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ESTIMATES OF THE UNITARITY INTEGRAL

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A B S T R A C T

The elastic unitarity integral is studied for amplitudes which satisfy a Mandelstam representation without subtraction. The double spectral functions are taken to belong to function spaces which allow local, even non-integrable, singularities. The existence of fixed point solutions is derived and the additional restrictions due to inelastic unitarity are discussed.

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1. INTRODUCTION

Recently, Atkinson has derived the existence of scattering amplitudes which satisfy a Mandelstam representation with a finite number of subtractions and exactly elastic unitarity, Refs. 1), 2). It was already known from Mandelstam's paper on the spectral representation ³⁾ that elastic unitarity can be formulated as a system of non-linear integral equations for the spectral functions. In Ref. 1), Atkinson studied this system for an unsubtracted Mandelstam representation and proved the existence of non-trivial solutions within a class of Hölder continuous double spectral functions. Moreover, it could be shown that some of these solutions satisfied inelastic unitarity bounds. These results have been partly extended to amplitudes with subtractions ^{2), 4), 5)}. In all these papers the proof of elastic unitarity is based on Ref. 1) and kept its restriction to Hölder continuous double spectral functions.

In this paper we study again the elastic unitarity integral, but now for a more general class of spectral functions, which even allows singularities like $\delta(t)$ or $\text{Pf} \frac{1}{t}$. There is some kind of regularity also for our double spectral functions: integration over t has to lead to a Hölder continuous function in s , at least within the elastic strip $4 \leq s \leq 16$.

This assumption is connected with our problem in a quite natural way. If we take the imaginary part of the partial waves

$$\text{Im } f_\ell(s) = \frac{2}{\pi} [s(s-4)]^{\frac{1}{2}} \int dt Q_\ell(1 + \frac{2t}{s-4}) S(s, t)$$

$\ell \geq \ell_0 \geq 0$, we know from unitarity that $\text{Im } f_\ell(s)$ is bounded. Hence, the integration over t smoothes all the singularities of $\rho(s, t)$. To guarantee in addition a bounded real part

$$\text{Re } f_\ell(s) \sim \frac{1}{\pi} \int \frac{\text{Im } f_\ell(s')}{s' - s} ds'$$

more information about $\varphi(s,t)$ is needed, e.g., Hölder continuity after the t integration for $\text{Im} f_\ell(s)$.

For our generalization it is sufficient to study the scattering of identical particles without subtractions in the elastic region. In Section 2, the elastic unitarity condition is formulated as a fixed point problem or a non-linear equation for the double spectral function. (Here we need only the double spectral function within a neighbourhood of the elastic strips in the s and t channels.) This non-linear mapping is discussed on several Banach spaces of functions. These spaces are introduced in Section 3. The estimates of the unitarity integral follow in Sections 4 and 5 (with some technical details in the Appendices A and B). As a simple application we prove the existence of fixed point solutions by iteration in Section 6. This iteration may start with a function in a rather general Banach space which allows $\delta(t)$ or $\text{Pf} \frac{1}{t}$ singularities. But any fixed point solution within this function space exhibits only Lebesgue-integrable singularities.

If we include inelastic unitarity bounds we are almost left with the old solutions ^{1),4),5)}; we can only prove the existence of such fixed point functions which are Hölder continuous in the elastic strip $4 \leq s \leq 16$ (singularities are possible for $s > 16$). There is no general argument which forbids singularities in this interval. But we need as input functions amplitudes which satisfy inelastic unitarity bounds and show rather strong singularities outside their double spectral region. Our methods, given in Appendix C, are not effective enough for such a construction.

