ON NIELSEN'S GENERALIZED POLYLOGARITHMS
AND THEIR NUMERICAL CALCULATION
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ABSTRACT

The generalized polylogarithms of Nielsen are studied, in particular their functional relations. New integral expressions are obtained, and explicit relations for function values of particular arguments are given. An Algol procedure for calculating 10 functions of lowest order is presented. The numerical values of the Chebyshev coefficients used in this procedure are tabulated. A table of the real zeros of these functions is also given.

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1. INTRODUCTION

At the beginning of this century, Nielsen [1] published a monograph "Der Eulersche Dilogarithmus und seine Verallgemeinerungen", in which he discussed a family of functions defined by

\[ S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1} t \log^p (1 - xt)}{t} \, dt \]  

(1.1)

for positive integers \(n\) and \(p\) and complex \(x\). The \(\log (1 - xt)\) is to be understood on its principal sheet. Consequently, we cut the \(x\)-plane from \(+1\) to \(\infty\). This formula embodies many particular cases treated separately by other authors, e.g., Euler's dilogarithm, Spence functions, Legendre's or Hurwitz's trilogarithm, polylogarithms, etc. For all these functions, however, the value \(p = 1\) is characteristic, so that for \(p > 1\) Eq. (1.1) defines functions not treated before the publication of Nielsen's monograph. We therefore call these functions \(S_{n,p}(x)\) Nielsen's generalized polylogarithms.

Although Nielsen provided an extensive theory of the \(S_{n,p}(x)\), his article (which unfortunately suffers from quite a number of misprints) does not seem to be sufficiently well-known to people working in areas where these functions appear. No reference could be traced in any of the relevant handbooks or integral tables.

Two of the authors (J.A.M. and E.R.) are engaged in calculations of higher order radiative corrections in quantum electrodynamics, where knowledge of the functions \(S_{n,p}(x)\) seems to play an essential role [2]. In the past, special cases of \(S_{n,p}(x)\) were discussed and some of their typical properties (functional relations) rediscovered by people working in the field mentioned [3],[4],[5]. Our interest in Nielsen's monograph was initiated by a reference given in the book of Lewin [6] who treats the special case \(p = 1\) in detail.

The aim of the present article is twofold: to provide a useful tool for people interested in problems where \(S_{n,p}(x)\) may appear, and to give an accurate method for its numerical evaluation, at least for some small values of \(n\) and \(p\).

Nielsen, referring to the dilogarithm, wrote in his paper: "Man darf also mit Recht sagen, dass ein trübes Schicksal über dem Dilogarithmus und den ihn behandelnden Arbeiten geschwebt hat, so dass e in nicht uninteressanter Abschnitt der elementaren Integralrechnung beinahe gans in Vergessenheit gesunken ist". It seems that, with these words, he unfortunately predicted the destiny of his own paper. It is therefore the hope of the authors that, by their small contribution to the problem, attention will again be drawn to Nielsen's important work on a useful family of functions.
2.1 Definitions and notation

The basic definition is given in Eq. (1.1). For \( p = 1, \ n \geq 2 \), one also writes

\[
S_n(x) = S_{n-1,1}(x) .
\]  

(2.1)

These functions \( S_n(x) \) are called polylogarithms. In particular, for \( n = 2 \) we have Euler's dilogarithm

\[
S_2(x) = - \int_0^x \frac{\log (1 - xt)}{t} \, dt = - \int_0^x \frac{\log (1 - t)}{t} \, dt ,
\]  

(2.2)

for \( n = 3 \) the trilogarithm, etc. It is, then, appropriate to introduce the term "generalized polylogarithm" for the case \( p > 1 \).

The polylogarithms are closely related to the Spence functions [7]

\[
L_n(x) = \int_0^x \frac{L_{n-1}(t)}{t-1} \, dt ; \quad L_0(t) = \frac{t - 1}{t} .
\]

(2.3)

In fact, one has

\[
S_n(x) = -L_n(1 - x) .
\]

The terms used are not consistent throughout the literature. Some times the Spence function for \( n = 2 \) as defined in Eq. (2.3) is called the dilogarithm [8] and vice-versa [9]. The notation for the functions also varies, e.g. Lewin [6] and other authors [10],[11] use

\[
\text{Li}_n(x) = S_n(x) .
\]

Integrating Eq. (1.1) for \( p = 1 \) by parts and using appropriate substitutions, one obtains

\[
S_n(x) = \frac{x}{(n-1)!} \int_0^\infty \frac{t^{n-1}}{e^t - x} \, dt = \frac{x}{(n-1)!} \int_0^\infty \frac{\log^{n-1} t}{t(t-x)} \, dt .
\]

(2.4)

For \( n = 1 \) it is easy to see from Eq. (1.1) that
\[ S_{1,p}(x) = \frac{1}{p!} \int_0^x \frac{t^p}{e^t - 1} \, dt = \frac{1}{p!} D_p(-\log(1-x)) \quad (\infty < x \leq 1). \]  

where

\[ D_p(x) = \int_0^x \frac{t^p}{e^t - 1} \, dt \]  

is known as the Debye function [8].

2.2 Some general properties

Again from the definition (1.1) of \( S_{n,p}(x) \) we obtain by differentiation and partial integration

\[ \frac{d}{dx} S_{n,p}(ax) = \frac{S_{n-1,p}(ax)}{x} \quad (n \geq 2). \]  

Hence

\[ S_{n,p}(ax) = \int \frac{x}{y} S_{n-1,p}(yt) \, dt. \]  

With the usual definition of the logarithm on the branch cut \(-\infty < x \leq 0\), namely

\[ \log(x + i\epsilon) = \log|x| + i\pi\theta(-x), \]  

where \( \theta(x) = 1 \) if \( x \geq 0 \) and \( \theta(x) = 0 \) if \( x < 0 \), we find that \( S_{n,p}(x) \) is real for \( x \leq 1 \). Across the cut from \(+1\) to \(-\infty\) the real part of \( S_{n,p}(x) \) is continuous, whereas the imaginary part changes sign.

For our purposes it is sufficient to consider \( x \) as real, if not otherwise stated, although many of the formulas given remain valid for complex \( x \), in particular the important relations (3.5) and (3.11).

The imaginary part of the boundary value of \( S_{n,p}(x) \) at the cut is studied in detail in Section 4.

The values of \( S_{n,p}(x) \) for the special arguments \( x = 1, x = -1, \) and \( x = \frac{1}{2} \) are of particular interest in the theory of the generalized polylogarithms. Following Nielsen, we set

\[ s_{n,p} = S_{n,p}(1) \]
\[ \sigma_{n,p} = (-1)^p S_{n,p}(-1) \]
\[ a_{n,p} = S_{n,p}(\frac{1}{2}). \]
Relations between these values are studied in Section 3.4. As in Eq. (2.1) we write

\[ s_n \equiv s_{n-1,1}, \quad \sigma_n \equiv \sigma_{n-1,1}, \quad a_n \equiv a_{n-1,1}. \]

We note already that for all \( n \) and \( p \)

\[ s_{n,p} = s_{p,n}. \]  \hspace{1cm} (2.11)

This can be deduced from Eq. (1.1) by the substitution \( t' = 1 - t \) and integration by parts. No similar relation exists for \( \sigma_{n,p} \) and \( a_{n,p} \).

### 2.3 Relation to the hypergeometric function

#### Power series expansions

The functions \( S_{n,p}(x) \) are related to derivatives of the hypergeometric function with special arguments. From the integral representation

\[ _2F_1(a, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{a} dt \]  \hspace{1cm} (2.12)

we obtain by comparison with Eq. (1.1)

\[ S_{n,p}(x) = \frac{(-1)^{n-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \beta^{n-1}\partial \alpha^{p}} \left[ t^{\beta-1} _2F_1(a, \beta; \beta + 1; x) \right]_{\beta=0} \]  \hspace{1cm} (2.13)

Recalling the power series expansion

\[ _2F_1(a, \beta; \gamma; z) = 1 + \frac{a\beta}{\gamma} \frac{z}{1!} + \frac{a(a+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \ldots \]  \hspace{1cm} (2.14)

we have

\[ S_{n,p}(x) = \frac{(-1)^n}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \beta^{n-1}\partial \alpha^{p}} \left[ (\alpha)_\beta \frac{x^\beta}{\gamma(\gamma+1)2!} \right]_{\beta=0} \]  \hspace{1cm} (2.15)

where

\[ (\alpha)_s = \Gamma(\alpha + s)/\Gamma(\alpha) = \alpha(\alpha + 1) \ldots (\alpha + s - 1) \]

\[ (\alpha)_0 = 1 \]
is the Pochhammer symbol. By differentiating with respect to $\beta$ it follows that

\[
S_{n,p}(x) = \frac{1}{p!} \frac{\partial^p}{\partial x^p} \left. \sum_{s=0}^{\infty} \frac{(a)_s x^s}{s^n s!} \right|_{a=0}.
\]  

(2.16)

Writing

\[
(a)_s = \sum_{\sigma=1}^{s} (-1)^{s+\sigma} S_{s,\sigma} (a)_\sigma
\]

(2.17)

where the $S_{k,j}$ are the Stirling numbers of the first kind, defined by [8]

\[
\log^s (1 + x) = s! \sum_{\sigma=0}^{\infty} S_{s,\sigma} \frac{x^\sigma}{\sigma!} \quad (|x| < 1)
\]

(2.18)

we see that

\[
\frac{\partial^p}{\partial x^p} (a)_s = \begin{cases} 
0 & \text{for } s < p \\
(-1)^{p+s} p! S_{s,p} & \text{for } s \geq p.
\end{cases}
\]

(2.19)

Introducing this result into Eq. (2.16), we obtain

\[
S_{n,p}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s S_{p+s}^{(p)}}{(p+s)! (p+s)^n} x^{p+s}
\]

(2.20)

which is the power series expansion of the generalized polylogarithm. Using the asymptotic relation [8]

\[
\left| S_{p+s}^{(p)} \right| \sim \frac{(p+s-1)! (\gamma + \log (p+s))^{p-1}}{(p-1)!} \quad (s \to \infty)
\]

(2.21)

one can show that this expansion is valid for $|x| \leq 1$. Of course, Eq. (2.20) can also be obtained directly from the definition (1.1) with the help of Eq. (2.18).

* ) These numbers are connected with the generalized Bernoulli numbers, see [7]. For recurrence relations of $S_{k,j}$ see Eq. (6.6) below.
Nielsen writes

\[ S_{n,p}(x) = \sum_{s=0}^{\infty} \frac{\omega_{p,s}^{p+s}}{(p+s)!} x^{p+s} \]  

(2.22)

so that in his notation

\[ \omega_{p,p+s} = \frac{(-1)^s S_p(p)}{(p+s-1)!} . \]  

(2.23)

We note as special cases of Eq. (2.20) for the dilogarithm

\[ S_{1,1}(x) = S_2(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^2} = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots , \]  

(2.24)

and in general, for the polylogarithm *,

\[ S_n(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^n} = \frac{x}{1^n} + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \cdots . \]  

(2.25)

For \( p > 1 \), the series become more complicated, e.g., one can show that

\[ S_{n-2,2}(x) = \sum_{s=2}^{\infty} \left( \sum_{r=1}^{s-1} \frac{1}{r} \right) \frac{x^s}{s^{n-1}} = \frac{x^2}{2^{n-1}} + \left( 1 + \frac{1}{2} \right) \frac{x^3}{3^{n-1}} + \left( 1 + \frac{1}{2} + \frac{1}{3} \right) \frac{x^4}{4^{n-1}} + \cdots , \]  

(2.26)

and

\[ S_{n-3,3}(x) = \sum_{s=3}^{\infty} \left( \sum_{r=1}^{s-2} \frac{1}{r} \left( \sum_{q=r+1}^{s-1} \frac{1}{q} \right) \right) \frac{x^s}{s^{n-2}} = \frac{x^3}{3^{n-2}} + \left[ \left( \frac{1}{2} + \frac{1}{3} \right) \frac{x^4}{4^{n-2}} + \left( \frac{1}{2} \times \frac{1}{3} \right) \frac{x^4}{4^{n-2}} \right] + \cdots . \]  

(2.27)

*) This series is also a special case of Lerch's transcendent \( \Phi(z,\alpha,\beta) = \sum_{s=0}^{\infty} \frac{z^s}{(\alpha + s)^\beta} \). In fact, one has \( S_n(x) = x^\Phi(x,n,1) = F(x,n) \), sometimes called Jonquières's function [12],[13]. Mitchell [14] writes \( S_n(x) = \zeta(1,n|x) \).
For $x \to 0$, we see from Eq. (2.20) that

$$S_{n,p}(x) = \frac{1}{p!} \sum_{k=0}^{n} \frac{x^k}{k!} + O(x^{p+1})$$

(2.28)

3. TRANSFORMATIONS OF THE ARGUMENT OF $S_{n,p}(x)$

Following Nielsen, it is convenient to introduce the auxiliary functions

$$L_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!} \int_0^x \frac{\log^{n-1} t \log^p (1-t)}{t} \, dt$$

(3.1a)

$$M_{n,p}(x) = \frac{(-1)^{n-1}}{(n-1)!} \int_0^x \frac{\log^{n-1} t \log^p (1+t)}{t} \, dt$$

(3.1b)

Their connection with $S_{n,p}(x)$ is easily obtained by performing the substitution $t' = tx$. One finds that

$$L_{n,p}(x) = \sum_{r=0}^{n-1} \frac{(-1)^r \log^r x}{r!} S_{n-r,p}(x)$$

(3.2a)

$$M_{n,p}(x) = (-1)^p \sum_{r=0}^{n-1} \frac{(-1)^r \log^r x}{r!} S_{n-r,p}(-x)$$

(3.2b)

The corresponding inverse relations are

$$S_{n,p}(x) = \sum_{s=0}^{n-1} \frac{\log^s x}{s!} L_{n-s,p}(x)$$

(3.3a)

$$S_{n,p}(-x) = (-1)^p \sum_{s=0}^{n-1} \frac{\log^s x}{s!} M_{n-s,p}(x)$$

(3.3b)

As is clear from their definitions, the functions $L_{n,p}(x)$ and $M_{n,p}(x)$ have a more complicated analytic structure than $S_{n,p}(x)$. They are, however, convenient
for the study of some transformations of the argument of $S_{n,p}(x)$.

