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A CONSTRUCTION OF FUNCTIONS THAT SATISFY CROSSING SYMMETRYAND UNITARITY

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A B S T R A C T

We construct $\pi^0\pi^0$ amplitudes with arbitrary polynomial increase in the double spectral region that satisfy crossing symmetry and the inelastic unitarity bounds $\text{Im } f_\ell(s) - |f_\ell(s)|^2 \geq 0$, $\ell = 0, 1, 2, \dots$, for all $s > 4$. Exact elastic unitarity is then obtained by an iteration procedure which was proposed by Mandelstam and discussed in detail by Atkinson. The resulting amplitude which fulfils crossing symmetry elastic unitarity and the unitarity bounds in the inelastic region is the sum of Regge terms with bounded trajectories and a once subtracted Mandelstam representation which guarantees the unitarity of the whole amplitude.

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1. - INTRODUCTION

The maximal analyticity domain of a two-particle scattering amplitude compatible with elastic unitarity was derived by Mandelstam ¹⁾. The further condition of the two-particle unitarity, $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$, $\ell = 0, 1, 2, \dots$, in the inelastic region restricts the growth of the scattering amplitude at high energy ²⁾. These bounds are even obtained under the weaker assumptions of axiomatic field theory. But also with Mandelstam analyticity the statements mainly concern the physical region, they do not affect the double spectral region. There is one exception, obtained by Martin ³⁾: if the amplitude has a positive double spectral function (and any finite number of subtractions) then its (locally averaged) values are bounded by $\text{const}(1+|s|+|t|)^{1+\epsilon}$ for all s and t on the first sheet. Hence the question arises whether an unrestricted double spectral function allows scattering amplitudes with arbitrary polynomial (or even stronger) increase which also satisfy the unitarity bounds. This problem becomes more difficult if in addition elastic unitarity is required. The convergence of the Mandelstam iteration ¹⁾ to obtain elastic unitarity has been derived by Atkinson ^{4), 5)} for any finite number of subtractions for a $\pi\pi$ amplitude. But including the unitarity bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$, $\ell = 0, 1, 2, \dots$, $s \geq 16$, a proof has only been given in the case without subtractions ⁴⁾ and, for $\pi^0\pi^0$ scattering, in the case with one subtraction ^{6), 7)}.

In the present paper we show how $\pi^0\pi^0$ amplitudes can be constructed which need $n (= 0, 1, 2, \dots)$ subtractions in their Mandelstam representation and satisfy elastic unitarity and the inelastic unitarity bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ for all energies $s \geq 16$.

In Section 2, the reduced problem without elastic unitarity is solved. The amplitudes have Regge asymptotics up to terms $\sim s^{-1+\epsilon}$, the partial waves are meromorphic in the half plane $\text{Re } \ell \geq -1+\epsilon$.

In Section 3, we apply the Mandelstam iteration to obtain also elastic unitarity. This problem is reduced to the case with one subtraction as treated in ⁷⁾. The strongly increasing parts of the resulting amplitudes show still Regge behaviour corresponding to poles in the half plane $\text{Re } \ell \geq 1-\epsilon$.

2. - THE CONSTRUCTION OF AMPLITUDES WHICH SATISFY CROSSING SYMMETRY AND UNITARITY BOUNDS

We consider $\pi^0\pi^0$ elastic scattering with pions of mass 1. The Mandelstam variables are then normalized to $s + t + u = 4$. The scattering angle in the s channel is denoted by $z = 1 + (2t/s-4)$; in addition the variable $x = \frac{1}{2}(s-4)$ $z = t + (s-4/2)$ is used.

We want to write down a function $A(s, t, u)$ which

- $\alpha)$ has Mandelstam analyticity,
- $\beta)$ satisfies crossing symmetry, i.e., $A(s, t, u)$ is symmetric in s, t and u ,
- $\gamma)$ does not violate the unitarity bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$, $\ell = 0, 1, 2, \dots$, for $s > 4$ and
- $\delta)$ shows Regge asymptotics up to terms $\sim s^{-1+\epsilon}$.

This amplitude is constructed as a sum

$$A(s, t, u) = R(s, u) + R(s, t) + R(t, u) + G(s, t, u) \quad (2.1)$$

where all functions are symmetric in their arguments.

The expression $R(s, u)$ is a term with Regge asymptotics in the s and in the u channel for fixed t

$$R(s, u) = \beta(t) \int_5^6 ds' \rho(s') \int_5^6 dt' \rho(t') \int_5^6 du' \rho(u') [(s'-s)(t'-t)(u'-u)]^{\gamma(t)} \quad (2.2)$$

with $\rho(s) \geq 0$, $\int ds \rho(s) = 1$, and a Hölder continuous

$$\gamma(t) = -\frac{A}{2} + \frac{1}{\pi} \int_4^\infty \frac{\text{Im } \gamma(t')}{t'-t} dt', \quad \text{Im } \gamma(t) \geq 0.$$

The analytic function $\beta(t)$ has a right-hand cut $4 < t < \infty$ and is real if $t < 4$.