2. THE UNITARITY MAPPING

We consider the elastic scattering of equal (pseudo) scalar particles of unit mass. The scattering amplitude $A(s,t)$ is decomposed as

$$A(s,t) = F(s,t) + H(s,t) \quad (2.1)$$

where both functions $F(s,t)$ and $H(s,t)$ are symmetric in s,t and $u=4-s-t$ and satisfy Mandelstam analyticity. Moreover $F(s,t)$ can be represented as

$$F(s,t) = F[\mathcal{P}(s,t)] = \frac{1}{\pi^2} \int \frac{\mathcal{P}(s',t')}{(s'-s)(t'-t)} ds' dt' \\ + \text{crossed terms ,}$$

$$\mathcal{P}(s,t) = \mathcal{P}(t,s) . \quad (2.2)$$

The function $H(s,t)$ is holomorphic in s up to $s=16$ and vanishes if $|t| \rightarrow \infty$ for values of s in the interval $4 \leq s \leq 19$. It is convenient to write

$$\mathcal{P}(s,t) = \psi(s,t) + \psi(t,s) \quad (2.3)$$

where $\psi(s,t)$ has no symmetry restrictions.

We then define a mapping $\psi(s,t) \rightarrow \psi'(s,t) = \Phi[\psi(s,t)]$ within the set of double spectral functions by the equations (2.1)-(2.3) and the unitarity integral

$$\Im F[\psi'(s,t)] = \lambda^2(s) \frac{1}{4\pi} [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 dt' \int_0^{2\pi} d\vartheta A^*(s,t') A(s,t''), \\ t'' = t+t' + 2 \frac{tt'}{s-4} - 2 \sqrt{tt'(1+\frac{t}{s-4})(1+\frac{t'}{s-4})} \cos \vartheta \quad (2.4)$$

Here, we have introduced a cut-off function $\lambda(s)$ with the properties: $\lambda(s)$ is Hölder continuous and

$$0 \leq \lambda(s) \leq 1 \quad \text{if } 4 \leq s < \infty ,$$

$$\lambda(s) = 1 \quad \text{if } 4 \leq s < 17 ,$$

$$\lambda(s) = 0 \quad \text{if } s > 19 .$$

The Eq. (2.4) can be transformed in the equivalent relation ³⁾

$$\psi'(s,t) = \lambda^2(s) \frac{2}{\pi} \int K(s,t,t_1,t_2) A_t^*(s,t_1) A_t(s,t_2) dt_1 dt_2 \quad (2.5)$$

where we use the absorptive part in the t channel

$$A_t(s,t) = d(s,t) + D(s,t) , \quad (2.6)$$

$$d(s,t) = \frac{1}{\pi} \int ds' \left[\frac{1}{s'-s} + \frac{1}{s'-4+s+t} \right] P(s',t) , \quad (2.7)$$

$$D(s,t) = \frac{1}{2i} \text{disc}_t H(s,t) . \quad (2.8)$$

The kernel $K(s,t,t_1,t_2)$ is defined as

$$K(s,t,t_1,t_2) = \begin{cases} L^{-\frac{1}{2}}(s,t,t_1,t_2) & \text{if } s > 4, t > 16, \\ & 4 < t_1 < \alpha(s,t,4) , \\ & 4 < t_2 < \alpha(s,t,t_1) , \\ 0 & \text{otherwise} \end{cases} \quad (2.9)$$

with the functions

$$L(s, t, t_1, t_2) = s(s-4) \left[t^2 + t_1^2 + t_2^2 - 2(tt_1 + tt_2 + t_1 t_2) - 4 \frac{t t_1 t_2}{s-4} \right] \quad (2.10)$$

and

$$\alpha(s, t, t_1) = t + t_1 + 2 \frac{t t_1}{s-4} - 2 \sqrt{t t_1 \left(1 + \frac{t}{s-4}\right) \left(1 + \frac{t_1}{s-4}\right)}. \quad (2.11)$$

We assume a support of $\psi(s, t)$ inside the region $4 \leq s \leq 19$, $t \geq 20$. Such a restriction is reproduced by the mapping $\psi(s, t) \rightarrow \psi'(s, t) = \Phi[\psi(s, t)]$ [the actual support of (2.5) is even smaller].

For any fixed point solution of this mapping $\psi(s, t) = \Phi[\psi(s, t)]$ the corresponding amplitude $A(s, t)$ satisfies crossing symmetry and elastic unitarity. To derive such solutions one has to study the mapping $\Phi[\psi(s, t)]$ in more detail. The first rigorous proof of existence was given by Atkinson¹⁾. He restricted the double spectral functions to be elements of a suitable Banach space of Hölder continuous functions and derived all the conditions to apply the Schauder fixed point principle or even the contraction mapping theorem.