3.1 The reflection $x \rightarrow 1 - x$

By performing the substitution $t' = 1 - t$ in Eq. (3.1a) and then integrating by parts, we have

$$L_{n,p}(1 - x) + L_{p,n}(x) = s_{n,p} - \frac{(-1)^{n+p}}{n!p!} \log^n x \log^n (1 - x). \quad (3.4)$$

Expressing $L_{n,p}(x)$ in terms of $S_{n,p}(x)$ by means of Eq. (3.2a) we obtain with the help of

$$\sum_{s=0}^{n-1} (-1)^s \binom{n}{s} = (-1)^{n-1}$$

and

$$L_{n,p}(1) = S_{n,p}(1) = s_{n,p}$$

the relation

$$S_{n,p}(x) = \sum_{s=0}^{n-1} \frac{\log^s x}{s!} \left( s_{n-s,p} - \sum_{r=0}^{p-1} \frac{(-1)^r \log^r (1 - x)}{r!} s_{p-r,n-s}(1 - x) \right)$$

$$+ \frac{(-1)^p}{n!p!} \log^n x \log^n (1 - x). \quad (3.5)$$

For $n = p = 1$, Eq. (3.5) reduces to the well-known formula for the dilogarithm

$$S_2(x) + S_2(1 - x) = s_2 - \log x \log (1 - x). \quad (3.6)$$

It follows from Eq. (3.5) that a similar relation, which would combine polylogarithms of arguments $x$ or $1 - x$ with elementary functions only, does not exist for $S_n(x)$ if $n \geq 3$. Thus, for the trilogarithm one has

$$S_3(x) = s_3 - S_2(1 - x) + \log x \{s_2 - S_2(1 - x)\} - \frac{1}{2} \log^2 x \log (1 - x). \quad (3.7)$$

*) The corresponding formulae of Nielsen [1] § 12(8), (9) contain a sign error.
This relation contains a function \( S_{n,p}(x) \) which is not a polylogarithm. However, if one considers the set of generalized polylogarithms \( S_{n,p}(x) \), no functions other than these and elementary functions are required for arbitrary \( n \) and \( p \) in Eq. (3.5).

### 3.2 The inversion \( x \to 1/x \)

In this case, Nielsen proceeds as follows. By substituting \( t' = 1/t \) in the integral of Eq. (3.1b) and using

\[
M_{n,p}(1) = (-1)^p S_{n,p}(-1) = \sigma_{n,p}
\]  

he finds after some manipulation

\[
M_{n,p}(x) = (-1)^n \left\{ \sum_{s=0}^{p-1} \binom{n + s - 1}{s} M_{n+s,p-s}(\frac{1}{x}) - \frac{\log^{n+p}x}{(n-1)!p!(n+p)} \right\} + C_{n,p}
\]  

where

\[
C_{n,p} = (1 - (-1)^n) \sigma_{n,p} = (-1)^n \sum_{r=1}^{p-1} \binom{n + r - 1}{r} \sigma_{n+r,p-r}.
\]

Applying Eq. (3.5b) and using the identity

\[
\sum_{s=0}^{n-1} \frac{a^s}{s!} \binom{n + r - s - 1}{r} y_{n+r-s} = \sum_{m=0}^{r} \binom{-1}{m} \frac{a^m}{m!} \binom{n + r - m - 1}{r-m} x_{n+r-m}
\]

where the quantities \( x_j \) and \( y_j \) are related by

\[
x_n = \sum_{p=0}^{n-1} \frac{(-1)^p}{p!} y_{n-p}, \quad y_n = \sum_{p=0}^{n-1} \frac{(-1)^p}{p!} x_{n-p},
\]

he obtains (after making the substitution \( x \to -x \)) the relation

\[
S_{n,p}(x) = (-1)^n \sum_{s=0}^{p-1} (-1)^s \sum_{r=0}^{s} \frac{\log^r(-x)}{r!} \binom{n + s - r - 1}{s - r} S_{n+s-r,p-s}(\frac{1}{x}) +
\]

\[+ (-1)^p \left\{ \sum_{r=0}^{n-1} \frac{\log^r(-x)}{r!} C_{n-r,p} + \frac{\log^{n+p}(-x)}{(n+p)!} \right\}.
\]
For the dilogarithm, this formula becomes

\[ S_2(x) = -S_2\left(\frac{1}{x}\right) - s_2 - \frac{1}{2} \log^2(-x), \]

(3.12)

and for the polylogarithm

\[ S_n(x) = (-1)^{n-1} S_n\left(\frac{1}{x}\right) - \sum_{r=0}^{n-2} \frac{\log^r(-x)}{r!} (1 + (-1)^{n-r}) \sigma_{n-r} - \frac{\log^n(-x)}{n!} \]

(3.13)

This relation was already known to Jonquière [15], [16]. Following [12], [13], this formula can also be written as

\[ S_n(x) + (-1)^n S_n\left(\frac{1}{x}\right) = -\frac{(2\pi i)^n}{n!} B_n\left(\frac{\log x}{2\pi i}\right) \]

(3.14)

where \( B_n(x) \) is the Bernoulli polynomial of order \( n \).

Contrary to the case of the reflection \( x \to 1 - x \), there is here a relation (3.13) which, for any \( n \), combines polylogarithms of arguments \( x \) and \( 1/x \) with elementary functions only. Moreover, only polylogarithms with the same \( n \) appear in this relation. For \( p \geq 2 \), several generalized polylogarithms \( S_{n,p}(1/x) \) are needed in Eq. (3.11), but only elementary functions are used in addition to these.

We insist on the fact that the generalized polylogarithms \( S_{n,p}(x) \) are a closed set, in the sense described above, under the transformations we have considered. \( S_{n,p}(1 - x) \) and \( S_{n,p}(1/x) \) can be expressed as linear combinations of \( S_{n,p}(x) \) with coefficients constructed from rational numbers and the functions \( \log(ix) \) and \( \log(1 - x) \). In addition, there appear terms composed of \( \sigma_{n,p}, \sigma_{n,p}', \log(ix), \log(1 - x) \). The Eqs. (3.5), (3.11) may be said to be homogeneous of degree \( n + p \) in the sense that if \( n + p \) is the degree of \( S_{n,p}(x) \), \( S_{n,p}', \sigma_{n,p} \), and if \( \log^q u \) is of degree \( q \), then each term in both equations is of degree \( n + p \).

3.3 Other transformations

If we consider the reflection \( P_1(x) = 1 - x \) and the inversion \( P_2(x) = 1/x \) as belonging to the group of the bilinear transformations

\[ P(x) = \frac{ax + \beta}{\gamma x + \delta} \]

with real coefficients and \( a\delta - \beta\gamma \neq 0 \), we see that they generate a subgroup
\[ P_0(x) = P_1 P_1(x) = P_2 P_2(x) = x \]
\[ P_1(x) = 1 - x \]
\[ P_2(x) = \frac{1}{x} \]
\[ P_3(x) = P_2 P_1(x) = \frac{1}{1 - x} \]
\[ P_4(x) = P_1 P_2(x) = \frac{x - 1}{x} \]
\[ P_5(x) = P_2 P_1 P_2(x) = \frac{x}{x - 1} \]  

(3.15)

By repeated use of Eq. (3.5) and (3.11) it is therefore easy to see that

\[ S_{n,p} \left( \frac{1}{1 - x} \right), \quad S_{n,p} \left( \frac{x - 1}{x} \right), \quad S_{n,p} \left( \frac{x}{x - 1} \right) \]

can be expressed as homogeneous combinations (in the above sense) of \( S_{n,p}(x) \) with the above-mentioned components and with additional terms involving \( \log{(x - 1)} \).

For reasons of simplicity we give explicitly only the known formulae for the dilogarithm:

\[ S_2(1 - x) = s_2 - S_2(x) - \log{x} \log{(1 - x)} \]
\[ S_2 \left( \frac{1}{x} \right) = -S_2(x) - s_2 - \frac{1}{2} \log^2{(-x)} \]
\[ S_2 \left( \frac{1}{1 - x} \right) = S_2(x) - 2s_2 + \log{x} \log{(1 - x)} - \frac{1}{2} \log^2{(x - 1)} \]  

(3.16)
\[ S_2 \left( \frac{x - 1}{x} \right) = S_2(x) + 2s_2 + \frac{1}{2} \log^2{(-x)} + \log{x} \log{(1 - x)} - \log^2{x} \]
\[ S_2 \left( \frac{x}{x - 1} \right) = -S_2(x) - \frac{1}{2} \log^2{(1 - x)} \]

where we have used Eq. (2.9) and Eq. (3.23) below.

From Eq. (3.15), one obtains as solution of the equations

\[ P_j(x) = x \quad (j = 1, \ldots, 5) \]

the six values

\[ x = \pm 1, \quad -1, \quad \frac{1}{2}, \quad 2, \quad \frac{1}{2} \pm \frac{1}{2} \sqrt{3} i \]

These numbers define the fixed points of the subgroup.
An interesting functional relation for $S_n(x)$ has been found by Rühl [17].
Using Eq. (2.4) one can prove by a partial fraction expansion that

$$S_n(x^m) = m^{n-1} \sum_{k=1}^{m} S_n(e_m^k x)$$  \hspace{1cm} (3.17)

where $e_m^k = e^{2\pi ik/m}$ are the $m^{th}$ roots of unity. In particular one has

$$S_n(x^2) = 2^{n-1} \left[ S_n(x) + S_n(-x) \right].$$  \hspace{1cm} (3.18)

We add here that other higher order functional relations, which combine polylogarithms of more complicated arguments, have been discussed recently by Maier and Zahn [11] for the trilogarithm $S_3(x)$ and by Wechsung [18] for the pentalogarithm $S_5(x)$.

3.4 Relations between numerical values for special arguments

In this section we investigate the relations between the constants $s_{n+p}$, $\sigma_{n+p}$ and $a_{n+p}$. These numbers play a special role in the theory of the generalized polylogarithms, since they are values of $S_{n+p}(x)$ for arguments which are invariant under the bilinear transformations discussed in the previous section. In particular, explicit relations between these constants will be given for $n + p = 2, \ldots, 5$.