From (2.2) we inspect that $R(s,u)$ has the asymptotic expansion

$$R(s,u) \cong \sum_{\nu=0}^{\infty} g_{\nu}(t) s^{\alpha(t)-\nu} \quad (2.3)$$

for $|s| > 6$, $|u| > 6$ and fixed t , where the leading Regge trajectory is given by

$$\alpha(t) = 2\gamma(t) = -A + \frac{1}{\pi} \int_4^{\infty} \frac{\text{Im} \alpha(t')}{t'-t} dt' \quad (2.4)$$

and

$$g_0(t) = \beta(t) \int dt' \rho(t') (t'-t)^{\gamma(t)} \quad (2.5)$$

is assumed to be a bounded function.

We restrict the values of A by

$$2n-1 < A \leq 2n \quad (2.6)$$

for some positive integer n .

If the function $\rho(s)$ is smooth enough $R(s,u)$ is bounded in any finite region. A sufficient condition is $\rho(s)$ has an infinite number of derivatives which vanish together with $\rho(s)$ at $s = 5$ and $s = 6$.

For $t < 4$ and $s > 6$ all functions in (2.2) are real except $(s'-s)^{\gamma(t)}$ which causes the imaginary part

$$\begin{aligned} \phi(s,t) = \text{Im} R(s,u) &= -\sin \pi \gamma(t) \cdot \beta(t) \\ &\times \int ds' \rho(s') \int dt' \rho(t') \int du' \rho(u') [(s-s')(t'-t)(u'-4+s+t)]^{\gamma(t)} \end{aligned} \quad (2.7)$$

and the real part

$$\begin{aligned} \psi(s,t) = \text{Re} R(s,u) &= \cos \pi \gamma(t) \cdot \beta(t) \\ &\times \int ds' \rho(s') \int dt' \rho(t') \int du' \rho(u') [(s-s')(t'-t)(u'-4+s+t)]^{\gamma(t)} \end{aligned} \quad (2.8)$$

(A) - The partial waves of $R(s,u)$

We now want to derive that the partial waves of $R(s,u)$ in the s channel

$$f_\ell^R(s) = \frac{1}{2} [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 R(s,u) P_\ell\left(1 + \frac{2t}{s-4}\right) dt$$

satisfy under certain conditions on $\alpha(t)$ and $\beta(t)$

$$\text{Im } f_\ell^R(s) \geq 4 |f_\ell^R(s)|^2, \quad \ell = 0, 1, 2, \dots \quad (2.9)$$

for $s > s_1$ with some finite s_1 .

For the estimates we remember the following simple statements :

- (a) - if the partial wave expansion coefficients of two functions $f_1(s,t)$ and $f_2(s,t)$ are positive then the function $f_1(s,t) \cdot f_2(s,t)$ has also positive partial waves ;
- (b) - if $f(s,t)$ has a power series in $x = t + (s-4)/2$ with positive coefficients then its partial waves are also positive.

(A.1) - In the first step we study the function

$$[(s-s')(t'-t)(u'-u)]^{\gamma(t)} = s^{-L} \exp[g(s,t)] \quad (2.10)$$

with

$$g(s,t) = \gamma(t) [\log(s-s') + \log(t'-t) + \log(u'-4+s+t)] + L \log s$$

and $L > -2A$. The Taylor coefficients of

$$g(s,t) = \sum c_n(s) x^n$$

are given by

$$c_0(s) = g\left(s, -\frac{s-4}{2}\right) > 0$$

and, if $n > 0$,

$$c_n(s) = \frac{1}{\pi} \int_4^{\infty} dt [\operatorname{Im} g(s, t) - (-1)^n \operatorname{Im} g(s, 4-s-t)] (t + \frac{s-4}{2})^{-n-1}$$

The imaginary part of $g(s, t)$ is easily seen as

$$\operatorname{Im} g(s, t) = \operatorname{Im} \gamma(t) [\log(s-s') + \log(t'-t) + \log(u'-4+s+t)] - \pi \theta(t-t') \operatorname{Re} \gamma(t)$$

and

$$\operatorname{Im} g(s, 4-s-t) = \pi \theta(t-u') \gamma(4-s-t)$$

for $t > 4$. If we now assume

$$\begin{aligned} -\pi \operatorname{Re} \gamma(t) + \pi \gamma(4-s-t) &\leq \\ &\leq \operatorname{Im} \gamma(t) [\log(s-4) + \log(t-5) + \log(s+t)] \end{aligned} \quad (2.11)$$

for all $t > 5$ and $s > s_2$ with some finite s_2 then the $c_n(s)$ are obviously positive for $s > s_2$. We may obtain the condition (2.11) for Hölder continuous functions $\operatorname{Im} \gamma(t)$ which are considered by Atkinson⁴⁾ (the class of functions $\mathcal{F}_0$ in Ref. ⁷⁾), i.e.,

$$\operatorname{Im} \gamma(t) \sim c \cdot t^{-\mu} \log^{-2} t, \quad c > 0, \quad 0 < \mu < \frac{1}{2}.$$