The subject of our paper is to study this mapping in more general spaces of spectral functions and to look for fixed point solutions which are not obtained in¹⁾.

We could include a finite number of subtractions for $F(s, t)$ in (2.2) and also take charged particles. But, the generalization we are interested in is independent of the number of subtractions; it can be explained most clearly in the simple case without subtractions.

We would like to mention that nevertheless the whole amplitude $A(s,t)$ may need an arbitrary number of subtractions since $H(s,t)$ is not restricted for $s > 19$, $t > 19$ and can increase in that region.

3. BANACH SPACES OF FUNCTIONS

The spectral functions will be submitted to several restrictions which are most clearly expressed by a norm condition. We therefore define the following Banach spaces of functions:

a. $\mathcal{L}^p(a)$, $1 \leq p < \infty$, $a \geq 0$

the space of all measurable functions $f(x)$, $4 \leq x < \infty$, with the finite norm

$$\|f(x)\|_{\mathcal{L}} = \left[\int_4^{\infty} |f(x) x^{-a}|^p dx \right]^{\frac{1}{p}} \quad (3.1)$$

b. $\mathcal{L}^p(a,b,\mu)$, $1 \leq p < \infty$, $a \geq 0$, $b \geq 0$, $0 < \mu < 1$

the space of all measurable functions $f(x,y)$, $4 \leq x,y < \infty$, with the finite norm

$$\begin{aligned} \|f(x,y)\|_{\mathcal{L}} = & \sup_{4 \leq x < \infty} x^{-b} \left[\int_4^{\infty} |f(x,y) y^{-a}|^p dy \right]^{\frac{1}{p}} + \\ & + \sup_{\substack{4 \leq x_1, x_2 < \infty \\ |x_1 - x_2| \leq 1}} x_1^{-b} |x_2 - x_1|^{-\mu} \left[\int_4^{\infty} |f(x_2,y) - f(x_1,y)|^p y^{-a} dy \right]^{\frac{1}{p}} \end{aligned} \quad (3.2)$$

The functions $f(x,y)$ of $\mathcal{L}^p(a,b,\mu)$ may be characterized as Hölder continuous functions in x with values $f(x,y) = f_x(y)$ in $\mathcal{L}^p(a)$.

The space $\mathcal{L}^p(a,0,\mu)$ is also denoted as $\mathcal{L}^p(a,\mu)$.

The functions $f(x,y) \in \mathcal{L}^p(a,b,\mu)$ with $f(4,y)=0$ for almost all y generate a closed subspace, $\mathcal{L}_0^p(a,b,\mu)$.

c. $\mathcal{E}(a,\mu)$, $-\infty < a < \infty$, $0 < \mu < 1$

the space of Hölder continuous functions $f(x)$, $4 \leq x < \infty$, with a finite norm

$$\begin{aligned} \|f(x)\|_{\mathcal{E}} = & \sup_{4 \leq x < \infty} x^{-a} |f(x)| + \\ & + \sup_{\substack{4 \leq x_1, x_2 < \infty \\ |x_2 - x_1| \leq 1}} x_1^{-a} |x_2 - x_1|^{-\mu} |f(x_2) - f(x_1)| \end{aligned} \quad (3.3)$$

d. $\mathcal{F}(a,\mu)$, $-\infty < a < \infty$, $0 < \mu < 1$

the space of Hölder continuous functions $f(x,y)$, $4 \leq x,y < \infty$ with the finite norm

$$\begin{aligned} \|f(x,y)\|_{\mathcal{F}} = & \sup_{4 \leq x,y < \infty} (xy)^{-a} |f(x,y)| + \\ & + \sup_{\substack{4 \leq x_1, x_2, y < \infty \\ |x_2 - x_1| \leq 1}} (x_1 y)^{-a} |x_2 - x_1|^{-\mu} |f(x_2, y) - f(x_1, y)| + \\ & + \sup_{\substack{4 \leq x, y_1, y_2 < \infty \\ |y_2 - y_1| \leq 1}} (x y_1)^{-a} |y_2 - y_1|^{-\mu} |f(x, y_2) - f(x, y_1)|. \end{aligned} \quad (3.4)$$

The functions $f(x,y) \in \mathcal{F}(a,\mu)$ which vanish if $x=4$, $f(4,y)=0$, generate a closed subspace $\mathcal{F}_0(a,\mu)$.