The $s_n$ can be expressed in terms of the Riemann zeta function with integer arguments

$$s_n = \zeta(n).$$  \hspace{1cm} (3.19)

As is well-known from the theory of this function, we have for even $n = 2m$

$$s_{2m} = \zeta(2m) = \frac{2^{2m-1} \pi^{2m} |B_{2m}|}{(2m)!}$$  \hspace{1cm} (3.20)

where the $B_{2m}$ are the Bernoulli numbers. In particular,

$$s_2 = \frac{\pi^2}{6}$$

$$s_4 = \frac{1}{2} s_2^2 = \frac{\pi^4}{90}.$$  \hspace{1cm}

No similar relations are known for odd values $n = 2m + 1$. 
We have noted already that for all \( n \) and \( p \)

\[ s_{n,p} = s_{p,n} \]

Furthermore, Nielsen proved that for all \( n \) and \( p \), \( s_{n,p} \) can be expressed as a polynomial in terms of \( s_q = s_{q-1,1} \) \((2 \leq q \leq n + p)\), with rational coefficients. This polynomial is homogeneous of degree \( n + p \), if we say as in Section 3.2 that \( s_{n,p} \) is of degree \( n + p \). A closed formula for this polynomial has been found by Rühl [17], namely

\[ s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)p!} \int_0^1 \log^{n-1} t \log^p (1 - t) \, dt \]  \hspace{1cm} (3.21)

\[ \sum_{k=1}^{\lfloor (n+p)/2 \rfloor} (-1)^{k+1} \sum_{m_i \geq 2} \frac{H_p(m_1, m_2, \ldots, m_k)}{m_1 m_2 \cdots m_k s_{m_1} s_{m_2} \cdots s_{m_k}} \]

where

\[ H_p(m_1, m_2, \ldots, m_k) = \sum_{\sum_{i=1}^k p_i = p} \binom{m_1}{p_1} \binom{m_2}{p_2} \cdots \binom{m_k}{p_k} \]  \hspace{1cm} (3.22)

The proof of this relation is given in Appendix 1. In particular, one finds that

\[ s_{2,2} = \frac{3}{2} s_4 - \frac{1}{2} s_2^2 = \frac{1}{10} s_2^5 = \frac{1}{4} s_4 \]

\[ s_{2,3} = 2 s_5 - s_2 s_3 \]

\[ s_{3,3} = \frac{10}{3} s_6 - \frac{3}{2} s_2 s_4 - s_3^2 + \frac{1}{6} s_2^3 \]

\[ s_{4,2} = \frac{5}{2} s_6 - s_2 s_4 - \frac{1}{2} s_3^2 \]

\[ s_{4,3} = 5 s_7 - 2 s_2 s_5 - \frac{5}{2} s_3 s_4 + \frac{1}{2} s_2^2 s_3 \]

\[ s_{4,4} = \frac{35}{4} s_8 - \frac{10}{3} s_2 s_6 - \frac{1}{4} s_3 s_5 - \frac{17}{8} s_4^2 + \frac{3}{4} s_2 s_4^2 + s_2 s_3^2 - \frac{1}{2} s_2^2 \]

Another general result is known for \( \sigma_n \), namely

\[ \sigma_n = (1 - 2^{1-n}) s_n \]  \hspace{1cm} (3.23)

*) Note that [1] §18(19) does not give the last term correctly.
which follows from Eq. (3.18). No similar relations seem to exist for the \( \sigma_{n,p} \) in the case \( p \geq 2 \) or for the \( a_{n,p} \).

By exploiting the information contained in Eq. (3.5), (3.11) for the special arguments \(-1, 0, \frac{1}{2}, 1, 2\), the following results are found:

For \( n + p = 2 \):

\[
\begin{align*}
    s_2 &= - \int_0^1 t^{-1} \log (1 - t) \, dt = \frac{s_2}{6} \\
    \sigma_2 &= \int_0^1 t^{-1} \log (1 + t) \, dt = \frac{1}{2} \sigma_2 \\
    a_2 &= - \int_0^1 t^{-1} \log \left(1 - \frac{1}{2} t\right) \, dt = \frac{1}{2} s_2 - \frac{1}{2} \log^2 2
\end{align*}
\]

For \( n + p = 3 \):

\[
\begin{align*}
    \sigma_{1,2} &= \frac{1}{2} \int_0^1 t^{-1} \log^2 (1 + t) \, dt = \frac{1}{6} s_3 \\
    \sigma_3 &= - \int_0^1 t^{-1} \log t \log (1 + t) \, dt = \frac{3}{4} s_3 \\
    a_{1,2} &= \frac{1}{2} \int_0^1 t^{-1} \log^2 \left(1 - \frac{1}{2} t\right) \, dt = \frac{1}{6} s_3 - \frac{1}{6} \log^3 2 \\
    a_3 &= \int_0^1 t^{-1} \log t \log \left(1 - \frac{1}{2} t\right) \, dt = \frac{7}{6} s_3 - \frac{1}{2} s_2 \log 2 + \frac{1}{6} \log^3 2
\end{align*}
\]

For \( n + p = 4 \):

\[
\begin{align*}
    s_4 &= - \frac{1}{2} \int_0^1 t^{-1} \log \log (1 - t) \, dt = \frac{2}{5} s_2^{**} \\
    s_{2,2} &= - \frac{1}{2} \int_0^1 t^{-1} \log t \log (1 - t) \, dt = \frac{1}{10} s_2^{**}
\end{align*}
\]

\(^*\) Note the misprint in \([1]\) §20(3).

\(^{**}\) Found by other methods, given here for completeness.
\( \sigma_{1,3} = \frac{1}{6} \int_0^1 t^{-1} \log^3 (1 + t) \, dt \)
\[ = \frac{2}{5} s^2 - a_4 - \frac{7}{8} s_3 \log 2 + \frac{1}{4} s_2 \log^2 2 - \frac{1}{24} \log^4 2 \]

\( \sigma_{2,2} = -\frac{1}{2} \int_0^1 t^{-1} \log t \log^2 (1 + t) \, dt \)
\[ = -\frac{3}{4} s_3^2 + 2 a_4 + \frac{7}{4} s_3 \log 2 - \frac{1}{2} s_2 \log^2 2 + \frac{1}{12} \log^4 2 \]

\( \sigma_4 = \frac{1}{2} \int_0^1 t^{-1} \log^2 t \log (1 + t) \, dt = \frac{7}{20} s_5^2 \)

\( s_{1,3} = -\frac{1}{6} \int_0^1 t^{-1} \log^3 \left( 1 - \frac{1}{2} t \right) \, dt \)
\[ = \frac{2}{5} s_3^2 - a_4 - \frac{7}{8} s_3 \log 2 + \frac{1}{4} s_2 \log^2 2 - \frac{1}{12} \log^4 2 \]  \hspace{1cm} (3,24)

\( s_{2,2} = -\frac{1}{2} \int_0^1 t^{-1} \log t \log^2 \left( 1 - \frac{1}{2} t \right) \, dt \)
\[ = \frac{1}{20} s_3^2 - \frac{1}{6} s_3 \log 2 + \frac{1}{24} \log^4 2 \]

\[ n + p = 5 \]

\( s_{2,3} = \frac{1}{6} \int_0^1 t^{-1} \log t \log^3 (1 - t) \, dt = 2 s_5 - s_2 s_3 \)

\( \sigma_{1,4} = \frac{1}{24} \int_0^1 t^{-1} \log^4 (1 + t) \, dt \)
\[ = s_5 - s_3 - a_4 \log 2 - \frac{7}{16} s_3 \log^2 2 + \frac{1}{6} s_2 \log^3 2 - \frac{1}{30} \log^5 2 \]

\[ \text{*) Found by other methods, given here for completeness.} \]
\[ \sigma_{2,3} = -\frac{1}{6} \int_0^1 t^{-1} \log t \log^3 (1 + t) \, dt \]
\[ = -\frac{33}{32} s_2 - \frac{1}{2} s_2 s_3 + 2 a_5 + 2 a_4 \log 2 + \frac{7}{6} s_2 \log^2 2 \]
\[ - \frac{1}{3} s_2 \log^3 2 + \frac{1}{15} \log^5 2 \]

\[ \sigma_{3,2} = \frac{1}{4} \int_0^1 t^{-1} \log^2 t \log (1 + t) \, dt = -\frac{29}{32} s_5 + \frac{1}{2} s_2 s_3 \]

\[ \sigma_5 = -\frac{1}{6} \int_0^1 t^{-1} \log^3 t \log (1 + t) \, dt = \frac{15}{4} s_5 \]

\[ a_{1,4} = \frac{1}{24} \int_0^1 t^{-1} \log^4 \left( 1 - \frac{1}{2} t \right) \, dt \]
\[ = s_5 - a_5 - a_4 \log 2 - \frac{7}{16} s_3 \log^2 2 - \frac{1}{6} s_2 \log^3 2 - \frac{1}{24} \log^5 2 \quad (3.24) \]

\[ a_{2,3} = \frac{1}{6} \int_0^1 t^{-1} \log t \log^3 \left( 1 - \frac{1}{2} t \right) \, dt \]
\[ = \frac{63}{32} s_5 - \frac{1}{2} s_2 s_3 - a_5 + \frac{7}{16} s_3 \log^2 2 - \frac{2}{5} s_5 \log 2 - \frac{1}{12} s_2 \log^3 2 \]
\[ + \frac{1}{60} \log^5 2 \]

\[ a_{3,2} = \frac{1}{4} \int_0^1 t^{-1} \log^2 t \log \left( 1 - \frac{1}{2} t \right) \, dt \]
\[ = \frac{1}{32} s_5 - \frac{1}{2} s_2 s_3 + a_5 + a_4 \log 2 - \frac{1}{20} s_2 \log 2 + \frac{1}{2} s_3 \log^2 2 - \]
\[ - \frac{1}{6} s_2 \log^3 2 + \frac{1}{40} \log^5 2 \]

We can summarize these results as follows. For \( n + p = 2 \) and 3 these constants can all be expressed in terms of \( s_2, s_3, \) and \( \log 2. \) For \( n + p = 4 \) and 5, however, not only \( a_4 \) and \( s_5 \) are needed, but also \( a_5 \) and \( s_5. \) No expression involving only \( a_4 \) and \( s_5 \) is known for \( a_4 \) and \( s_5. \)
We remark that, according to Eq. (2.25), for \( n \geq 2 \)

\[
a_n = \sum_{r=1}^{\infty} \frac{1}{2^r r^2} \tag{3.25}
\]

so that we could write here for \( n = 1 \)

\[
\log 2 = a_1 = \sum_{r=1}^{\infty} \frac{1}{2^r r^2} \tag{3.26}
\]

It should be noted that the general structure of the relations (3.24) for higher \( n + p \) remains unknown.

Many of the above relations are contained in Nielsen's paper, in most cases implicitly. Furthermore, the following general relations, which can be deduced by using properties of the function \( L_{n,p}(x) \), are satisfied by the above expressions

\[
a_{1,p} = \sigma_{1,p} - \frac{\log^{p+1} 2}{(p + 1)!} \tag{3.27}
\]

and

\[
a_{n,p} = \sum_{q=1}^{n} \binom{n + p - q - 1}{p - 1} \sigma_{q,n+p-q} + \sum_{q=1}^{p} \binom{n + p - q - 1}{n - 1} \sigma_{q,n+p-q} \tag{3.28}
\]

This last formula corrects Eq. §19 (12) of Nielsen *. Nielsen actually claims in this connection that for all odd \( n + p \), \( \sigma_{n,p} \) can be expressed as a homogeneous polynomial (in the sense of Section 3.2) in terms of \( \sigma_q (2 \leq q \leq n + p) \) alone, with rational coefficients, e.g.

\[
\begin{bmatrix} \sigma_{1,4} \\ \sigma_{2,3} \end{bmatrix} = r' s_2 s_3 + r'' s_3
\tag{3.29}
\]

However, using Eq. (3.26) for \( n + p = 5 \), one finds

\[
\begin{align*}
2 \sigma_{1,4} + \sigma_{2,3} &= s_{1,4} = \sigma_{3,2} - \sigma_{4,1} \\
6 \sigma_{1,4} + 3 \sigma_{2,3} &= s_{2,3} = \sigma_{3,2}
\end{align*} \tag{3.30}
\]

* Note that [1], Eqs. §19(11) and (13) are also wrong, as can be easily verified numerically for small \( n, p \).
The appearance of the combination $2 \sigma_{1,4} + \sigma_{2,3}$ prevents us from finding independent separate expressions for $\sigma_{1,4}$ and $\sigma_{2,3}$. Because of this, and in view of the fact that Nielsen's proof is erroneous, we doubt whether his statement is correct.

4. DISPERSION RELATIONS

It was noted above that $S_{n,p}(x)$ has a branch cut along the real axis from $1$ to $\infty$. In this section we want to exploit the information contained in the integral representation of $S_{n,p}(x)$ in terms of its discontinuity across the branch cut. Relations obtained through this procedure are usually called dispersion relations by theoretical physicists. In this way one can express a number of integrals involving $S_{n,p}(x)$ in terms of $S_{n,p}(x)$ of higher order and logarithms.