Other simple examples are functions

$$\gamma(t) = c \cdot t^{-\mu} + O\left(\frac{1}{t}\right).$$

If we choose $\gamma(t)$ such that (2.11) is satisfied, the coefficients $c_n(s)$, $n = 0, 1, \dots$, of $g(s, t)$ are positive and also

$$s^{-L} \exp[g(s, t)] = s^{-L} \sum_{\nu=0}^{\infty} (\nu!)^{-1} [g(s, t)]^{\nu} = \sum_{n=0}^{\infty} A_n(s) x^n$$

develops positive Taylor coefficients $A_n(s)$ for $s > s_2$. From our statement (b), we see that the function (2.10) has then positive partial waves, a result which also applies to $\int ds' \rho(s') \int dt' \rho(t') \times \int du' \rho(u') [\dots] \gamma^{(t)}$ since $\rho(s) \gg 0$.

(A.2) - In the next step we discuss $\beta(t) \cdot \cos \pi \gamma(t)$. We choose $\beta(t)$ in such a way that $\beta(t) \cdot \cos \pi \gamma(t)$ has negative partial waves and $R(s,u)$ shows the correct particle poles on the second sheet (the residues are polynomials in s).

We define a function $\eta(t)$ by $\gamma(t) = -N + \eta(t)$ where N is the minimal integer $N \geq \frac{A}{2}$. The condition (2.6) restricts the limiting values of $\eta(t)$ for $|t| \rightarrow \infty$ to $0 \leq \eta(-\infty) < \frac{1}{2}$. The partial waves of $\eta(t)$ are positive. If we now expand $2(-1)^N (2L-2N-\alpha(t))^{-1} \cos \pi \gamma(t) = (L-\eta(t))^{-1} \cos \pi \eta(t)$ into a series of $\eta(t)$, L is an even integer greater than N , the coefficients are

$$c_n = L^{-n-1} \sum_{\ell=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^\ell}{(2\ell)!} (\pi L)^{2\ell}$$

The sum

$$\sigma_n = \sum_{\ell=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^\ell}{(2\ell)!} (\pi L)^{2\ell}$$

and therefore c_n are positive for $n = 0$ and for large values of n , since σ_n converges to $\cos \pi L = 1$. Hence we can find a polynomial in $\eta(t)$, $P(\eta)$, such that $P(\eta)(L-\eta)^{-1} \cos \pi \eta$ has a positive power series in η . To introduce all the (finite) particle poles we may multiply this expression with terms $(M-\eta(t))^{-1} = 2(2M-2N-\alpha(t))^{-1}$ where M is any positive integer. The result is still a positive power series in $\eta(t)$ which then also develops positive partial waves. This construction defines $-\beta(t) \cos \pi \gamma(t)$, hence $\beta(t) \cos \pi \gamma(t)$ has negative partial waves. The functions $\beta(t)$ and $g_0(t)$ in (2.5) are bounded on the physical sheet. We may easily alter $\beta(t)$ by multiplication with a function that has positive partial waves. An interesting example is $\exp[-(4-t)^\delta]$, $0 < \delta < \frac{1}{2}$. With this factor, $\beta(t)$ and therefore $R(s,u)$ do not show up in the t channel asymptotic expansion, $|t| \rightarrow \infty$.

(A.3) - The imaginary part (2.7) is related to the real part (2.8) by

$$\phi(s,t) = -\operatorname{tg} \pi \gamma(t) \psi(s,t) \quad (2.12)$$

From our construction we have obtained a real part $\psi(s,t)$ with negative partial waves. It is now easily shown that $\operatorname{tg} \pi \gamma(t) = \sum c_n(s) x^n$ as a power series in $x = t + (s-4/2)$ has positive coefficients and hence

positive partial waves. By condition (2.6) the function $c_0(s) = \operatorname{tg} \pi \gamma(-\frac{s-4}{2})$ is positive for s greater than some value s_3 . On the other hand, all the functions

$$c_n(s) = \frac{1}{\pi} \int_4^{\infty} (t + \frac{s-4}{2})^{-n-1} \operatorname{Im} \operatorname{tg} \pi \gamma(t) dt, \quad n \geq 1,$$

have positive values since

$$\operatorname{tg} z = \operatorname{tg}(x+iy) = \frac{\operatorname{tg} x (1 - \operatorname{th}^2 y) + i \operatorname{th} y (1 + \operatorname{tg}^2 x)}{1 + (\operatorname{tg} x \operatorname{th} y)^2}$$

is a Herglotz function, i.e., $\operatorname{Im} \operatorname{tg} z > 0$ if $\operatorname{Im} z > 0$.

