2} \ln^2 \left(\frac{2}{y} \right) \cdot \left(\frac{2}{y} \right)^{|n|} \theta(x-y).
 \end{aligned}$$

The integral of this quantity from 0 to 1 tends to zero as $|n| \rightarrow \infty$.

Lemma 2

The operator

$$x^{-N} H_n(x, y) y^{N-1}$$

with

$$H_n(x, y) = -\frac{\lambda}{2|n|} [1 - x^{-n}(2-x)^n y^n (2-y)^{-n}] \Theta(x-y),$$

N positive integer, for $\operatorname{Re} n > -N$ is bounded and analytic and we have

$$\mathcal{B}_N(n) \equiv \|x^{-N} H_n y^{N-1}\| \leq \frac{\lambda}{2|n|} \left\{ \frac{1}{N} + \binom{1}{2^{-\operatorname{Re} n}} \frac{1}{N + \operatorname{Re} n} \right\},$$

where the upper value in the round bracket refers to $\operatorname{Re} n > 0$

and the lower to $-N < \operatorname{Re} n < 0$.

Proof

$$\begin{aligned} \int_0^1 |x^{-N} H_n(x, y) y^{N-1}| dy &\leq \frac{\lambda}{2|n|} \left\{ x^{-N} \int_0^x y^{N-1} dy + \right. \\ &+ \left. \binom{1}{2^{-\operatorname{Re} n}} x^{-N-\operatorname{Re} n} \int_0^x y^{N-1+\operatorname{Re} n} dy \right\} = \\ &= \frac{\lambda}{2|n|} \left\{ \frac{1}{N} + \binom{1}{2^{-\operatorname{Re} n}} \frac{1}{N + \operatorname{Re} n} \right\}. \end{aligned}$$

The proof of analyticity is similar to that given in Lemma 1.

The terms containing logarithms have, however, to be treated in more detail. For example with $\delta = N + \operatorname{Re} n > 0$

$$x^{-\delta} \int_0^x \ln \frac{x(2-y)}{y(2-x)} y^{\delta-1} dy \leq x^{-\delta} \int_0^x \ln \frac{x}{y} y^{\delta-1} dy + \\ + x^{-\delta} \ln 2 \int_0^x y^{\delta-1} dy = \frac{1}{\delta^2} + \frac{\ln 2}{\delta}.$$

We recall from the text that with g_n we denote a vector which is regular analytic in n for $-N < \operatorname{Re} n < 0$, with $p(n)$ a polynomial in n and with a, b, c, d non-negative integers.

We prove now the following:

Lemma 3

For $0 \leq a \leq N-1$, $\operatorname{Re} n > -N$, $K_n(x^a)$ is equal to a finite sum of terms of the type (I) and (II) below:

$$(I) \quad \frac{p(n)}{n} (\eta^2)^b x^{a+b+c+1} \quad \text{and} \quad \frac{p(n)}{n} \frac{x^{a+b+c+d+1}}{n+a+b+c+1} (\eta^2)^b$$

$$(II) \quad x^N g_n(x, \eta^2) \quad \text{with} \quad \|g_n(x, \eta^2)\| \leq \frac{\operatorname{const}(n) |\eta^2|^r}{\min(1-\eta^2, 1)} \\ 0 \leq r \leq N-a-1$$

and

$$\frac{x^N g_n(x) (\eta^2)^b}{n+a+b+c+1}$$

with

$$0 \leq b \leq N-a-2$$

$$0 \leq c \leq N-a-2$$

Proof

$$K_n(x^a) = - \frac{\lambda}{2n} \int_0^x [1 - x^{-n}(2-x)^n y^n (2-y)^{-n}] y^a \cdot (1 + \eta^2 y(2-y) + \dots + [\eta^2 y(2-y)]^{N-a-2} + \frac{[\eta^2 y(2-y)]^{N-a-1}}{1 - \eta^2 y(2-y)}) dy.$$

The term

$$x^N \int_0^1 x^{-N} H_n(x, y) y^{N-1} \frac{[\eta^2 y(2-y)]^{N-a-1}}{1 - \eta^2 y(2-y)} dy$$

is equal to

$$x^N g_n(\eta^2, x) \quad \text{with}$$

$$\|g_n(\eta^2, x)\| \leq \mathcal{H}_N(n) \frac{|2\eta^2|^{N-a-1}}{\min(1-\eta^2, 1)}.$$

This term is of type (II).