From the definition (1.1) or from Eq. (3.11), which are both valid for complex arguments $z = x$, we see that for $z \to \infty$

$$S_{n,p}(z) = O(\log^{n-1} z)$$

or

$$\lim_{z \to \infty} \frac{S_{n,p}(z)}{z} = 0$$

in the whole of the cut $z$-plane. We note further from Eq. (2.28) that $S_{n,p}(z)/z$ is regular at $z = 0$, because of $p \geq 1$. Since $S_{n,p}(z)/z$ has no singularities other than the branch cut along $1 < x < \infty$, it is possible to apply the Cauchy theorem to this function, giving

$$\frac{1}{z} S_{n,p}(z) = \frac{1}{2\pi i} \int_{C} \frac{S_{n,p}(\zeta)}{\zeta(\zeta - z)} d\zeta \quad (4.1)$$

for any contour not crossing the branch cut. If the contour is as shown in Fig. 1, the integral over the circle vanishes for $\zeta \to \infty$ and one obtains the "dispersion relation"

$$\frac{1}{z} S_{n,p}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{S_{n,p}(\zeta + i\epsilon) - S_{n,p}(\zeta - i\epsilon)}{\zeta(\zeta - z)} d\zeta \quad (4.2)$$

$$= \frac{1}{\pi} \int_{\gamma} \frac{\text{Im} S_{n,p}(\zeta)}{\zeta(\zeta - z)} d\zeta .$$
The boundary value along the cut is obtained from formula (4.2) if we set \( z \) real plus a small positive imaginary part \( i\varepsilon (\varepsilon \to 0^+) \), which means

\[
\int \frac{f(\zeta) d\zeta}{\zeta - z - i\varepsilon} = \text{P} \int \frac{f(\zeta) d\zeta}{\zeta - z} + \text{Im} f(z) .
\]

Changing the variable of integration to \( t = 1/\zeta \) and replacing \( z \) by \( x \), we finally have

\[
S_{n,p}(x) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im} S_{n,p}(1/t)}{t - \frac{1}{x}} \, dt . \tag{4.3}
\]

The value of \( \text{Im} S_{n,p}(1/t) \) can be found from relation (3.11), noting that on the right-hand side the imaginary parts come from \( \log (-x) \) only. In general, we have
\(- \frac{1}{\pi} \text{Im} \ S_{n,p}\left( \frac{1}{t} \right) \)

\[
\begin{align*}
&= -(-1)^n \sum_{s=1}^{p-1} (-1)^s \sum_{m=1}^{s} (-1)^m \left( \binom{n+s-m-1}{s-m} \right) S_{n-s,m,p-s}(t) \times \\
&\quad \times \sum_{k=0}^{[\frac{(n-1)/2}]} (-1)^k \frac{2k \log \frac{\pi}{2k+1} \frac{m-2k-1}{(2k+1)!(n-2k-1)!}}{t} - \tag{4.4} \\
&\quad - (-1)^p \sum_{r=1}^{n-1} (-1)^r C_{n-r,p} \sum_{k=0}^{[\frac{(r-1)/2}]} (-1)^k \frac{2k \log \frac{\pi}{2k+1} \frac{r-2k-1}{(2k+1)!(r-2k-1)!}}{t} - \\
&\quad - (-1)^n \sum_{k=0}^{[\frac{(n+p-1)/2}]} (-1)^k \frac{2k \log \frac{\pi}{2k+1} \frac{n+p-2k-1}{(2k+1)!(n+p-2k-1)!}}{t}.
\end{align*}
\]

where the sum over \(s\) or \(r\) is replaced by zero if \(p = 1\) or \(n = 1\), respectively.

In particular, we obtain for \(p = 1\) either from Eq. (4.4) or directly from the definition (1.1)

\[
\begin{align*}
- \frac{1}{\pi} \text{Im} \ S_n\left( \frac{1}{t} \right) &= \frac{(-1)^n}{(n-1)!} \log^{n-1} t \quad \star \tag{4.5} \\
\end{align*}
\]

so that

\[
S_n(x) = \frac{(-1)^n}{(n-1)!} \int_0^x \log^{n-1} \frac{t}{t - \frac{1}{x}} \quad \star \star \tag{4.6}
\]

which can be found also from Eq. (2.4).

For \(p = 2\), one can start from Eq. (4.4) with \(n = 1\), and by applying the operation \(\int t \text{d}t/t\) repeatedly one obtains, with the help of Eq. (2.7), the expression

---

*) Lewin [6] page 2 and following uses a different convention for the imaginary parts.

**) This relation is also true for \(n = 1\), if we define \(S_1(x) = -\log (1 - x)\).
\[-\frac{1}{x} \Im S_{n-2,2}\left(\frac{1}{x}\right) = (-1)^{n-1} \left\{ S_{n-1}(t) = \sum_{r=0}^{n-2} s_{n-r} \log^{r} t + \log^{n-1} t \right\} \] (4.7)

Inserting this into Eq. (4.3) and using Eq. (4.6) gives

\[\int_{0}^{1} \frac{S_{n-1}(t)}{t - \frac{1}{x}} \, dt = (-1)^{n-1} \left( S_{n-2,2}(x) + S_{n}(x) \right) + \sum_{r=2}^{n-2} (-1)^{r} s_{n-r} S_{r}(x) + s_{n-1} \log(1 - x). \] (4.8)

Hence in particular

\[\int_{0}^{1} \frac{S_{2}(t)}{t - \frac{1}{x}} \, dt = S_{1,2}(x) + S_{3}(x) + s_{2} \log(1 - x) \] (4.9)

\[\int_{0}^{1} \frac{S_{3}(t)}{t - \frac{1}{x}} \, dt = -S_{2,2}(x) - S_{4}(x) + s_{2} S_{2}(x) + s_{3} \log(1 - x) \] (4.10)

\[\int_{0}^{1} \frac{S_{4}(t)}{t - \frac{1}{x}} \, dt = S_{3,2}(x) + S_{5}(x) + s_{2} S_{3}(x) - s_{3} S_{2}(x) + s_{4} \log(1 - x). \] (4.11)

With the help of the derivative relation (2.7), and after several integrations by parts, the integral on the left-hand side of Eq. (4.8) can be written as

\[\int_{0}^{1} \frac{S_{n-1}(t)}{t - \frac{1}{x}} \, dt = s_{n-1} \log(1 - x) + \sum_{r=2}^{m} (-1)^{r} s_{n-r} S_{r}(x) + \]

\[+ (-1)^{m+1} \int_{0}^{1} \frac{S_{n-m-1}(t) S_{m}(x t)}{t} \, dt. \] (4.12)
which yields, by comparison with Eq. (4.6), an expression for an integral containing a product of two polylogarithms, namely

\[
\int_0^1 \frac{S_{n-m-1}(t) S_m(x)}{t} \, dt = (-1)^m \left\{ \left( -1 \right)^n \left[ S_{n-2,2}(x) + S_n(x) \right] - \sum_{r=m+1}^{n-2} (-1)^r S_{n-r} S_r(x) \right\} \quad (m \geq 2, \quad n - m \geq 3)
\]

This formula was found by Nielsen in a different way*).

The procedure used for deriving formula (4.3) can also be applied to the function

\[ \log \left( 1 - x \right) S_n(x) \]

This permits us to write the representation

\[
\log \left( 1 - x \right) S_n(x) = \frac{1}{\pi} \int_0^1 \frac{\text{Im} \left[ \log \left( 1 - \frac{1}{t} \right) S_n \left( \frac{1}{t} \right) \right]}{t - \frac{1}{x}} \, dt \quad .
\]

Using

\[ \text{Im} \{ f(z) g(z) \} = \text{Im} f(z) \text{ Re} g(z) + \text{ Re} f(z) \text{ Im} g(z) \]

we obtain, using Eq. (4.5),

\[ \log \left( 1 - x \right) S_n(x) \]

\[
= \int_0^1 \frac{dt}{t - \frac{1}{x}} \left[ \text{Re} \ S_n \left( \frac{1}{t} \right) + \left( -1 \right)^n \frac{1}{(n - 1)!} \left( \log^{n-1} t \log \left( 1 - t \right) - \log^n t \right) \right].
\]

From Eq. (3.13) we find

\[ (-1)^{n-1} \text{Re} \ S_n \left( \frac{1}{t} \right) = S_n(t) - \sum_{r=2}^{n} \binom{n}{r} S_r \frac{n-r-1}{(n-r)!} \frac{\log^{r-1} t}{n!} \]

\[ + \sum_{r=2}^{n} \binom{n}{r} S_r \frac{n-r}{(n-r)!} \frac{\log^{r-2} t}{n!} \quad . \]

*\) Note the misprint in [1] §16(4). The right-most term should read \( S_{n+q-1,2}(x) \).
Inserting this result into Eq. (4.15) and using Eqs. (4.6), (4.8), we get

$$\frac{(-1)^n}{(n-1)!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^{n-1} t \log (1-t)$$

$$= \log (1-x) \left( S_n(x) - s_n \right) + S_{n-1,2}(x) - n S_{n+1}(x) + \sum_{r=2}^{n-1} s_r S_{n-r+1}(x)$$

\text{(4.17)}

This formula is valid for $n \geq 2$, the sum on the right-hand side being equal to zero for $n = 2$. In particular, we have

$$\int_0^1 \frac{dt}{t - \frac{1}{x}} \log t \log (1-t)$$

$$= \log (1-x) \left( S_2(x) - s_2 \right) + S_{1,2}(x) - 2 S_3(x)$$

$$- \frac{1}{2!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^2 t \log (1-t)$$

$$= \log (1-x) \left( S_3(x) - s_3 \right) + S_{2,2}(x) - 3 S_4(x) + s_2 S_3(x)$$

\text{(4.18)}

$$\frac{1}{3!} \int_0^1 \frac{dt}{t - \frac{1}{x}} \log^3 t \log (1-t)$$

$$= \log (1-x) \left( S_4(x) - s_4 \right) + S_{3,2}(x) - 4 S_5(x) + s_2 S_4(x) + s_3 S_3(x)$$

\text{(4.19)}

For $n = 1$, the relation degenerates into

$$- \int_0^1 \frac{dt}{t - \frac{1}{x}} \log (1-t) = \frac{1}{2} \log^2 (1-x) - S_2(x)$$

\text{(4.19)}
We note that from the last two equations (4.18) one has again
\[ s_{2,2} = \frac{1}{10} s_2^2; \quad s_{3,2} = 2 s_3 - s_2 s_3. \]

It is also possible to derive from Eq. (4.17) the relation
\[ s_{n-1,2} = \frac{n}{2} s_{n+1} - \frac{1}{2} \left( s_2 s_{n-1} + s_3 s_{n-2} + \cdots + s_{n-1} s_2 \right) \quad (4.20) \]
which was found by Nielsen in a different way.

Integral representations similar to Eq. (4.3) and (4.14) can be written not only for \( S_{n,p}(x) \) but also for any product of these functions. These relations, however, become more and more complicated and we shall not enter into a systematic exposition here.

5. ZEROS OF \( S_{n,p}(x) \)

A numerical tabulation of \( S_{n,p}(x) \) for real \( x \) and given values \( n, p \) indicates that \( \text{Re} \ S_{n,p}(x) \) has \( p \) zeros on the positive real axis, whereas \( \text{Im} \ S_{n,p}(x) \) has \( p - 1 \) zeros on this axis. It is likely that this behaviour is true for all integer values \( n > 0, p > 0 \).

The zeros are listed in the following table. The power of 10 is given in brackets.
<table>
<thead>
<tr>
<th></th>
<th>Zeros of Re $S_{n,p}(x)$</th>
<th>Zeros of Im $S_{n,p}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1, p = 1$</td>
<td>1.25951 70369 845 (1)</td>
<td>-</td>
</tr>
<tr>
<td>$n = 1, p = 2$</td>
<td>1.52632 71090 716</td>
<td>5.50374 16162 127</td>
</tr>
<tr>
<td></td>
<td>2.02726 44791 049 (2)</td>
<td></td>
</tr>
<tr>
<td>$n = 2, p = 1$</td>
<td>8.51716 73342 884 (1)</td>
<td>-</td>
</tr>
<tr>
<td>$n = 1, p = 3$</td>
<td>1.13731 31682 930</td>
<td>1.72742 72723 880</td>
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<td></td>
<td>4.05240 89793 124</td>
<td>2.18159 46206 047 (1)</td>
</tr>
<tr>
<td></td>
<td>1.90643 92968 251 (3)</td>
<td></td>
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<tr>
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6. THE NUMERICAL CALCULATION OF $S_{n,p}(x)$

6.1 Existing tables and computational procedures

A review of tables of polylogarithms and related functions is given by Fletcher et al. [7]. No tables seem to exist for the generalized polylogarithms $S_{n,p}(x)$ with $p > 1$.