We now write the partial wave expansions for ψ , ϕ and

$$\begin{aligned} -\operatorname{tg} \pi \gamma \\ \psi(s, t) &= 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} \sum_{\ell} (2\ell+1) a_{\ell}(s) P_{\ell}(z), \\ \phi(s, t) &= 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} \sum_{\ell} (2\ell+1) b_{\ell}(s) P_{\ell}(z) \end{aligned}$$

and

$$-\operatorname{tg} \pi \gamma(t) = \sum_{\ell} (2\ell+1) d_{\ell}(s) P_{\ell}(z)$$

where the functions $b_{\ell}(s)$ and $d_{\ell}(s)$ are negative. From the relation (2.12) we conclude

$$\begin{aligned} b_{\ell}(s) &\geq d_{\ell}(s) a_{\ell}(s) \geq 0 \\ \ell &= 0, 1, 2, \dots, \quad s > \max(s_2, s_3) \end{aligned} \quad (2.13)$$

All the partial waves $f_{\ell}^R(s) = a_{\ell}(s) + ib_{\ell}(s)$ of $R(s, u)$ are estimated by

$$|f_{\ell}^R(s)| \leq \frac{1}{2} [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 |R(s, u)| dt$$

Since the function (2.5) is bounded, it remains to investigate the integral $\operatorname{const} \cdot s^{-1} \int_0^s \alpha(t) dt$. The trajectory $\alpha(t)$ is a monotonously increasing convex function for $-\infty < t < 4$. To obtain bounded partial waves we assume

$$\alpha(0) = \alpha_0 \leq 1 \quad (2.14)$$

in addition to $A > 0$ in (2.6). The values of $\alpha(t)$ are negative for $t \rightarrow -\infty$, $\alpha(t) \leq -A + \varepsilon < 0$ if $t < -t_1(\varepsilon)$, $0 < \varepsilon < A$. We majorize $\alpha(t)$ by $\alpha_0 + c \cdot t$ in the interval $-t_1 < t < 0$, $c = t_1^{-1}(\alpha_0 - \alpha(-t_1))$, hence the above integral does not exceed

$$\begin{aligned} \left| s^{-1} \int_{4-s}^0 s^{\alpha(t)} dt \right| &\leq s^{\alpha_0-1} \int_{-t_1}^0 s^{c \cdot t} dt + s^{-A+\varepsilon} \\ &\leq (c \cdot \log s)^{-1} s^{\alpha_0-1} + s^{-A+\varepsilon} \end{aligned}$$

Therefore the partial waves $f_\ell^R(s)$ become arbitrarily small for large values of s , uniformly for all $\ell = 0, 1, 2, \dots$

If $A \neq 2, 4, 6, \dots$, the function $d_0(s) = -(s-4)^{-1} \int_0^{\infty} \operatorname{tg} \pi \gamma(t) dt$ in (2.13) has the negative limit $\operatorname{tg} \pi \frac{A}{2} < 0$ for $s \rightarrow \infty$ and we can estimate the partial waves by

$$|f_\ell^R(s)| \leq \frac{1}{8} \min \{ 1, |d_0(s)| \} \quad (2.15)$$

for s greater than some energy s_4 . From the inequality (2.13), we then obtain

$$b_\ell(s) \geq d_0(s) a_\ell(s) \geq 8 a_\ell^2(s)$$

or

$$2b_\ell(s) \geq 8a_\ell^2(s) + 8b_\ell^2(s)$$

which is the wanted result (2.9).

If $A = 2, 4, 6, \dots$, the function $d_0(s)$ vanishes as $\alpha(-s-4/2) - A$ for $s \rightarrow \infty$. But also in this case we can go through the proof if only the intercept α_0 is small enough to allow the estimate (2.15) (this is always possible for $\alpha_0 \leq 0$).

(B) - The crossed channel contributions

We have now to look for the other terms in Eq. (2.1), The function $R(s, t)$ is just obtained by the interchange $t \leftrightarrow u$ from $R(s, u)$, therefore its partial waves in the s channel are $(-1)^\ell f_\ell^R(s)$. The sum

$R(s,t) + R(s,u)$ contributes to the even partial waves with $\hat{f}_\ell(s) = 2 f_\ell^R(s)$, $\ell = 0, 2, 4, \dots$, and instead of (2.9) we get

$$\operatorname{Im} \hat{f}_\ell(s) \geq 2 |\hat{f}_\ell(s)|^2 \quad (2.16)$$

$\ell = 0, 2, 4, \dots, \quad s > s_1 \geq 4$

The third term in Eq. (2.1) does not exceed $\operatorname{const} |g_0(s)| (1+|t|)^{-A/2} (1+|s|+|t|)^{-A/2 + \epsilon}$ for great s since then $\alpha(s) \rightarrow -A < -1$. We can apply the Froissart-Gribov integral representation for its partial waves $f_\ell^{(3)}(s)$ and obtain, since $g_0(s)$ is bounded,

$$|f_\ell^{(3)}(s)| \leq \operatorname{const} s^{-\chi + \epsilon} Q_\ell(1 + \frac{10}{s-4}) \quad (2.17)$$

$$\chi = \min(A, 1 + \frac{A}{2}) > 1$$

for $\ell = 0, 2, 4, \dots$, and all $s > 4$, with $\epsilon > 0$ arbitrarily small.