The unitarity mapping can be extended to some classes of generalized functions. In the following we use

e. $\mathcal{E}'(a,\mu)$

the continuous linear functionals $\tau(y)$ on the space $\mathcal{E}(a,\mu)$,

$$f(\gamma) \in E(a, \mu) \quad , \quad \tau(\gamma) \in E'(a, \mu)$$

$$|\langle \tau(\gamma) | f(\gamma) \rangle| \leq C_\tau \|f\|_E$$

f. $\tilde{E}(a, \mu_1, \mu_2)$, $-\infty < a < \infty$, $0 < \mu_1, \mu_2 < 1$

the continuous linear functionals $T(x, y) \equiv T_x(y)$ on $E(a, \mu)$ which depend Hölder continuously on a parameter x , $4 \leq x < \infty$,

$$f(\gamma) \in E(a, \mu_1) \quad , \quad T_x(\gamma) \in \tilde{E}(a, \mu_1, \mu_2) \quad ,$$

$$\langle T_x(\gamma) | f(\gamma) \rangle = F(x) \in E(0, \mu_2) \quad ,$$

$$\|F(x)\|_{E(0, \mu_2)} \leq C_T \|f\|_{E(a, \mu_1)}.$$

We denote by $\tilde{E}_c(a, \mu_1, \mu_2)$ the subset of all functionals $T(x, y) \in \tilde{E}(a, \mu_1, \mu_2)$ which have a compact support in x .

g. $\mathcal{F}'(a, \mu)$

the space of the continuous linear functionals on $\mathcal{F}(a, \mu)$.

With the usual definitions of the norm of a mapping the spaces e.-g. are also Banach spaces.

Finally, we want to note some simple relations:

$$\mathcal{L}^1(a) \subset E'(-a, \mu) \quad ,$$

$$\mathcal{L}^1(a, \mu) \subset \tilde{E}(-a, \mu, \mu') \quad ,$$

$$\mathcal{F}(a, \mu) \subset \mathcal{L}^1(1+a', a, \mu) \quad , \quad a' > a \quad ,$$

$$E(a, \mu) \times E(a, \mu) \subset \mathcal{F}(a, \mu) \quad .$$

If $\tau_1(x)$ and $\tau_2(x) \in E'(a, \mu)$ then the product $\tau_1(x) \tau_2(y)$ defines a continuous functional on $\mathcal{F}(a, 2\mu)$, i.e.,

$$E'(a, \mu) \times E'(a, \mu) \subset \mathcal{F}'(a, 2\mu).$$

4. THE REDUCED UNITARITY MAPPING

As a first step we consider the mapping $f \times g \rightarrow B$

$$B(s, t) = \int K(s, t, t_1, t_2) f(t_1) g(t_2) dt_1 dt_2 \quad (4.1)$$

where f and g depend only on one variable and K is the kernel (2.9). Since our norms (3.2) are asymmetric in the first and in the second variable we also introduce $\hat{B}(t, s) \equiv B(s, t)$.

In Appendix A the necessary estimates on integrals over $K(s, t, t_1, t_2)$, $K(s', t, t_1, t_2) - K(s, t, t_1, t_2)$ and $K(s, t', t_1, t_2) - K(s, t, t_1, t_2)$ are evaluated to apply the integration theorem of Appendix B. In the following we list some results.

4.1 If $f(t) \in \mathcal{L}^v(a + \frac{1}{v})$ and $g(t) \in \mathcal{L}^w(a + \frac{1}{w})$ then $B(s, t)$ and $\hat{B}(t, s)$ are elements of $\mathcal{L}_0^r(a + \frac{1}{r}, a, \mu)$ for the range of the quantities v, w, r, a, μ : $v \geq 1, w \geq 1, r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geq 0$ with some $q, 1 \leq q \leq 2, 0 < \mu < \frac{1}{q} - \frac{1}{2}$ and $a \geq -\frac{1}{2} + \mu$.