An Algol procedure [19] is available for the dilogarithm $S_2(x)$. This procedure calculates the function for real $x$ to about 13 significant digits.

6.2 A method for the calculation of $S_{n,p}(x)$

We discuss a method for the numerical evaluation of the generalized polylogarithm function $S_{n,p}(x)$ for some small positive integers $n$, $p$ and arbitrary real $x$.

In the interval $-1 \leq x \leq \frac{1}{2}$, we define the function

$$
\tilde{S}_{n,p}(x) = p^n p^n x^p S_{n,p}(x)
$$

which has the property

$$
\tilde{S}_{n,p}(0) = 1.
$$

For a fixed pair of integers $n$, $p$ we approximate this normalized function $\tilde{S}_{n,p}(x)$ in the given interval by a Chebyshev series

$$
\tilde{S}_{n,p}(x) \approx \sum_{r=0}^{\infty} c^{n,p}_r T_r \left( \frac{4x + 1}{3} \right)
$$

where

$$
T_r(x) = \cos (r \arccos x)
$$

is the Chebyshev polynomial of degree $r$.

For $-1 \leq x \leq \frac{1}{2}$, $S_{n,p}(x)$ is then obtained directly from Eq. (6.1) by means of this approximation. In the case $\frac{1}{2} < x \leq 2$, we use in addition the reflection formula (3.5). For $-\infty < x < -1$ and $2 < x < \infty$ the inversion formula (3.11) is used.

We have restricted ourselves to the calculation of the $c^{n,p}_r$ for the integers $1 \leq n \leq 4$, $1 \leq p \leq 4$, $n + p \leq 5$. We present in this paper Chebyshev coefficients
for the 10 functions $S_{1,1}; S_{1,2}; S_{2,1}; S_{1,3}; S_{2,2}; S_{3,1}; S_{1,4}; S_{2,3}; S_{3,2}; S_{4,1}$.

The method may, however, be used in principle for the calculation of $S_{np}(x)$ for other positive integer values of $n$ and $p$.

Because of the fact that $S_{np}(x)$ has a branch point at $x = 1$ and all its derivatives higher than the $(n - 1)$st become infinite at $x = 1$, it is clear that the function cannot be approximated by a polynomial near this point. We have therefore chosen $x = \frac{1}{2}$ empirically as the upper limit of the basic interval for the approximation.

6.2.1 The calculation of the Chebyshev coefficients

For the calculation of the $c_{np}^m$, we used the method developed by Pavie [20], which is available as a Fortran program written in double precision mode.

In order to calculate the required input values $S_{np}(x)$ in $-1 \leq x \leq \frac{1}{2}$, we proceeded as follows. In the interval $\frac{-1}{2} \leq x \leq \frac{1}{2}$, we used the power series expansion (2.20), which gives

$$S_{np}(x) = \sum_{s=0}^{\infty} (-1)^s \frac{p!}{(p + s)!} \left( \frac{p}{p + s} \right)^n S_{p+s}(x). \tag{6.5}$$

This sum was calculated in double precision mode until an accuracy of $\sim 10^{-20}$ was reached. The Stirling numbers $S^{(p)}_m$ were generated beforehand by their stable recurrence relation [8]

$$S^{(p)}_0 = S^{(p-1)}_0 - \sigma S^{(p)}_0,$$

$$S^{(p)}_1 = 0, \quad S^{(p)}_p = 1 \tag{6.6}$$

and the quotients $S^{(p)}_{\sigma+1}/S^{(p)}_\sigma$ were stored for $p = 1(1)4, \sigma = 1(1)100$.

In the interval $-1 \leq x < \frac{-1}{2}$, the evaluation of the power series becomes difficult when $x$ approaches $-1$. A direct numerical integration of the definition (1.1) is not advisable for $n > 1$, since the integrand behaves like

$$t^{-1} \log^{n-1} t \log^p (1 - xt) = 0(t^{-1} \log^{n-1} t) \tag{6.7}$$

for $t \to 0$. For $n = 1$, however, the logarithmic factor in the right-hand side disappears and in this case we have integrated
\[ 3_{n,p}(x) = (-1)^p p \int_0^t t^{-1} \log^p (1 - xt) \, dt \quad (6.8) \]

Numerically, using an accurate double precision Gaussian integration routine [24]. Special care has been taken in the computation of the logarithm near \( t = 0 \), and a relative error of \( \sim 10^{-20} \) was reached in the result \( 3_{n,p}(x) \).

In the case \( n > 1 \), we write Eq. (1.1) as

\[ S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)! p!} \left( \int_0^5 + \int_5^t \right) \frac{\log^{n-1} t \log^p (1 - xt)}{t} \, dt \quad (6.9) \]

Substituting \( t = 5t' \) in the first integral, we obtain the relation

\[ \frac{(-1)^{n+p-1}}{(n-1)! p!} \int_0^5 \frac{\log^{n-1} t \log^p (1 - xt)}{t} \, dt = \sum_{s=0}^{n-1} \frac{(-1)^s \log^s 5}{s!} S_{n-s,p}(5x) . \quad (6.10) \]

For \( \delta = \frac{1}{2} \) this leads to

\[ 3_{n,p}(x) = 2^{-p} \sum_{s=0}^{n-1} \frac{(p \log 2)^s}{s!} 3_{n-s,p}\left(\frac{x}{2}\right) + \quad (6.11) \]

\[ + \frac{(-1)^{n+p-1}}{(n-1)! p} x^{-p} \int_{t/2}^t \frac{\log^{n-1} t \log^p (1 - xt)}{t} \, dt . \]

Because we now have \(-\frac{1}{2} \leq x/2 \leq -\frac{1}{4}\), the functions \( 3_{n-s,p}(x/2) \) may now be calculated from the power series (6.5). The integral on the right-hand-side was integrated numerically in the same way as in Eq. (6.8), since the integrand has no singularities in the region of integration.

6.3 The Algol procedure

We give an Algol procedure for the computation of the real and imaginary parts of \( S_{n,p}(x) \) for arbitrary real \( x \) and the values of \( n \) and \( p \) given above. The procedure makes use of an auxiliary procedure Cheby already described by Clenshaw et al. [22]
procedure \text{Snp}(n,p,x,\text{ReSnp},\text{ImSnp});

\textbf{value} n,p,x; \textbf{integer} n,p; \textbf{real} x,\text{ReSnp},\text{ImSnp};

\textbf{comment} This procedure calculates the real part \text{ReSnp} and the imaginary part \text{ImSnp} of Nielsen's generalized polylogarithm function

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1} t \log^p (1-xt)}{t} \, dt$$

for given integers \( n \) and \( p \) satisfying \( 1 \leq n \leq 4, 1 \leq p \leq 4, n + p \leq 5 \). Thus, any of the 10 functions \( S_{1,1}, S_{1,2}, S_{2,1}, S_{1,3}, S_{2,2}, S_{3,1}, S_{1,4}, S_{2,3}, S_{3,2}, S_{4,1} \) can be evaluated. The argument \( x \) can be any real number. For \( x \leq 1 \), \( S_{n,p}(x) \) is real valued and \text{ImSnp} will be set to zero. In most cases, 13 to 14 significant digits are correct, for exceptions see Section 6.5. An error exit \text{La} is provided for illegal values of \( n \) or \( p \);

\begin{verbatim}
begin integer r,s,n1,p1,j,k;
  real x1,y,rr,rr1,rr0,rr1,rr4,h,q,hs;
  array Sat1[1:4,1:4],fact,vr,vi[0:5],w[0:4],pi[1:5];
  switch Chebyw := L11,L12,L13,L14,La,La,La,La,La,L111,L21,L22,
                  L23,La,La,La,La,La,La,L31,L32,La,La,La,
                  La,La,La,La,La,La1;
  switch retsw := R1,R2,R3,R4,R5; 
  switch vsw := V1,V2,V3,V4,V5;
  if n < 1 \lor n > 4 \lor p < 1 \lor p > 4 \lor n + p > 5 then go to La;

comment All constants needed in the different branches of the procedure are given in the following statements, although for certain values \( n,p,x \) it may happen that only some of them are used in the relevant branch. The following notations hold:
fact[r] := r, \; \pi[r] := \pi^r, \; Sat[r,s] := S_{r,s}(4) = s_{r,s}, \; C[r,s] = C_{r,s};
\text{fact}[0] := \text{fact}[1] := 1; \; \pi[1] := 3.14159 26535 89793;
for r := 2 step 1 until 5 do
begin
  fact[r] := r \times \text{fact}[r-1]; \; \pi[r] := \pi[1] \times \pi[r-1]
end;

comment The numerical values of \( s_{r,s} \) are given by

$$\begin{pmatrix}
  \zeta(2) & \zeta(3) & \zeta(4) & \zeta(5) \\
  \zeta(3) & \frac{1}{2}\zeta(4) & 2\zeta(5)-\zeta(2)\zeta(3) & * \\
  \zeta(4) & 2\zeta(5)-\zeta(2)\zeta(3) & * & * \\
  \zeta(5) & * & * & * 
\end{pmatrix}$$
\end{verbatim}
and of \( C_{r, a} \) by

\[
(C_{r, a}) = \begin{pmatrix}
\zeta(2) & \zeta(3) & \zeta(4) & \zeta(5) \\
0 & -\frac{1}{2}\zeta(4) & -\zeta(5) - \zeta(2)\zeta(3) & * \\
\frac{1}{4}\zeta(4) & \zeta(5) + \zeta(2)\zeta(3) & * & * \\
0 & * & * & * \\
\end{pmatrix}
\]

where \( \zeta(s) \) is the Riemann zeta function;

\[
\begin{aligned}
\text{Sat1}[1,1] & := \text{C}[1,1] := 1.64\,493\, 40\,668\, 48226; \\
\text{Sat1}[1,2] & := \text{Sat1}[2,1] := \text{C}[1,2] := 1.20\,205\, 690\,317\, 59594; \\
\text{Sat1}[1,3] & := \text{Sat1}[3,1] := \text{C}[1,3] := 1.08\,232\, 32\,337\, 11138; \\
\text{Sat1}[1,4] & := \text{Sat1}[4,1] := \text{C}[1,4] := 1.03\,692\, 77\,551\, 43\,370; \\
\text{Sat1}[2,2] & := 0.27\,058\, 08\,084\, 27\,785; \\
\text{Sat1}[2,3] & := \text{Sat1}[3,2] := 0.09\,655\, 11\,599\, 89\,444; \\
\text{C}[2,1] & := \text{C}[4,1] := 0; \\
\text{C}[3,1] & := 1.69\,406\, 56\,689\, 94\,492; \text{C}[2,2] := -\text{C}[3,1]; \\
\text{C}[3,2] & := 3.04\,423\, 21\,054\, 40\,666; \text{C}[2,3] := -\text{C}[3,2]; \\
\end{aligned}
\]

**if** \( x = 1 \) **then**

\[
\begin{aligned}
\text{ReSn}p & := \text{Sat1}[n, p]; \text{ImSn}p := 0; \text{go to Lo} \\
\end{aligned}
\]

**else if** \( x < -1 \land x \leq 0.5 \) **then**

\[
\begin{aligned}
n1 & := n; p1 := p; x1 := x; \\
y & := (4 \times x1 + 1)/3; j := 1; \text{go to Lch}; \\
R1: \text{ReSn}p & := \text{chasum}; \text{ImSn}p := 0; \text{go to Lo} \\
\end{aligned}
\]

**else if** \( x < -1 \lor x > 2 \) **then**

\[
\begin{aligned}
k & := n + p; x1 := x; j := 4; \\
V: \text{vr}[0] & := 1; \text{vr}[1] := \ln(\text{abs}(x1)); \\
\text{vi}[0] & := \text{vi}[1] := 0; \\
\text{for} \ r := 2 \text{ step} 1 \text{ until} k \text{ do} \\
\text{begin} \\
\text{vr}[r] & := \text{vr}[1] \times \text{vr}[r-1]; \text{vi}[r] := 0 \\
\text{end} \\
\end{aligned}
\]