Since the exponent $\chi(s)$ in $R(t,u)$ develops an imaginary part the double spectral function of $R(t,u)$ admits also negative values near the boundary of its support at $t = 5$ for arbitrarily large s . Therefore we do not even obtain a positive $\operatorname{Im} f_\ell^{(3)}(s)$ if $s \rightarrow \infty$.

But we can compensate $f_\ell^{(3)}(s)$ by the partial waves of an amplitude $G(s,t,u)$ which satisfies a Mandelstam representation with one or even without subtraction. If the spectral functions of $G(s,t,u)$ are sufficiently positive at the boundary of their support (at $t = 4$) the partial waves of $R(t,u) + G(s,t,u)$ satisfy

$$\operatorname{Im} (f_\ell^{(3)}(s) + f_\ell^G(s)) \geq 2 |f_\ell^{(3)}(s) + f_\ell^G(s)|^2 \quad (2.18)$$

for $\ell = 0, 2, \dots$, and $s > 4$ (compare Ref. 7).

If $g_0(s)$, Eq. (2.5), falls off stronger than $s^{-\lambda}$, $\lambda > 0$, the estimate (2.17) is improved by the factor $s^{-\lambda}$ and we can choose an unsubtracted double dispersion representation for $G(s, t, u)$ with an asymptotic behaviour $\sim \max[s^{-1+\varepsilon}, s^{-\lambda-A+1+\varepsilon}]$. The amplitude (2.1) $A(s, t, u)$ has then a meromorphic partial wave projection at least for $\text{Re } \ell \geq -\ell_0$, $\ell_0 = \min(1, \lambda + A - 1) + \varepsilon$. For all values of s with $\text{Re } \alpha(s) > -\ell_0$ the leading asymptotic term as $t \rightarrow \infty$ is a Regge term.

Summing up the contributions to the partial waves $f_\ell(s)$ of $A(s, t, u)$ we obtain from (2.16) and (2.18) the equality

$$\text{Im } f_\ell(s) \geq |f_\ell(s)|^2 \quad (2.19)$$

$\ell = 0, 2, 4, \dots$ and $s > s_1$.

An estimate like (2.17) holds for all the contributions of the Regge terms $\tilde{f}_\ell(s) = f_\ell^{(1)}(s) + f_\ell^{(2)}(s) + f_\ell^{(3)}(s)$, $|\tilde{f}_\ell(s)| < \text{const} \int_4^5 (t-4)^\mu Q_\ell(1 + \frac{2t}{s-4}) dt$, $\ell = 0, 2, 4, \dots$, within the finite interval $4 \leq s \leq s_1$ (we assume Hölder continuity with index μ , $0 < \mu < \frac{1}{2}$, near the threshold $t = 4$).

Therefore $G(s, t, u)$ can be chosen in such a way that (2.19) applies to all energies $s > 4$ [without affecting the asymptotic behaviour of $G(s, t, u)$].

3. - THE ITERATION FOR ELASTIC UNITARITY

In this Section we want to derive a crossing symmetric amplitude that satisfies in addition to the unitarity bounds $\text{Im } f_\ell(s) \geq |f_\ell(s)|^2$ exactly elastic unitarity. These functions are constructed by an iteration which was first given by Mandelstam ¹⁾, a rigorous proof of the convergence is due to Atkinson ^{4),5)}. For our purposes it is most convenient to use this iteration in the form of Ref. ⁸⁾ where the problem is reduced to the case of a Mandelstam representation with one subtraction which has been studied in detail in Refs. ⁶⁾ and ⁷⁾.

Let $F[\rho, \varphi]$ denote the once subtracted Mandelstam representation

$$F[\rho, \varphi] = g(s) + g(t) + g(u) + \frac{s+t}{\pi^2} \int \frac{\rho(s', t') ds' dt'}{(s'-s)(t'-t)(s'+t')} + \text{crossed terms} \quad (3.1)$$

with
$$g(s) = \frac{1}{\pi} \int \frac{\varphi(s')}{s'-s} ds' .$$

The iteration is then given by

$$A_{n+1}(s, t) = F_n(s, t) + A_0(s, t), \quad n = 0, 1, 2, \dots \quad (3.2)$$

where we define

$$F_n(s, t) = F[\rho_n^{\text{el}}(s, t) + \rho_n^{\text{el}}(t, s), \varphi_n^{\text{el}}(s)] \quad (3.3)$$