The terms

$$- \frac{\lambda}{2n} \int_0^x y^{a+b+c} (\eta^2)^b dy = - \frac{\lambda}{2n} \frac{x^{a+b+c+1}}{a+b+c+1} (\eta^2)^b$$

are of type (I).

We are left then with the terms

$$\frac{\lambda}{2n} x^{-n} (2-x)^n (\eta^2)^b \int_0^x y^{n+a+b} (2-y)^{-n+b} dy$$

which can be written as

$$\begin{aligned} & \frac{\lambda}{2n} (\eta^2)^b x^{-n} (2-x)^n 2^{-n+b} \int_0^x y^{n+a+b} (p_0(n) + p_1(n)y + \dots \\ & \dots + p_n(n)y^{N-2-a-b} + g_n(y)y^{N-1-a-b}) dy. \end{aligned}$$

For the last term one has

$$\begin{aligned} & \left| \frac{\lambda}{2n} (\eta^2)^b x^{-n} (2-x)^n 2^{-n+b} \int_0^x y^{n+N-1} g_n(y) dy \right| \leq \\ & \leq (\eta^2)^b \frac{\lambda}{2n!} 2^b \binom{1}{2^{-Re n}} \|g_n\| \frac{x^N}{Re n + N}. \end{aligned}$$

Therefore this term is of type (II).

The remaining terms

$$\frac{\lambda}{2n} (2\eta^2)^b p(n) x^{-n} \left(1 - \frac{x}{2}\right)^n \int_0^x y^{n+a+b+c} dy$$

are equal to

$$\begin{aligned} & \frac{\lambda}{2n} (2\eta^2)^b p(n) \left(1 - \frac{x}{2}\right)^n \frac{x^{a+b+c+1}}{n+a+b+c+1} = \\ & = \frac{\lambda}{2n} (2\eta^2)^b p(n) \left\{ \sum_{d=0}^{N-b-a-c-2} \frac{x^{a+b+c+d+1}}{n+a+b+c+1} p_d(n) + \right. \\ & \left. + \frac{x^N g_n(x)}{n+a+b+c+1} \right\}. \end{aligned}$$

These terms are of type (I) or of type (II).

We prove now the following two Lemmas:

Lemma 4

For $-N < \operatorname{Re} n < 0$, N positive integer, $K_n(x^{\sim-1} g_n(x)) = x^{\sim} g_n^{(1)}(x)$
with $g_n^{(1)}(x)$ regular analytic and

$$\|g_n^{(1)}(x)\| \leq \frac{\mathcal{H}_N(n) \|g_n(x)\|}{\min(1-\eta^2, 1)}.$$

Lemma 5

For $-N < \operatorname{Re} n < 0$, N positive integer, $K_n(x^{\sim} g_n(x)) = x^{\sim} g_n^{(1)}(x)$
with $g_n^{(1)}(x)$ regular analytic and

$$\|g_n^{(1)}(x)\| \leq \frac{\mathcal{H}_N(n) \|g_n(x)\|}{1-\eta^2}.$$

Proof

The inequalities are directly derived from the definition of the norm of $x^{\sim} H_n(x, y) y^{\sim-1}$ and from the following inequalities

$$\frac{1}{1-\eta^2 y(2-y)} \leq \frac{1}{\min(1-\eta^2, 1)}$$

$$\frac{1}{1-\eta^2 y(2-y)} \leq \frac{y^{-1}}{1-\eta^2}.$$

The analyticity is a consequence of Lemma 2.

In the text, to prove the dominance of the first iteration

as $\eta^2 \rightarrow \infty$ also for $\text{Re } n$ equal to a negative integer, we use the following variations of Lemmas 3, 4 and 5, which are valid for $-N + \frac{1}{2} < \text{Re } n$, $\eta^2 < 0$.

3.1. In Lemma 3 the term $x^N g_n(x, \eta^2)$ can be replaced by

$$x^{N-1/2} g_n(x, \eta^2) \quad \text{with} \quad \|g_n(x, \eta^2)\| < \text{const}(n) (-\eta^2)^{N-a-3/2}.$$