**if** \( x > 1 \) **then**

\[
\begin{aligned}
\text{begin} \text{comment} \text{ If } \xi < 0, \text{ the real and imaginary parts of } \log^\pi \xi \text{ are computed} \\
in the following statements. The definition } \log (\xi) = \log |\xi| - i\pi \\
\text{is used for } \xi > 0; \\
\end{aligned}
\]
go to vsw[k];


    vi[2] := -2 \times p1[1] \times vr[1];

end;

for r := 2 step 1 until k do
    begin
        vr[r] := vr[r]/fact[r];
        vi[r] := vi[r]/fact[r]
    end;
    go to retsw[j];

E1: x1 := 1/x; y := (4 \times x1 + 1)/3; j := 2;
    sr1r := vr[k]; sr1i := vi[k];

for r := 0 step 1 until n - 1 do
    begin
        h := G[n-r,p];
        sr1r := sr1r + vr[r] \times h;
        sr1i := sr1i + vi[r] \times h
    end;
ReSmp := ImSmp := 0; q := 1;

for s := 0 step 1 until p - 1 do
    begin
        arr := sri := 0; p1 := p - s;
        for r := 0 step 1 until s do
            begin
                n1 := n + s - r; go to Loh;
            end;
E2: h := (fact[n1-1]/(fact[s-r] \times fact[n-s-1])) \times chaum;
    arr := arr + vr[r] \times h;
    sri := sri + vi[r] \times h
end;
ReSmp := ReSmp + q \times arr; ImSmp := ImSmp + q \times sri;
q := -q
end;

h := (-1)^n; q := (-1)^p;
ReSmp := h \times ReSmp + q \times sr1r;
ImSmp := h \times ImSmp + q \times sr1i;
go to Lo
end

else
begin comment The reflection formula (3.5) is used;
\[
\begin{align*}
& w[0] := 1; w[1] := \ln(x); \\
& \text{for } r := 2 \text{ step } 1 \text{ until } n \text{ do} \\
& \quad w[r] := w[1] \times w[r-1]/r; \\
& \quad k := p; x1 := 1 - x; j := 5; \text{go to V}; \\
& R5: \quad y := (4 \times x1 + 1)/3; j := 3; \\
& \quad h := (-1)^{i} p \times w[n]; \\
& \quad ReSnP := vr[p] \times h; ImSnP := vi[p] \times h; \\
& \text{for } s := 0 \text{ step } 1 \text{ until } n - 1 \text{ do} \\
& \begin{align*}
& \quad \text{begin} \\
& \quad \text{arr} := sri := 0; q := 1; p1 := n - s; \\
& \quad \text{for } r := 0 \text{ step } 1 \text{ until } p - 1 \text{ do} \\
& \begin{align*}
& \quad \text{begin} \\
& \quad \text{n1} := p - r; \text{go to Lch}; \\
& \quad R3: \quad h := q \times chaun; \\
& \quad \text{arr} := \text{arr} + vr[r] \times h; \text{sri} := \text{sri} + vi[r] \times h; \\
& \quad q := -q \\
& \text{end}; \\
& \quad \text{ReSnP} := \text{ReSnP} + w[s] \times (Sat[p1,p] - \text{arr}); \\
& \quad \text{ImSnP} := \text{ImSnP} - w[s] \times \text{sri} \\
& \text{end}; \\
& \text{go to Lo} \\
& \text{end}; \\
\end{align*}
\end{align*}
\]
\end{align*}
\]
\] Lch: \begin{align*}
& \text{begin comment The appropriate Chebyshev approximation is calculated;} \\
& \text{go to Chebysw[10 \times n1-10+p1];} \\
& \text{L11: chaun := Cheby(y,cs11,ns11); go to L;} \\
& \text{L12: chaun := Cheby(y,cs12,ns12); go to L;} \\
& \text{L21: chaun := Cheby(y,cs21,ns21); go to L;} \\
& \text{L13: chaun := Cheby(y,cs13,ns13); go to L;} \\
& \text{L22: chaun := Cheby(y,cs22,ns22); go to L;} \\
& \text{L31: chaun := Cheby(y,cs31,ns31); go to L;} \\
& \text{L14: chaun := Cheby(y,cs14,ns14); go to L;} \\
& \text{L23: chaun := Cheby(y,cs23,ns23); go to L;} \\
& \text{L32: chaun := Cheby(y,cs32,ns32); go to L;} \\
& \text{L41: chaun := Cheby(y,cs41,ns41);} \\
& \text{L: chaun := chaun \times x11p1/(fact[p1] \times p11n1);} \\
& \text{go to retsw[j]} \\
& \text{end;}
\end{align*}
\] Lo: \text{end Snp}
6.4 Numerical values of the Chebyshev coefficients

The coefficients $c_n^D$ are presented in the same format as given by Clenshaw et al. [21], where further explanations can be found. The numbers were calculated in double precision mode on a CDC 6600 computer, and then rounded to 15 decimal digits. All the expansions are given for the interval $-1 \leq x \leq \frac{1}{2}$.
$$S_{1,1}(x) = x \sum_{r} c_r R_r \left( \frac{4x + 1}{3} \right)$$

$$S_{1,2}(x) = \frac{x^2}{4} \sum_{r} c_r R_r \left( \frac{4x + 1}{3} \right)$$

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$$\Sigma c_r = 1.16448 10529 30025$$

$$\Sigma (-1)^r c_r = 582246 70334 24113$$

$$\Sigma c_r = 516048 60676 82043$$

$$\Sigma (-1)^r c_r = 60102 84515 79798$$
\[ S_{2,1}(x) = x \sum_{r} c_r \frac{4x + 1}{3} \]

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\[ S_{1,3}(x) = \frac{x^3}{18} \sum_{r} c_r \frac{4x + 1}{3} \]

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\[ \Sigma'_{\sigma_r} = 1.07442 63872 16080 \]

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\[ \Sigma'_{\sigma_r} = 2.03533 01589 58561 \]

\[ \Sigma'(-1)_{\sigma_r} = 42754 25938 07133 \]
\[ S_{2,2}(x) = x^2 \sum_r c_r T_r \left( \frac{4x + 1}{3} \right) \]

\[ S_{3,1}(x) = x \sum_r c_r T_r \left( \frac{4x + 1}{3} \right) \]

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\[ \Sigma c_r = 1.30426 \]

\[ \Sigma (-1)^r c_r = 670228 \]
\[ S_{1,4}(x) = \frac{x^4}{96} \sum_{r} c_r \frac{\Gamma(\frac{4x+1}{3})}{\Gamma(\frac{1}{3})} \]

\[ S_{2,3}(x) = \frac{x^3}{94} \sum_{r} c_r \frac{\Gamma(\frac{4x+1}{3})}{\Gamma(\frac{1}{3})} \]

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\[ \Sigma' (-1)^r c_r = 0.30096 09217 61363 \]
\[ S_{3,2}(x) = \frac{x^2}{16} \sum r c_r T_r \left( \frac{4x + 1}{3} \right) \]

\[ S_{4,1}(x) = x \sum r c_r T_r \left( \frac{4x + 1}{3} \right) \]

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\[ \Sigma c_r = 1.000000 11584 84537 \]

\[ \Sigma'(-1)c_r = 0.97211 97704 46910 \]

\[ \Sigma c_r = 1.000000 11584 84537 \]

\[ \Sigma'(-1)c_r = 0.97211 97704 46910 \]
6.5 Numerical tests and accuracy

The Algol procedure has been tested on a CDC 3800 computer at the University of Geneva. A FORTRAN version has been written using complex arithmetic and tested for the following cases on a CDC 6600 computer at CERN.

i) Evaluation of $S_{1,p}^p(x) = p! \int_0^x t^{-p} S_{1,p}(t) \, dt$ for $p = 1(1)4$, $x = -100(10)-10(1)-1(0.1)0.9(0.01)0.99$, $x = 1 - 10^{-m}$ [m = 3(1)13]. The results have been compared with those obtained from

\[ S_{1,p}^p(x) = p! x^{-p} \int_0^{x} \frac{t^{-p}}{e^t - 1} \, dt \]

by numerical integration. Agreement was found in most cases to 13 or 14 significant digits, except in the case $p > 2$ for $m > 8$, where in the worst case, $p = 4$, at least 10 significant digits agreed.

ii) Evaluation of $S_{n,1}(x) = x^{-1} \sum_{s=0}^{\infty} \frac{x^s}{(s + 1)^{n+1}}$ for $n = 1(1)4$, $x = -0.7(0.1)0.9(0.01)0.99$. The results have been compared with those found from

\[ S_{n,1}(x) = \sum_{s=0}^{\infty} \frac{x^s}{(s + 1)^{n+1}} \]

by direct summation. Agreement was found to 13 or 14 significant digits in all cases.

iii) Evaluation of

\[ S_{n,p}(x) = \int_0^x \frac{S_{n-1,p}(t)}{t} \, dt \]

for $x < 1$ and

\[ S_{n,p}(x) = S_{n,p} + \int_1^x \frac{S_{n-1,p}(t)}{t} \, dt \]

for $x > 1$ by numerical integration for $n = 2(1)4$, $p = 1(1)4$, $n + p \leq 5$, $x = \pm 20000, \pm 10000, \pm 1000, \pm 100, \pm 10, \pm 5, -2, -1, 0, 0.5, 1.5, 1 \pm 10^{-m}$ with $m = 6, 11$ for Re $S_{n,p}(x)$ and $x = 1.01, 1.1, 1.5, 3.5, 10, 100, 1000, 10000, 20000$ for Im $S_{n,p}(x)$. Agreement to 13 or 14 significant digits was found in practically all cases.
Of course, accuracy is lost for values of $x$ in the neighbourhood of a zero. One or two digits of $\operatorname{Re} S_{n,p}(x)$ may also be lost for $p > 3$ in the regions $-2 \leq x \leq -1$ and $0.5 \leq x \leq 0.6$. In the case of $\operatorname{Im} S_{n,p}(x)$, accuracy will be lost for $1 < x < 1 + \delta$ ($\delta > 0$), because of $\operatorname{Im} S_{n,p}(1) = 0$. $\delta$ varies between $< 0.01$ for $n = 1$, $p$ arbitrary, and approximately $0.19$ for $n = 3$, $p = 2$. Nevertheless, an absolute accuracy of around $10^{-12}$ is likely to be obtained in all these cases.

The results obtained for the polylogarithms $S_n(x)$ were also checked against the 10-figure table of the Amsterdam Mathematisch Centrum [23], occasional discrepancies of one or two units in the tenth figure were found.

The authors had no access to the 20 figure table of Shafer [24].

Acknowledgement

We thank Dr. W. Rühl for reading the manuscript and for helpful comments, in particular for giving the proofs of Eq. (3.17) and (3.21).
The proof of Eq. (3.21) uses some of the ideas of Nielsen's proof and proceeds as follows. Using Eq. (2.12) with $-\alpha$ instead of $\alpha$, one obtains, analogously to Eq. (2.13),

\[
\sigma_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \alpha^{n-1} \partial \beta^p} \left[ \beta^{-1} \Gamma_1 (-\alpha, \beta; \beta + 1; \cdot) \right]_{\alpha=\beta=0}
\]

which can be written as [25]

\[
\sigma_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \alpha^{n-1} \partial \beta^p} \left[ \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta)}{\beta \Gamma(1 + \alpha + \beta)} \right]_{\alpha=\beta=0}
\]

Using [25]

\[
\log \Gamma(1 + z) = -\gamma z + \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} z^m \quad (|z| < 1)
\]

we can write

\[
\sigma_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \alpha^{n-1} \partial \beta^p} \left[ \beta^{-1} e^\varphi(\alpha, \beta) \right]_{\alpha=\beta=0}
\]

where

\[
\varphi(\alpha, \beta) = -\sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} \left[ (\alpha + \beta)^m - \alpha^m - \beta^m \right].
\]

Nielsen's theorem follows by noting that any $\zeta(q)$ in this expression is multiplied by a homogeneous polynomial of degree $q$ in $\alpha$ and $\beta$. Expanding the exponential function yields

\[
\sigma_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \frac{\partial^{n+p-1}}{\partial \alpha^{n-1} \partial \beta^p} \left\{ \beta^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[ \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} \left[ (\alpha + \beta)^m - \alpha^m - \beta^m \right] \right] \right\}_{\alpha=\beta=0}
\]
In order to carry out the differentiations with respect to $\alpha$, we apply the Leibniz formula
\[
\frac{d^p}{dx^p} \left[ \prod_{i=1}^{k} f_i(x) \right] = \left[ \sum_{i=1}^{k} f_i(x) \right]^{(p)}
\]
\[
= \sum_{p_1, p_2, \ldots, p_k} \frac{p_1! \cdots p_k!}{p_1 \cdots p_k} f_1^{(p_1)}(x)f_2^{(p_2)}(x) \cdots f_k^{(p_k)}(x)
\]
so that each sum over $m_i$ is differentiated $p_i$ times with respect to $\alpha$. The $p_i$ can therefore be restricted by
\[
1 \leq p_i \leq m_i - 1 ; \quad \sum_{i=1}^{k} p_i = p .
\]