The spectral functions $\rho_n^{\text{el}}(s, t)$ and $\varphi_n^{\text{el}}(s)$ are determined by the unitarity equation

$$\begin{aligned}
 \text{Im } F[s_n^{\text{el}}(s, t), \varphi_n^{\text{el}}(s)] &= \\
 &= \lambda(s) \frac{1}{8\pi} \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \int_{-1}^{+1} dz' \int_0^{2\pi} d\vartheta \hat{A}_n^*(s, z') \times \\
 &\quad \times \hat{A}_n(s, z z' - (1-z^2)^{\frac{1}{2}}(1-z'^2)^{\frac{1}{2}} \cos \vartheta) \quad (3.4) \\
 &(\hat{A}_n(s, z) = A_n(s, t))
 \end{aligned}$$

Here we have introduced a cut-off function $\lambda(s)$ with

$$\begin{aligned}
 \lambda(s) &\quad \text{differentiable ;} \\
 \lambda(s) &= 1 \quad \text{if } 4 \leq s \leq s_\lambda \quad (> 16) ; \\
 0 \leq \lambda(s) \leq 1 &\quad \text{if } s_\lambda \leq s \leq s_\lambda < \infty ; \\
 \lambda(s) &= 0 \quad \text{if } s_\lambda < s < \infty .
 \end{aligned}$$

Equation (3.4) is equivalent to (3.5) and (3.7) (we drop the subscript n),

$$\begin{aligned}
 \rho^{\text{el}}(s, t) &= \lambda(s) \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \times \\
 &\times \int dt_1 dt_2 \left[t^2 + t_1^2 + t_2^2 - 2(t t_1 + t t_2 + t_1 t_2 - 4 \frac{t t_1 t_2}{s-4}) \right]^{-\frac{1}{2}} \times \quad (3.5) \\
 &\times D^*(s, t_1) D(s, t_2)
 \end{aligned}$$

The integration extends over the region $t_1, t_2 \geq 4$ and

$$t_1^{\frac{1}{2}} \left(1 + \frac{t_2}{s-4}\right)^{\frac{1}{2}} + t_2^{\frac{1}{2}} \left(1 + \frac{t_1}{s-4}\right)^{\frac{1}{2}} \leq t^{\frac{1}{2}}$$

The function $D(s, t)$ is the absorptive part in the t channel

$$D(s, t) = \frac{1}{2i} \text{disc}_t A(s, t) \quad (3.6)$$

The spectral function $\varphi^{\text{el}}(s)$ is determined by

$$\begin{aligned} \varphi^{\text{el}}(s) + 2(s-4)^{-1} \int_{4-s}^0 dt \frac{s+t}{\pi} \int \frac{\varphi^{\text{el}}(s,t) dt'}{(t'-t)(s+t')} &= \\ &= \frac{1}{2} \lambda(s) s^{-\frac{1}{2}} (s-4)^{-\frac{3}{2}} \left| \int_{4-s}^0 dt A(s,t) \right|^2 \end{aligned} \quad (3.7)$$

The input function $A_0(s,t)$ has to be specified. We assume the following properties :

- a) $A_0(s,t)$ is symmetric in s, t and u ;
- b) $A_0(s,t)$ satisfies Mandelstam analyticity and is holomorphic up to $s = 16$;
- c) $|A_0(s,t)| \leq c \cdot t^{1-\mu} \log^{-2} t$ if $4 \leq s \leq s_\lambda$ and $t \geq 4$ with $0 < \mu < \frac{1}{2}$. For technical details of the proof we need some more information, the discontinuity $\text{disc}_t A_0(s,t) = 2iD_0(s,t)$ has to be Hölder continuous with index μ , or, more explicitly, $(\lambda(s))^{\frac{1}{2}} D_0(s,t) \in \mathcal{B}$ where the Banach space $\mathcal{B}$ has been defined in ⁶⁾ and ⁷⁾ ;
- d) The imaginary part of the partial waves $f_{0\ell}(s)$ of $A_0(s,t)$ is bounded below, $\text{Im} f_{0\ell}(s) \geq c \cdot h_\ell(s) > 0$ for $\ell = 0, 2, 4, \dots$, and $s > 16$. The functions $h_\ell(s)$ are defined by

$$h_\ell(s) = \begin{cases} 0 & \text{if } s < 16 \\ (s-16)^\mu s^{e-2\mu} \log^{-2} s, & 0 < 2e < \mu \text{ if } \ell = 0, s > 16 \\ (s-16)^\mu s^{-2\mu} \log^{-2} s \int_{16}^{17} (t-16)^\mu Q_\ell(1+\frac{2t}{s-4}) dt & \text{if } \ell > 0, s > 16 \end{cases}$$

This implies that the support of the double spectral function of $A_0(s,t)$ extends to $s = 16$ and $t = 16$. The following proof actually allows a somewhat smaller support as discussed at the end of Section 4 in Ref. ⁷⁾.

- e) The partial waves satisfy

$$\text{Im} f_{0\ell}(s) \geq 4 |f_{0\ell}(s)|^2$$

for $\ell = 0, 2, 4, \dots$ and $s > s_\lambda$.