4.1. $K_n(x^{N-1} g_n(x)) = x^{N-1/2} g_n^{(1)}(x)$

with

$$\|g_n^{(1)}(x)\| \leq \frac{\text{const}(n)}{\sqrt{-\eta^2}}.$$

5.1. $K_n(x^{N-1/2} g_n(x)) = x^{N-1/2} g_n^{(1)}(x)$

with

$$\|g_n^{(1)}(x)\| \leq \frac{\text{const}(n)}{-\eta^2}.$$

Proof

The proof of 3.1 and 4.1 is obtained by noting that for $\eta^2 < 0$

$$\frac{1}{1 - \eta^2 y(2-y)} \leq \frac{1}{\sqrt{-\eta^2} \sqrt{y}}.$$

The proof of 5.1 is obtained by noting that for $\eta^2 < 0$

$$\frac{1}{1 - \eta^2 y(2-y)} \leq \frac{1}{-\eta^2 y}.$$

APPENDIX C

We prove now the following Lemma

Lemma 1' The operator $x^{-N} K_n^{(c)}(\rho, \rho') y^N$ with

$$x = \frac{1 - e^{-i\rho}}{2}, \quad y = \frac{1 - e^{-i\rho'}}{2} \quad \text{and}$$

$$K_n^{(c)}(\rho, \rho') = -\frac{\lambda}{4n} \left[1 - \left(\frac{(1 - e^{-i\rho'})(3 + e^{-i\rho})}{(1 - e^{-i\rho})(3 + e^{-i\rho'})} \right)^n \right] \cdot \frac{e^{i(\frac{\pi}{2} - \rho')} \Theta(\rho - \rho')}{1 - \frac{n^2}{4} (1 - e^{-i\rho})(3 + e^{-i\rho'})}$$

with N non-negative integer and for $\operatorname{Re} n > -N-1$, is (a) bounded, (b) analytic and (c) its norm tends to zero as $n^2 \rightarrow +\infty$.

Proof

Boundedness and behaviour of the norm.

We shall again distinguish two cases: (i) $|n| > \frac{1}{2}$;

(ii) $|n| \leq \frac{1}{2}$.

We begin with

(i) $|n| > \frac{1}{2}$.

First of all we notice that with

$$y = \frac{1 - e^{-i\rho'}}{2}, \quad x = \frac{1 - e^{-i\rho}}{2}, \quad \text{for } \rho \geq \rho' \quad \text{one has}$$

$$\left| \frac{y}{x} \right| \leq 1, \quad \left| \frac{2-x}{2-y} \right| \leq 1 \quad \text{and}$$

$$-\frac{\pi}{4} < \arg \left(\frac{y(2-x)}{x(2-y)} \right) < \frac{3\pi}{4}.$$

Then we have, using the same notations as in Appendix B

$$|x^{-N} K_n^{(\omega)} y^N| \leq \frac{\lambda}{4|n|} \left\{ 1 + e^{\frac{3\pi}{4} |\operatorname{Im} n|} \left[\left(\frac{\rho'}{\pi} \right)^{\operatorname{Re} n + N} \right] \begin{pmatrix} 1 \\ 2^{-\operatorname{Re} n} \end{pmatrix} \right\}.$$

$$\frac{\Theta(\rho - \rho')}{|1 - \eta^2 y(2-y)|} = \Theta(\rho - \rho') k_n^{(\omega)}(\rho').$$

(ii) In this region we use again the same method as in Appendix B.

$$\begin{aligned} & \left| -\frac{\lambda}{4n} \left(\frac{x}{y} \right)^{-N} \left[1 - \left(\frac{y(2-x)}{x(2-y)} \right)^n \right] \frac{\Theta(\rho - \rho')}{1 - \eta^2 y(2-y)} \right| \leq \\ & \leq \frac{\lambda}{4} \max_{|n| \leq \frac{1}{2}} \left| \left(\frac{y(2-x)}{x(2-y)} \right)^n \ln \frac{y(2-x)}{x(2-y)} \Theta(\rho - \rho') \right| \frac{1}{|1 - \eta^2 y(2-y)|} \leq \\ & \leq \frac{\lambda}{4} e^{\frac{3\pi}{4} |\operatorname{Im} n|} \sqrt{2} \left(\frac{\rho'}{\pi} \right)^{-\frac{1}{2}} \left\{ \ln \frac{2\pi}{\rho'} + \frac{3\pi}{4} \right\} \frac{\Theta(\rho - \rho')}{|1 - \eta^2 y(2-y)|} = \\ & = k_n^{(\omega)}(\rho') \Theta(\rho - \rho'). \end{aligned}$$

In all cases we have, for $\operatorname{Re} n > -N-1$

$$\int_0^\pi k_n^{(\omega)}(\rho') d\rho' < \infty \quad (\text{C.1})$$

and furthermore, for η^2 sufficiently large, $\eta^2 > \bar{\eta}^2$, one easily sees that

$$\frac{1}{|1 - \eta^2 y(2-y)|} \leq \frac{1}{\left[1 + \frac{\eta^4}{32} \left\{ (3 - 2\cos p' - \cos 2p')^2 + (2\sin p' + \sin 2p')^2 \right\} \right]^{\frac{1}{2}}} \quad (C.2)$$

Such inequality implies that the integral written above, Eq. (C.1), as $\eta^2 \rightarrow +\infty$ tends to zero. We have thus proved the statements a and c. The analyticity is proved by reasonings similar to those in Appendix B.