We obtain
\[
S_{n,p} = \frac{(-1)^n p^{-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial \beta^{n-1}}
\]
\[
\left\{ \beta^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{p_1 \geq 1}^{k} \sum_{m_1 \geq 2p_1+1}^{\infty} (-1)^{m_1} \zeta(m_1) \frac{m_1}{m_1} \left( \begin{array}{c}
\frac{m_1}{p_i} \\
p_i
\end{array} \right)^{m_1-p_i} \right\}_{\beta=0}
\]

Applying the Leibniz formula again for differentiation with respect to $\beta$ and setting $\beta = 0$ gives the condition
\[
\sum_{i=1}^{k} m_i = n + p
\]
so that
\[
S_{n,p} = \sum_{k} \frac{(-1)^{k+1}}{k!} \sum_{p_1 \geq 1}^{k} \sum_{m_1 \geq 2p_1+1}^{\infty} \sum_{i=1}^{k} \zeta(m_1) \frac{m_1}{m_1} \left( \begin{array}{c}
\frac{m_1}{p_i} \\
p_i
\end{array} \right)
\]
\[
\sum_{p_1 \geq 1}^{k} \sum_{m_1 \geq 2p_1+1}^{\infty} \sum_{i=1}^{k} \left( \begin{array}{c}
\frac{m_1}{p_i} \\
p_i
\end{array} \right)
\]
\[
S_{n,p} = \sum_{k} \frac{(-1)^{k+1}}{k!} \sum_{p_1 \geq 1}^{k} \sum_{m_1 \geq 2p_1+1}^{\infty} \sum_{i=1}^{k} \zeta(m_1) \frac{m_1}{m_1} \left( \begin{array}{c}
\frac{m_1}{p_i} \\
p_i
\end{array} \right)
\]
\[
\sum_{p_1 \geq 1}^{k} \sum_{m_1 \geq 2p_1+1}^{\infty} \sum_{i=1}^{k} \left( \begin{array}{c}
\frac{m_1}{p_i} \\
p_i
\end{array} \right)
\]
from which Eq. (3.21) follows.
As an example we give the calculation of $s_{4,4}$.

\[ n = 4, \ p = 4; \quad \Sigma P_i = 4; \quad \Sigma m_i = 8; \quad \zeta(q) = s_q \]

\[ s_{4,4} = H_4(8) \frac{9}{8} - \frac{1}{2} \left[ H_4(2,6) + H_4(6,2) \right] \frac{s_2 s_4}{2 \times 6} + \frac{1}{3} \left[ H_4(3,5) + H_4(5,3) \right] \frac{s_3 s_4}{3 \times 5} + \]

\[ + H_4(4,4) \frac{s_4^2}{4^2} + \frac{1}{6} \left[ H_4(2,4,4) + H_4(2,4,2) + H_4(4,2,2) \right] \frac{s_2^2 s_4}{2^2 \times 4} + \]

\[ + \left[ H_4(2,3,3) + H_4(3,2,3) + H_4(3,3,2) \right] \frac{s_3 s_3 s_4}{2 \times 3^2} \right] - \frac{1}{24} H_4(2,2,2,2) \frac{s_4^4}{2^4}. \]

We obtain

\[ H_4(8) = \binom{8}{4} = 70 \]

\[ H_4(2,6) = H_4(6,2) = \binom{2}{1} \binom{6}{3} = 40 \]

\[ H_4(3,5) = H_4(5,3) = \binom{3}{2} \binom{5}{2} + \binom{3}{1} \binom{5}{3} = 60 \]

\[ H_4(4,4) = \binom{4}{2} \binom{4}{2} + 2 \binom{4}{1} \binom{4}{3} = 68 \]

\[ H_4(2,2,4) = H_4(2,4,2) = H_4(4,2,2) = \binom{2}{1} \binom{2}{1} \binom{4}{2} = 24 \]

\[ H_4(2,3,3) = H_4(3,2,3) = H_4(3,3,2) = 2 \binom{2}{1} \binom{3}{1} \binom{3}{1} = 36 \]

\[ H_4(2,2,2,2) = \binom{2}{1} \binom{2}{1} \binom{2}{1} \binom{2}{1} = 16 \]

so that

\[ s_{4,4} = \frac{35}{4} s_6 - \frac{10}{3} s_2 s_4 - 4 s_3 s_5 - \frac{17}{8} s_4^2 + \frac{3}{4} s_2^2 s_4 + s_2 s_3^3 - \frac{1}{24} s_4^4. \]
We give here the explicit forms of the reflection formula (3.5) and the inversion formula (3.11) for $1 \leq n \leq 4$, $1 \leq p \leq 4$, $n + p \leq 5$.

The reflection formulae (3.5) for $x < 1$

\begin{align*}
S_{1,1}(x) &= s_2 - S_{1,1}(1 - x) - \log x \log (1 - x) \\
S_{1,2}(x) &= s_3 - S_{2,1}(1 - x) + \log (1 - x) S_{1,1}(1 - x) + \frac{1}{2} \log x \log^2 (1 - x) \\
S_{2,1}(x) &= s_3 - S_{1,2}(1 - x) + \log x [s_2 - S_{1,1}(1 - x)] - \frac{1}{2} \log^2 x \log (1 - x) \\
S_{1,3}(x) &= s_4 - S_{3,1}(1 - x) + \log (1 - x) S_{2,1}(1 - x) - \frac{1}{2} \log^2 (1 - x) S_{1,1}(1 - x) - \frac{1}{6} \log x \log^3 (1 - x) \\
S_{2,2}(x) &= \frac{1}{4} s_4 - S_{2,2}(1 - x) + \log (1 - x) S_{1,2}(1 - x) + \log x [s_3 - S_{2,1}(1 - x)] + \log (1 - x) S_{1,1}(1 - x) + \frac{1}{4} \log^2 x \log^2 (1 - x) \\
S_{3,1}(x) &= s_4 - S_{3,1}(1 - x) + \log x [s_3 - S_{1,2}(1 - x)] + \frac{1}{2} \log^2 x [s_2 - S_{1,1}(1 - x)] - \frac{1}{6} \log^3 x \log (1 - x) \\
S_{1,4}(x) &= s_5 - S_{4,1}(1 - x) + \log (1 - x) S_{3,1}(1 - x) - \frac{1}{2} \log^2 (1 - x) S_{2,1}(1 - x) + \frac{1}{6} \log^2 (1 - x) S_{1,1}(1 - x) + \frac{1}{24} \log x \log^4 (1 - x) \\
S_{2,3}(x) &= 2 s_5 - s_2 s_3 - S_{3,2}(1 - x) + \log (1 - x) S_{2,2}(1 - x) - \frac{1}{2} \log^2 (1 - x) S_{1,2}(1 - x) + \log x [s_4 - S_{3,1}(1 - x)] + \log (1 - x) S_{2,1}(1 - x) - \frac{1}{2} \log^2 (1 - x) S_{1,1}(1 - x) - \frac{1}{12} \log^2 x \log^3 (1 - x)
\end{align*}
\[ S_{2,1}(x) = 2 s_2 - s_1 s_3 (1 - x) + \log (1 - x) S_{1,3}(1 - x) + \]
\[ + \log x \left[ \frac{1}{2} s_3 - S_{2,1}(1 - x) + \log (1 - x) S_{1,1}(1 - x) \right] + \]
\[ + \frac{1}{2} \log^2 x \left[ s_3 - S_{1,1}(1 - x) + \log (1 - x) S_{1,1}(1 - x) \right] + \]
\[ + \frac{1}{12} \log^3 x \log^2 (1 - x) \right] = - \frac{1}{6} \log x . \]

\[ S_{4,1}(x) = s_4 - S_{1,4}(1 - x) + \log x \left[ s_4 - S_{1,4}(1 - x) \right] + \]
\[ + \frac{1}{2} \log^2 x \left[ s_4 - S_{1,4}(1 - x) \right] + \frac{1}{6} \log^3 x \left[ s_2 - S_{1,1}(1 - x) \right] - \]
\[ - \frac{1}{24} \log^4 x \log (1 - x) \right] = - \frac{1}{6} \log x . \]

The reflection formulae (3.5) for \( x \geq 1 \)

The formulae for the real part \( \text{Re} \; S_{n,p}(x) \) can easily be found by replacing in the above expressions:

\[ \log (1 - x) \text{ by } \log (x - 1) \]
\[ \log^2 (1 - x) \text{ by } \log^2 (x - 1) - \pi^2 \]
\[ \log^3 (1 - x) \text{ by } \log^3 (x - 1) - 3 \pi^2 \log (x - 1) \]
\[ \log^4 (1 - x) \text{ by } \log^4 (x - 1) - 6 \pi^2 \log^2 (x - 1) + \pi^4 . \]

For the imaginary part \( \text{Im} \; S_{n,p}(x) \), the following expressions hold:

\[ - \frac{1}{n} \text{ Im } S_{1,1}(x) = - \log x . \]
\[ - \frac{1}{n} \text{ Im } S_{1,2}(x) = S_{1,1}(1 - x) + \log x \log (x - 1) \]
\[ - \frac{1}{n} \text{ Im } S_{2,1}(x) = - \frac{1}{2} \log^2 x \]
\[ - \frac{1}{n} \text{ Im } S_{1,3}(x) = S_{1,1}(1 - x) - \log (x - 1) S_{1,1}(1 - x) - \]
\[ - \frac{1}{2} \log x \left[ \log^2 (x - 1) - 2 s_2 \right] \]
\[ - \frac{1}{n} \text{ Im } S_{2,2}(x) = S_{1,2}(1 - x) + \log x S_{1,1}(1 - x) + \frac{1}{2} \log^2 x \log (x - 1) \]
\[ - \frac{1}{n} \text{ Im } S_{3,1}(x) = - \frac{1}{6} \log^3 x \]
\[-\frac{1}{\pi} \text{Im } S_{1,4}(x) = S_{3,1}(1 - x) - \log (x - 1) \cdot S_{2,1}(1 - x) + \]
\[+ \frac{1}{2} \left[ \log^2 (x - 1) - 2 s_2 \right] S_{1,1}(1 - x) + \]
\[+ \frac{1}{6} \log x \left[ \log^3 (x - 1) - 6 s_2 \log (x - 1) \right] \]
\[-\frac{1}{\pi} \text{Im } S_{2,3}(x) = S_{2,2}(1 - x) - \log (x - 1) \cdot S_{1,2}(1 - x) + \]
\[+ \log x \left[ S_{2,1}(1 - x) - \log (x - 1) \cdot S_{1,1}(1 - x) \right] - \]
\[-\frac{1}{4} \log^2 x \left[ \log^2 (x - 1) - 2 s_2 \right] \]
\[-\frac{1}{\pi} \text{Im } S_{3,2}(x) = S_{1,3}(1 - x) + \log x \cdot S_{1,2}(1 - x) + \frac{1}{2} \log^2 x \cdot S_{1,1}(1 - x) + \]
\[+ \frac{1}{6} \log^3 x \cdot \log (x - 1) \]
\[-\frac{1}{\pi} \text{Im } S_{1,1}(x) = -\frac{1}{24} \log^4 x \]