The preceding Section allows to construct these functions $A_0(s, t)$ in the case of any finite number of subtractions. To obtain b), we may shift all the thresholds in (2.2) beyond $s = 16$, $t = 16$ or $u = 16$. But this implies also for the fixed point solution $A(s, t)$ Regge trajectories which are holomorphic up to $s = 16$. The methods of Section 2 suggest another construction without this unwanted property. If $\alpha(s)$ and $\beta(s)$ are chosen as Hölder continuous functions ($\text{Im } \alpha(s) \in \mathcal{F}_0$ in the notation of ⁷⁾) with the additional requirement

$$\text{Re } \alpha(s) < 1 - \mu \quad \text{if } s < s_\Lambda \quad (3.9)$$

The sum $R(s, u) + R(s, t) + R(t, u)$ satisfies a) and c). It develops a double spectras function $\mathcal{P}^R(s, t)$ and one single spectral function $\varphi^R(s)$ for $4 < s < s_\Lambda$. Let $\Lambda(s, t) = \Lambda(t, s)$ be a differentiable cut-off function

$$\Lambda(s, t) = \begin{cases} 1 & \text{if } 4 \leq s \leq 16 \text{ or if } 4 \leq t \leq 16 \\ 0 & \text{if } s > s_\Lambda \text{ and } t > s_\Lambda \end{cases}$$

then $G(s, t, u)$ is taken as

$$G(s, t, u) = -F[\Lambda(s, t) \mathcal{P}^R(s, t), \lambda(s) \varphi^R(s)] + F[\omega(s, t), \chi(s)]$$

where the support of $\omega(s, t)$ lies within $s \geq 16$ and $t \geq 16$ and $\chi(s) = 0$ if $s < 16$. The amplitude (2.1) is then holomorphic up to $s = 16$ and the asymptotics is still dominated by the Regge terms in regions where $\text{Re } \alpha(t) > 1 - \mu$. Following the arguments of Section 2 we can find Hölder continuous functions $\chi(s)$ and $\omega(s, t)$ (elements of the Banach space $\mathcal{B}$ ⁷⁾) such that $A_0(s, t) = \frac{1}{4} A(s, t, u)$ fulfils a), b), c) and e). The further condition d) is easily obtained by a once subtracted Mandelstam representation $F[\omega, \chi]$, we can also assume it for $A_0(s, t)$ [see ⁷⁾ Section 4, this property d) applies to all amplitudes constructed there].

The condition c) restricts the functions $F_n(s, t)$ which are generated by the iteration (3.2)-(3.7) to a once subtracted Mandelstam representation hence the proof of convergence in ⁶⁾ can be applied. The function $(\lambda(s))^{\frac{1}{2}} A_0(s, t)$ which enters the equations (3.3)-(3.7) has to be small enough [more precisely the $\mathcal{B}$ -norm of $(\lambda(s))^{\frac{1}{2}} A_0(s, t)$ has to be small ⁶⁾] to guarantee the convergence. If necessary this can be achieved by a substitution, we take $c \cdot A_0(s, t)$ with some positive $c < 1$ instead of the original $A_0(s, t)$. The properties a)-e) are maintained by this multiplication.

The fixed point solution $A(s, t)$ of the iteration (3.2) satisfies crossing symmetry and elastic unitarity. If $A_0(s, t)$ needs $n \geq 1$ subtractions in its Mandelstam representation then also $A(s, t)$ has to be subtracted n times. We have now to show that the conditions a)-e) also imply the unitarity bounds

$$\operatorname{Im} f_\ell(s) \geq |f_\ell(s)|^2, \quad \ell = 0, 2, 4, \dots \quad \text{for } s > 16. \quad (3.10)$$

The function $A(s, t)$ splits into, (3.2), (3.3),

$$A(s, t) = F(s, t) + A_0(s, t)$$

with

$$F(s, t) = F[s^{\text{el}}(s, t), \varphi^{\text{el}}(s)] + F[s^{\text{el}}(t, s), 0] \quad (3.11)$$

The partial waves are correspondingly written as

$$f_\ell(s) = f_\ell^{\text{el}}(s) + f_{0\ell}(s)$$

with

$$f_\ell^{\text{el}}(s) = f_{1\ell}(s) + f_{3\ell}(s)$$

The elastic unitarity equation (3.4) is equivalent to $\text{Im } f_{1l}(s) = \lambda(s) |f_l(s)|^2$, $s \geq 4$ [$\text{Im } f_{0l}(s) = \text{Im } f_{2l}(s) = 0$ for $s < 16$]. So we have to show

$$\text{Im } (f_{0e}(s) + f_{2e}(s)) \geq (1 - \lambda(s)) |f_e(s)|^2$$

for $l = 0, 2, 4, \dots$, and $s > 16$. The arguments to derive these inequalities are similar to those in Ref. ⁷⁾ Section 4, and therefore we do not give all the details. First we mention that condition d) implies $D_0(s, t) \geq \text{const } t^{1-\mu} \log^{-2} t$ if $4 \leq s \leq 16$ and $t > s_2$ which is sufficient to obtain $\mathcal{P}^{el}(s, t) \geq 0$ for $4 \leq s \leq 16$ and all $t > 16$ (see ⁶⁾ Section 4.2), if $t \leq 16$ the function $\mathcal{P}^{el}(s, t)$ vanishes. Therefore the partial waves $f_{2l}(s)$ of $F[\mathcal{P}^{el}(t, s), 0]$ can be estimated by $\text{Im } f_{2l}(s) \geq -c_1 h_l(s)$, $c_1 > 0$. Following ⁷⁾ Section 4, we even obtain