Furthermore, we obtain the estimates (with the corresponding norms)

$$\|B(s, t)\|_{(r)} \leq C \|f(t)\|_{(v)} \cdot \|g(t)\|_{(w)},$$

$$\|\hat{B}(t, s)\|_{(r)} \leq C \|f(t)\|_{(v)} \cdot \|g(t)\|_{(w)}.$$

We note particularly two limiting cases. For $v=w=1$ this is a mapping between the spaces

$$\mathcal{L}^1(b) \times \mathcal{L}^1(b) \rightarrow \mathcal{L}_0^q(b-1+\frac{1}{q}, b-1, \mu)$$

with $1 \leq q < 2$, $0 < \mu < \frac{1}{q} - \frac{1}{2}$, $b \geq \frac{1}{2} + \mu$. If $v^{-1} + w^{-1} < \frac{3}{2}$ the function $B(s, t)$ becomes locally bounded. But it is also continuous, and $\mathcal{L}^\infty(a, a, \mu)$ can be read as $\mathcal{F}(a, \mu)$. For $\frac{4}{3} < q \leq 2$, $b \geq 1 - \frac{1}{q}$ we obtain a mapping $\mathcal{L}^q(b) \times \mathcal{L}^q(b) \rightarrow \mathcal{F}_0(b - \frac{1}{q}, \mu)$ with $0 < \mu < \frac{3}{2} - \frac{2}{q}$.

4.2 Let us consider the case $\mathcal{L}^1(b) \times \mathcal{L}^1(b) \rightarrow \mathcal{L}_0^q(\dots)$ in more detail. Since $\mathcal{L}^\infty$ is the dual space of $\mathcal{L}^1$ we know that $B_{t_1 t_2}(s, t) = K(s, t, t_1, t_2)(t_1 t_2)^b$ is a bounded function of t_1 and t_2 with values $B(s, t) \in \mathcal{L}_0^q$ for almost all t_1 and t_2 . In Appendix A the kernel $K(s, t, t_1, t_2)$ is shown to be Hölder continuous in the following sense:

$$|t_1 - t_1'|^{-\mu} |K(s, t, t_1, t_2) - K(s, t, t_1', t_2)| = \tilde{K}_{t_1'}(s, t, t_1, t_2)$$

allows estimates of the same kind as $K(s, t, t_1, t_2)$ if only the range of the Hölder indices is restricted. Therefore, the function $B_{t_1 t_2}(s, t)$ is also Hölder continuous in t_1 and t_2 ; and $f(t)$ and $g(t)$ can be continued from $\mathcal{L}^1(b)$ to linear functionals on a space of Hölder continuous functions. More precisely we infer that Eq. (4.1) defines a bounded mapping

$$\mathcal{E}'(-b, \mu_1) \times \mathcal{E}'(-b, \mu_1) \rightarrow \mathcal{L}_0^q(b-1+\frac{1}{q}, b-1, \mu)$$

if $1 \leq q < 2$, $0 < \mu + 2\mu_1 < \frac{1}{q} - \frac{1}{2}$, $b \geq \frac{1}{2} + \mu + 2\mu_1$. By these conditions μ_1 is restricted to $0 < \mu_1 < \frac{1}{4}$.

5. THE UNITARITY MAPPING

The integral transform (4.1) is the most subtle part of the unitarity mapping introduced in Section 2. For a complete discussion we have to include an s dependence of the functions f and g in (4.1) and to investigate Hilbert transform in (2.7).