**The inversion formulae (3.11) for } x > 1 \]

\[\text{Re } S_{1,1}(x) = 2 \cdot s_2 - S_{1,1} \left( \frac{1}{x} \right) - \frac{1}{2} \log^2 x \]
\[\text{Re } S_{1,2}(x) = s_3 - S_{1,2} \left( \frac{1}{x} \right) + S_{2,1} \left( \frac{1}{x} \right) + \log x \cdot S_{1,1} \left( \frac{1}{x} \right) + \frac{1}{6} \log^3 x - 3 \cdot s_2 \log x \]
\[\text{Re } S_{2,1}(x) = S_{2,1} \left( \frac{1}{x} \right) - \frac{1}{6} \log^3 x + 2 \cdot s_2 \log x \]
\[\text{Re } S_{1,3}(x) = -\frac{12}{4} \cdot s_4 - S_{1,3} \left( \frac{1}{x} \right) + S_{2,2} \left( \frac{1}{x} \right) - S_{3,1} \left( \frac{1}{x} \right) \]
\[+ \log x \left[ S_{1,2} \left( \frac{1}{x} \right) - S_{2,1} \left( \frac{1}{x} \right) \right] + \frac{1}{2} \left( 6 \cdot s_2 - \log^2 x \right) \cdot S_{1,1} \left( \frac{1}{x} \right) - \]
\[-\frac{1}{24} \log^4 x + \frac{3}{2} \cdot s_2 \log^2 x \]
\[\text{Re } S_{2,3}(x) = 2 \cdot s_4 + S_{2,2} \left( \frac{1}{x} \right) - 2 \cdot S_{3,1} \left( \frac{1}{x} \right) - \log x \cdot S_{2,1} \left( \frac{1}{x} \right) + \frac{1}{24} \log^4 x - \]
\[-\frac{3}{2} \cdot s_2 \log^2 x + s_3 \log x \]
\[\text{Re } S_{3,1}(x) = 2 \cdot s_4 - S_{3,1} \left( \frac{1}{x} \right) - \frac{1}{24} \log^4 x + s_2 \log^2 x \]
\[ \text{Re } S_{1,4}(x) = s_2 - S_{1,4} \left( \frac{1}{x} \right) + S_{2,3} \left( \frac{1}{x} \right) - S_{3,2} \left( \frac{1}{x} \right) + S_{4,1} \left( \frac{1}{x} \right) + \]
\[ + \log x \left[ S_{1,3} \left( \frac{1}{x} \right) - S_{2,2} \left( \frac{1}{x} \right) + S_{3,1} \left( \frac{1}{x} \right) \right] + \]
\[ + \frac{1}{2} \left( \log^2 x - 6 s_2 \right) \left[ S_{2,1} \left( \frac{1}{x} \right) - S_{1,2} \left( \frac{1}{x} \right) \right] + \]
\[ + \frac{1}{6} \left( \log^3 x - 18 s_2 \log x \right) S_{1,4} \left( \frac{1}{x} \right) + \frac{1}{120} \log^5 x - \frac{1}{2} s_2 \log^3 x + \]
\[ + \frac{1}{4} s_4 \log x . \]

\[ \text{Re } S_{2,3}(x) = s_2 + s_2 s_3 + S_{2,3} \left( \frac{1}{x} \right) - 2 S_{3,2} \left( \frac{1}{x} \right) + 3 S_{4,1} \left( \frac{1}{x} \right) + \]
\[ + \log x \left[ 2 S_{3,1} \left( \frac{1}{x} \right) - S_{2,2} \left( \frac{1}{x} \right) \right] + \]
\[ + \frac{1}{2} \left( \log^2 x - 6 s_2 \right) S_{2,1} \left( \frac{1}{x} \right) - \frac{1}{120} \log^5 x + \frac{1}{2} s_2 \log^3 x - \]
\[ - \frac{1}{4} s_4 \log x . \]

\[ \text{Re } S_{3,2}(x) = s_2 - 2 s_2 s_3 - S_{3,2} \left( \frac{1}{x} \right) + 3 S_{4,1} \left( \frac{1}{x} \right) + \log x S_{3,1} \left( \frac{1}{x} \right) + \]
\[ + \frac{1}{120} \log^5 x - \frac{1}{2} s_2 \log^3 x + \frac{1}{2} s_3 \log^2 x + 2 s_4 \log x . \]

\[ \text{Re } S_{4,1}(x) = S_{4,1} \left( \frac{1}{x} \right) - \frac{1}{120} \log^5 x + \frac{1}{3} s_2 \log^3 x + 2 s_4 \log x . \]

\[ - \frac{1}{\pi} \text{Im } S_{1,1}(x) = - \log x . \]

\[ - \frac{1}{\pi} \text{Im } S_{1,2}(x) = - s_2 + S_{1,1} \left( \frac{1}{x} \right) + \frac{1}{2} \log^2 x . \]

\[ - \frac{1}{\pi} \text{Im } S_{2,1}(x) = - \frac{1}{2} \log^2 x . \]

\[ - \frac{1}{\pi} \text{Im } S_{1,3}(x) = S_{1,2} \left( \frac{1}{x} \right) - S_{2,1} \left( \frac{1}{x} \right) - \log x S_{3,1} \left( \frac{1}{x} \right) - \frac{1}{6} \log^3 x + s_2 \log x . \]

\[ - \frac{1}{\pi} \text{Im } S_{2,2}(x) = s_3 - S_{2,1} \left( \frac{1}{x} \right) + \frac{1}{6} \log^3 x - s_2 \log x . \]

\[ - \frac{1}{\pi} \text{Im } S_{3,1}(x) = - \frac{1}{6} \log^3 x . \]
\[ \begin{align*}
\frac{1}{\pi} \text{Im } S_{1,4}(x) &= \frac{3}{4} a_4 + S_{1,3}(\frac{1}{x}) - S_{2,2}(\frac{1}{x}) + S_{3,4}(\frac{1}{x}) + \\
&+ \log x \left[ S_{2,4}(\frac{1}{x}) - S_{1,2}(\frac{1}{x}) \right] + \left( \frac{1}{2} \log^2 x - s_2 \right) S_{1,1}(\frac{1}{x}) + \\
&+ \frac{1}{24} \log^4 x - \frac{1}{2} s_2 \log^3 x \\
\frac{1}{\pi} \text{Im } S_{2,3}(x) &= -\frac{7}{4} a_4 + 2 S_{3,3}(\frac{1}{x}) - S_{2,2}(\frac{1}{x}) + \log x S_{2,1}(\frac{1}{x}) - \\
&- \frac{1}{24} \log^4 x + \frac{1}{2} s_2 \log^3 x \\
\frac{1}{\pi} \text{Im } S_{3,2}(x) &= -a_4 + S_{3,3}(\frac{1}{x}) + \frac{1}{24} \log^4 x - \frac{1}{2} s_2 \log^3 x + a_3 \log x \\
\frac{1}{\pi} \text{Im } S_{4,1}(x) &= -\frac{1}{24} \log^4 x
\end{align*} \]

The inversion formulae (3.11) for \( x < -1 \)

\[ \begin{align*}
S_{1,1}(x) &= -s_2 - S_{1,1}(\frac{1}{x}) - \frac{1}{2} \log^2 |x| \\
S_{1,2}(x) &= s_3 - S_{1,2}(\frac{1}{x}) + S_{2,1}(\frac{1}{x}) + \log |x| S_{1,1}(\frac{1}{x}) + \frac{1}{6} \log^3 |x| \\
S_{2,1}(x) &= S_{2,1}(\frac{1}{x}) - \frac{1}{6} \log^3 |x| - s_2 \log |x| \\
S_{1,3}(x) &= -s_4 - S_{1,3}(\frac{1}{x}) + S_{2,2}(\frac{1}{x}) - S_{3,4}(\frac{1}{x}) + \\
&+ \log |x| \left[ S_{3,4}(\frac{1}{x}) - S_{2,4}(\frac{1}{x}) \right] - \frac{1}{2} \log^2 |x| S_{1,1}(\frac{1}{x}) - \frac{1}{24} \log^4 |x| \\
S_{2,2}(x) &= -\frac{7}{4} s_4 + S_{2,2}(\frac{1}{x}) - 2 S_{3,3}(\frac{1}{x}) - \log |x| S_{2,1}(\frac{1}{x}) + \\
&+ \frac{1}{24} \log^4 |x| + s_3 \log |x| \\
S_{3,1}(x) &= -\frac{7}{4} s_4 - S_{3,1}(\frac{1}{x}) - \frac{1}{24} \log^4 |x| - \frac{1}{2} s_2 \log^3 |x|
\end{align*} \]
\[ S_{1,4}(x) = a_5 - S_{1,4}\left(\frac{1}{x}\right) + S_{2,2}\left(\frac{1}{x}\right) - S_{3,2}\left(\frac{1}{x}\right) + S_{4,1}\left(\frac{1}{x}\right) + \]
\[ + \log |x| \left[ S_{1,3}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) + S_{3,1}\left(\frac{1}{x}\right) \right] + \]
\[ + \frac{1}{2} \log^2 |x| \left[ S_{2,1}\left(\frac{1}{x}\right) - S_{1,3}\left(\frac{1}{x}\right) \right] + \frac{1}{6} \log^3 |x| S_{1,4}\left(\frac{1}{x}\right) + \]
\[ + \frac{1}{40} \log^5 |x| \]

\[ S_{2,3}(x) = a_5 + a_3 a_2 + S_{2,3}\left(\frac{1}{x}\right) - 2 S_{3,2}\left(\frac{1}{x}\right) + 3 S_{4,1}\left(\frac{1}{x}\right) + \]
\[ + \log |x| \left[ 2 S_{3,1}\left(\frac{1}{x}\right) - S_{2,2}\left(\frac{1}{x}\right) \right] + \frac{1}{2} \log^2 |x| S_{2,2}\left(\frac{1}{x}\right) - \]
\[ - \frac{1}{40} \log^5 |x| - a_4 \log |x| \]

\[ S_{3,2}(x) = a_5 + a_3 a_2 - S_{3,2}\left(\frac{1}{x}\right) + 3 S_{4,1}\left(\frac{1}{x}\right) + \log |x| S_{3,1}\left(\frac{1}{x}\right) + \frac{1}{120} \log^5 |x| + \]
\[ + \frac{1}{2} a_3 \log^2 |x| - \frac{7}{4} a_4 \log |x| \]

\[ S_{4,1}(x) = S_{4,1}\left(\frac{1}{x}\right) - \frac{1}{120} \log^5 |x| - \frac{1}{6} a_2 \log^3 |x| - \frac{7}{4} a_4 \log |x| \]
\[
\begin{pmatrix}
1.64493 & 4.0608 & 4.8226 \\
1.20205 & 6.9031 & 5.9594 \\
1.08232 & 3.2337 & 1.1138 \\
1.03692 & 7.7551 & 4.3370
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.20205 & 6.9031 & 5.9594 \\
2.70580 & 8.0842 & 7.7845 \text{ (-1)} \\
9.65111 & 5.9989 & 4.4373 \text{ (-2)} \\
9.65111 & 5.9989 & 4.4373 \text{ (-2)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.08232 & 3.2337 & 1.1138 \\
9.65111 & 5.9989 & 4.4373 \text{ (-2)} \\
4.05368 & 9.7271 & 5.1974 \text{ (-2)} \\
4.12316 & 5.1524 & 3.2536 \text{ (-3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.08232 & 3.2337 & 1.1138 \\
4.05368 & 9.7271 & 5.1974 \text{ (-2)} \\
4.12316 & 5.1524 & 3.2536 \text{ (-3)} \\
6.02891 & 5.3283 & 3.1914 \text{ (-4)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
8.22467 & 0.3342 & 4.1132 \text{ (-1)} \\
9.01542 & 6.7736 & 9.6957 \text{ (-1)} \\
9.47032 & 8.2949 & 7.2459 \text{ (-1)} \\
9.72119 & 7.7044 & 6.9093 \text{ (-1)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.50257 & 1.1289 & 4.9493 \text{ (-1)} \\
8.77856 & 7.1568 & 6.5350 \text{ (-2)} \\
4.89363 & 9.7049 & 9.6906 \text{ (-2)} \\
9.60156 & 8.4431 & 2.9833 \text{ (-3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
2.37523 & 6.6322 & 6.1849 \text{ (-2)} \\
9.60156 & 8.4431 & 2.9833 \text{ (-3)} \\
4.89363 & 9.7049 & 9.6906 \text{ (-2)} \\
3.13500 & 9.6016 & 8.0862 \text{ (-3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
5.82240 & 5.2646 & 5.0125 \text{ (-1)} \\
5.37213 & 1.9360 & 8.0402 \text{ (-1)} \\
5.17479 & 0.6167 & 3.8994 \text{ (-1)} \\
5.08400 & 5.7924 & 2.2687 \text{ (-1)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
9.47530 & 0.4230 & 12.771 \text{ (-2)} \\
4.07582 & 3.9159 & 3.0925 \text{ (-2)} \\
1.85307 & 8.6065 & 4.6661 \text{ (-2)} \\
1.41342 & 3.7214 & 9.9001 \text{ (-2)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.0413 & 3.7214 & 9.9001 \text{ (-2)} \\
3.87606 & 7.3446 & 6.5264 \text{ (-3)} \\
1.80165 & 3.7870 & 3.8018 \text{ (-3)} \\
* & * & *
\end{pmatrix}
\]
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Re $S_{11}(x)$  Re $S_{21}(x)$
Re $S_{41}(x)$  Re $S_{61}(x)$
$-2 \leq x \leq 3.5$

FIG. 3