We note that, in $\text{Re } n > 0$,

$$|K_n^{(c)}| \leq \frac{\lambda}{2|n|} e^{\frac{3\pi}{4} |\text{Im } n|} \frac{\Theta(p-p')}{|1 - \eta^2 y(2-y)|} \quad (C.3)$$

This implies that, as $\eta^2 \rightarrow +\infty$, the norm of $K_n^{(c)}$ goes to zero uniformly in the region to the right of that branch of the curve

$$\frac{e^{\frac{3}{4}\pi |\text{Im } n|}}{|n|} = \text{const} \quad (C.4)$$

which lies in $\text{Re } n > 0$.

Lemma 2'

The operator $x^{-N} H_n^{(c)}(p, p') y^{N-1}$, where

$$H_n^{(c)}(p, p') = -\frac{\lambda}{4n} \left[1 - x^{-n} (2-x)^n y^n (2-y)^{-n} \right] e^{i\left(\frac{\pi}{2} - p'\right)} \Theta(p-p')$$

and N is a positive integer, for $\operatorname{Re} n > -N$, is bounded and analytic and we have

$$\begin{aligned} \mathcal{H}_N^{(n)} &\equiv \left\| x^{-N} H^{(n)}(\rho, \rho') y^{N-1} \right\| \leq \\ &\leq \frac{\lambda}{4|n|} \left\{ \frac{2}{N} \left(\frac{\pi}{2} \right)^N + e^{\frac{3\pi}{4} |\operatorname{Im} n|} \binom{1}{2^{-\operatorname{Re} n}} \operatorname{Max} \left(\pi, 2 \left(\frac{\pi}{2} \right)^{N+\operatorname{Re} n} \right) \frac{1}{N+\operatorname{Re} n} \right\} \end{aligned}$$

Proof

$$\begin{aligned} \int_0^\rho |x^{-N} H_n^{(n)} y^{N-1}| d\rho' &\leq \frac{\lambda}{4|n|} \left\{ \left(\frac{\rho}{\pi} \right)^{-N} \int_0^\rho \left(\frac{\rho'}{2} \right)^{N-1} d\rho' + \right. \\ &+ e^{\frac{3\pi}{4} |\operatorname{Im} n|} \binom{1}{2^{-\operatorname{Re} n}} \operatorname{Max} \left(\pi, 2 \left(\frac{\pi}{2} \right)^{N+\operatorname{Re} n} \right) \rho^{-N-\operatorname{Re} n} \int_0^\rho (\rho')^{N+\operatorname{Re} n-1} d\rho' \left. \right\}. \end{aligned}$$

The analyticity is proved by reasonings similar to those in Appendix B.

By $f_n(x)$, $x = x(\rho)$ we denote a function continuous in the interval $[0, \pi]$ of ρ which as a vector is analytic in n for $-N < \operatorname{Re} n < 0$.

We now prove the following Lemma:

Lemma 3'

For $0 \leq a \leq N-1$, $K_n^{(c)}(x^a)$ is equal to a finite sum of terms of type (I) and (II) below.

$$(I) \quad \frac{p(n)}{n} (\eta^2)^b x^{a+b+c+1} \quad \text{and} \quad \frac{p(n)}{n} (\eta^2)^b \frac{x^{a+b+c+d+1}}{n+a+b+c+1}$$

$$(II) \quad x^r f_n(x, \eta^2), \quad \text{with} \quad \|f_n(x, \eta^2)\| < \frac{\text{const}(n) |\eta^2|^r}{\min_{y \in C} |1 - \eta^2 y(2-y)|}$$

$$0 \leq r \leq N-a-1$$

and

$$\frac{x^r f_n(x) (\eta^2)^b}{n+a+b+c+1}$$

with

$$0 \leq b \leq N-a-2$$

$$0 \leq c \leq N-a-2.$$

Proof

$$K_n^{(c)}(x^a) = - \frac{\lambda}{2n} \int_0^x [1 - x^{-n} (2-x)^n y^n (2-y)^{-n}] y^a \cdot$$

$$\cdot (1 + \eta^2 y(2-y) + \dots + [\eta^2 y(2-y)]^{N-a-2} + \frac{[\eta^2 y(2-y)]^{N-a-1}}{1 - \eta^2 y(2-y)}) dy.$$