$$\begin{aligned} \text{Im } f_{2e}(s) - 2(1 - \lambda(s)) |f_{1e}(s) + f_{2e}(s)|^2 &\geq \\ &\geq -c_2 h_e(s) \end{aligned} \quad (3.13)$$

where the positive constant c_2 depends quadratically on $A_0(s, t)$. With a small norm for $(\lambda(s))^{\frac{1}{2}} A_0(s, t)$, condition d) then implies $\text{Im } f_{0l}(s) \geq 2c_2 h_l(s)$, or including e) $\text{Im } f_{0l}(s) \geq 2|f_{0l}(s)|^2 + c_2 h_l(s)$. From (3.13) we infer

$$\begin{aligned} \text{Im } f_{0e}(s) &\geq (1 - \lambda(s)) (2|f_{0e}(s)|^2 + 2|f_{1e}(s) + f_{2e}(s)|^2) - \\ &\quad - \text{Im } f_{2e}(s) \end{aligned}$$

and immediately we get the wanted result (3.12).

Finally we want to make some comments about our solution. The Regge behaviour of the input function $A_0(s, t)$ still shows up in $A(s, t)$ for all contributions which increase stronger than $t^{1-\mu}$ if $t \rightarrow \infty$. Therefore the partial waves become meromorphic functions in l if $\text{Re } l > 1 - \mu$. In the region $\text{Re } l < 1 - \mu$ the analytic structure is obscured by the iteration (3.2), at least cuts are introduced.

The iteration (3.2) may be generalized such that also $F_n(s, t)$ satisfies a Mandelstam representation with $m(> 1)$ subtractions, the condition (3.9) is then weakened to $\text{Re } \alpha(s) < m - \mu$ if $s < s_\mu$. The proof of Atkinson ⁵⁾ shows that convergence also follows in this case. But for the discussion of the unitarity bounds in the inelastic region the situation becomes much more difficult. The contribution of the partial waves of $F(s, t)$ in (3.11) is then estimated by $s^{m-1} h_\ell(s)$ which is no longer a bounded function for $m \geq 2$. Hence, to derive the unitarity bounds (3.10), if possible at all, one needs a more detailed knowledge of the different terms to account cancellations.

This discussion shows a shortcoming of our solutions, the amplitude is bounded by $|A(s, t)| < \text{const } s^{1-\mu} \log^{-2} s$ for fixed $t < s_\mu$ and arbitrary values of s , we do not reach an asymptotically constant total cross-section. The behaviour $\text{Im } A(s, 0) \sim c \cdot s$ for large s was still possible in the construction of Section 2 [see condition (2.14)]. But to obtain elastic unitarity, using Atkinson's proof ⁴⁾, we needed Hölder continuous functions bounded by $\text{const } s^{1-\mu} \log^{-2} s$ with a positive μ .

There is an interesting generalization to non-polynomially bounded functions ⁸⁾, e.g., a modified Veneziano amplitude ⁹⁾. The iteration (3.2) goes through if only the conditions a)-c) are satisfied. So Atkinson and Balász ¹⁰⁾ have taken the Veneziano amplitude with a complex trajectory as input function. But they do not know whether the inelastic unitarity bounds are satisfied. A way to investigate this problem has been shown by Martin ¹¹⁾ but there are still no final results.

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REFERENCES

- 1) S. Mandelstam - Phys.Rev. 112, 1344 (1958).
- 2) M. Froissart - Phys.Rev. 123, 1053 (1961) ;
T. Kinoshita, J.J. Loeffel and A. Martin - Phys.Rev. 135, B1464 (1964) ;
For a summary of the corresponding literature, see :
R.J. Eden - "High Energy Collisions of Elementary Particles",
Cambridge University Press (1967).
- 3) A. Martin - Nuovo Cimento 61A, 56 (1969) ;
- 4) D. Atkinson - Nuclear Phys. B7, 375 (1968) ; B8, 377 (1968).
- 5) D. Atkinson - Argonne Preprint ANL/HEP 6917 (July 1969).
- 6) J. Kupsch - Nuclear Phys. B12, 155 (1969).
- 7) J. Kupsch - Nuclear Phys. B11, 573 (1969).
- 8) J. Kupsch - CERN Preprint TH. 1038 (May 1969).
- 9) G. Veneziano - Nuovo Cimento 57A, 190 (1968).
- 10) D. Atkinson and L.A.P. Balász - Argonne Preprint (July 1969).
- 11) A. Martin - Phys.Letters 29B, 431 (1969).