The term

$$x^r \int_0^1 x^N H_n^{(c)}(p, p') y^{N-1} \frac{[\eta^2 (2-y)]^{N-a-1}}{1 - \eta^2 y(2-y)} dy$$

has

the property

$$\left| \int_0^1 x^{-N} H_n^{(c)}(\eta, \eta') y^{N-1} \frac{[\eta^2 (2-y)]^{N-a-1}}{1-\eta^2 y(2-y)} dy \right| \leq$$

$$\leq \frac{\mathcal{H}_N^{(c)}(n) |2\eta^2|^{N-a-1}}{\min_{y \in C} |1-\eta^2 y(2-y)|}.$$

For $\eta^2 > \bar{\eta}^2$ we have, $|1-\eta^2 y(2-y)| \geq 1$.

The terms

$$-\frac{\lambda}{2n} \int_0^x y^a (\eta^2)^b y^{b+c} dy = -\frac{\lambda}{2n} (\eta^2)^b \frac{x^{a+b+c+1}}{a+b+c+1}$$

are of type (I).

We are left with

$$\frac{\lambda}{2n} \int_0^x x^{-n} (2-x)^n y^n (2-y)^{-n} y^{a+b} (\eta^2)^b (2-y)^b dy =$$

$$= \frac{\lambda}{2n} (2\eta^2)^b x^{-n} \left(1 - \frac{x}{2}\right)^n \int_0^x y^{n+a+b} (p_0(n) + p_1(n)y + \dots$$

$$\dots + p(n) y^{N-1-a-b} p_n(y)) dy.$$

The last term gives

$$\frac{\lambda}{2n} x^N (2\eta^2)^b x^{-n-N} \left(1 - \frac{x}{2}\right)^n \int_0^x p(n) y^{N-1+n} p_n(y) dy$$

and

$$\left| x^{-n-N} \left(1 - \frac{x}{2}\right)^n \int_0^x y^{N-1+n} f_n(y) dy \right| \leq \\ \leq \frac{e^{\frac{3}{4}\pi |\operatorname{Im} n|}}{2} \|f_n\| \left(\begin{matrix} 1 \\ 2^{-\operatorname{Re} n} \end{matrix} \right) \operatorname{Max} \left(\pi, 2 \left(\frac{\pi}{2} \right)^{N+\operatorname{Re} n} \right).$$

$$\int_0^{\rho} \rho'^{\operatorname{Re} n + N - 1} d\rho' = \frac{e^{\frac{3}{4}\pi |\operatorname{Im} n|}}{2} \|f_n\|.$$

$$\left(\begin{matrix} 1 \\ 2^{-\operatorname{Re} n} \end{matrix} \right) \operatorname{Max} \left(\pi, 2 \left(\frac{\pi}{2} \right)^{N+\operatorname{Re} n} \right) \frac{1}{\operatorname{Re} n + N}.$$

Thus this term is of type (II).

The remaining terms are of the form

$$\frac{\lambda}{2n} (2\eta^2)^b \left(1 - \frac{x}{2}\right)^n p(n) \frac{x^{a+b+c+1}}{a+b+c+1} = \\ = \frac{\lambda}{2n} p(n) (2\eta^2)^b \left\{ \sum_{d=0}^{N-b-a-c-2} \frac{x^{a+b+c+d+1}}{a+b+c+n+1} p_d(n) + \frac{x^N f_n(x)}{a+b+c+n+1} \right\}.$$

These terms are of type (I) or of type (II). We prove now the following two lemmas.

Lemma 4'

Given a positive integer N

$$K_n^{(c)} (x^{N-1} f_n(x)) = x^N f_n^{(1)}(x)$$

with $\|f_n^{(1)}(x)\|$ finite for $\eta^2 > 1$ and

$$\|f_n^{(1)}(x)\| \leq \mathcal{H}_N^{(1)}(n) \|f_n(x)\|, \quad \text{for } \eta^2 > \bar{\eta}^2.$$

Lemma 5'

Given a positive integer N

$$K_n^{(1)}(x^\sim f_n(x)) = x^\sim f_n^{(1)}(x)$$

with $\|f_n^{(1)}(x)\|$ finite for $\eta^2 > 1$ and

$$\|f_n^{(1)}(x)\| \leq \frac{\mathcal{H}_N^{(1)}(n) \|f_n(x)\|}{\eta^2 - 1} \text{ const, for } \eta^2 > \bar{\eta}^2.$$

Proof

The result follows from Lemma 2' and by noting in the first case that for $\eta^2 > \bar{\eta}^2$

$$\frac{1}{|1 - \eta^2 y(2-y)|} \leq 1$$

and in the second case that for $\eta^2 > \bar{\eta}^2$, from (C.2) we have

$$\frac{1}{|1 - \eta^2 y(2-y)|} < \frac{\text{const}}{(\eta^2 - 1)y} \quad (\text{C.5})$$

The analogues of Lemmas 3.1, 4.1 and 5.1 are proved similarly as we did in Appendix B, noticing that for $\eta^2 > \bar{\eta}^2$ in addition to Eq. (C.5) holds also

$$\frac{1}{|1 - \eta^2 y(2-y)|} < \frac{\text{const}}{\eta \sqrt{y}}.$$

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- (14) Here and in the following we shall always indicate with p the modulus of the 3-momentum, $p = |\vec{p}|$.

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- (17) R.G. Newton, see ref. 16.

- (18) Eq. (13) has been recently applied by J. Arafune, H. Ezawa, K. Nakamura and N. Murai, Lett. al Nuovo Cimento 2 394 (1969), to numerical calculations of Regge trajectories, confirming the rigorous results that were obtained in ref. (5).

- (19) We use the relations 2.8.(22) and 2.8.(51) of ref. (15), Chpt. 2.

- (20) We note that the pole at $n = 0$ is actually not present in the solution as we have already seen in the treatment of the solution for $\operatorname{Re} n > -1$, given in subsection A.

- (21) See E.C. Titchmarsh, "The theory of functions" (Oxford 1932), Chpt. III.
- (22) At the threshold, $\eta^2 = 1$ we have a singularity which is clearly independent of the integration contour. Such a singularity is to be expected on physical grounds.
- (23) Here we correct a statement made in the second paper of ref. (5). The uniform retreat of the poles to the left of $\text{Re } n =$
 $= \alpha + 1$ (even), $\text{Re } n = \alpha$ (odd),
 $\alpha > -1$, holds only for $\lambda \rightarrow 0$ for $\eta^2 < 1$
or as $\eta^2 \rightarrow -\infty$.
- (24) For the square roots we choose the following determinations
 $\eta = i\sqrt{-\eta^2}$ for $\eta^2 < 0$ and $\sqrt{\eta^2 - 1} = i\sqrt{1 - \eta^2}$ for $\eta^2 < 1$.
- (25) See e.g. R.G. Newton "The Complex J-plane" ref. (17).
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Figure Captions

Fig. 1. Singularities of the integrand in the generic term of the iteration series and deformed integration contour (c). The structure of the singularities is different in the three cases; I ($\eta^2 < 0$); II ($0 < \eta^2 < 1$); III ($\eta^2 > 1$).

Fig. 2. Behaviour of the even time parity trajectories on the complex n -plane for small λ in the region $\eta^2 < 0$ ($\eta^2 = \frac{t}{4}$). Two trajectories (except the leading even trajectory) start at $\eta^2 = 0$ from the neighbourhoods of two consecutive negative integers $(-1, -2)$ $(-3, -4)$... move one against the other on the real axis as η^2 decreases, collide and turn at right angle into the complex plane terminating for $\eta^2 = -\infty$ at $n = -1$. The leading trajectory moves on the real axis from $n = \sqrt{\frac{1}{4} + \lambda} - \frac{1}{2}$ at $\eta^2 = 0$ to $n = 0$ at $\eta^2 = -\infty$.

Fig. 3. Behaviour of the odd time parity trajectories on the n -plane, for small λ , in the region $\eta^2 < 0$. The behaviour is qualitatively similar to that of the even time parity trajectories of Fig. 2 except that the pairs of colliding trajectories originate here from the neighbourhoods of the integer $(-2, -3)$ $(-4, -5)$ The leading odd trajectory moves on the real axis from $n = \sqrt{\frac{1}{4} + \lambda} - \frac{3}{2}$ to $n = -1$.

Fig. 4. Behaviour of the even time parity trajectories
for $\eta^2 < 1$

Fig. 5. Behaviour of the odd time parity trajectories for
 $\eta^2 < 1$

FIG.1

FIG.2

FIG.3

FIG.4