The generalization of Section 4 to a mapping

$$\begin{aligned} B(s,t) &= \int K(s,t,t_1,t_2) f(s,t_1) g(s,t_2) dt_1 dt_2, \\ \hat{B}(t,s) &= B(s,t) \end{aligned} \tag{5.1}$$

is straightforward. If $f(s,t)$ and $g(s,t)$ are elements of a space $\mathcal{L}^p(a,0,\mu)$ or $\tilde{E}(-a,\mu,\mu')$ we again derive norm conditions of the type (3.2) or (3.4) for $B(s,t)$ and $\hat{B}(t,s)$. The additional s dependence of f and g is easily estimated by

$$\begin{aligned} B(s_2,t) - B(s_1,t) &= \\ &= \int dt_1 dt_2 \left\{ (K(s_2,t,t_1,t_2) - K(s_1,t,t_1,t_2)) f(s_2,t_1) g(s_2,t_2) + \right. \\ &\quad + K(s_1,t,t_1,t_2) (f(s_2,t_1) - f(s_1,t_1)) g(s_2,t_2) + \\ &\quad \left. + K(s_1,t,t_1,t_2) f(s_1,t_1) (g(s_2,t_2) - g(s_1,t_2)) \right\}, \end{aligned}$$

and we can reduce the problem to that solved in Section 4.

5.1 The result of subsection 4.1 can now be extended to the following statement:

if $f(s,t) \in \mathcal{L}^v(a+\frac{1}{v}, \mu)$ and $g(s,t) \in \mathcal{L}^w(a+\frac{1}{w}, \mu)$ then $B(s,t)$ and $\hat{B}(t,s)$ are elements of $\mathcal{L}_0^r(a+\frac{1}{r}, a, \mu)$ for $v \geq 1, w \geq 1, r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geq 0$ with some $q, 1 \leq q < 2, 0 < \mu < \frac{1}{q} - \frac{1}{2}$ and $a \geq -\frac{1}{2} + \mu$. The norms of these functions are related by

$$\|B(s,t)\|_{(\tau)} \leq C \|f(s,t)\|_{(v)} \cdot \|g(s,t)\|_{(w)}$$

$$\|\hat{B}(t,s)\|_{(\tau)} \leq C \|f(s,t)\|_{(v)} \cdot \|g(s,t)\|_{(w)}$$

(5.3)

[If $r = \infty$ we can take $\mathcal{F}(a, \mu)$ instead of $\mathcal{L}^\infty(a, a, \mu)$].

5.2 As in 4.2 the functions $f(s,t)$ and $g(s,t)$ can be continued to functionals in the variable t . The corresponding result reads:

the equation (5.1) defines a continuous mapping

$$\tilde{E}(-b, \mu_1, \mu) \times \tilde{E}(-b, \mu_1, \mu) \rightarrow \mathcal{L}_0^q(b-1+\frac{1}{q}, b-1, \mu)$$

if $1 \leq q < 2, 0 < \mu + 2\mu_1 < \frac{1}{q} - \frac{1}{2}, b \geq \frac{1}{2} + \mu + 2\mu_1$.

5.3 The unitarity mapping of Section 2 also involves the linear transform (2.7) for the absorptive part in the t channel. Let us first assume that the s dependence of the double spectral function is described by $\psi(s,t) \in \mathcal{L}_0^p(a, b, \mu)$, $b < 0, 0 < \mu < 1$. The mapping

$$d[\psi(s,t)] = \frac{1}{\pi} \int ds' \left[\frac{1}{s' - s} + \frac{1}{s' - 4 + s + t} \right] \psi(s', t)$$

(5.4)

can then be estimated as in the case of ordinary Hölder continuous functions. It turns out to be a bounded transform from $\mathcal{L}_0^p(a, b, \mu)$ into $\mathcal{L}^p(a, 0, \mu)$ if $b < 0$ and $0 < \mu < 1$.

The contribution of the crossed term $\hat{\psi}(s,t) = \psi(t,s)$ is evaluated in the same way if also $\hat{\psi}(s,t) \in \mathcal{L}_0^p(a,b,\mu)$. Such a property applies to double spectral functions which are obtained by the elastic iteration (5.1). But the weaker condition $\psi(s,t) \in \mathcal{L}_0^p(a,\mu)$ is sufficient (also for the crossed term) if we know in addition that the support of $\psi(s,t)$ lies within $4 \leq s \leq 19$, $t \geq 20$ (see Section 2). We denote the corresponding subspace of $\mathcal{L}^p(a,\mu)$ by $\mathcal{L}_c^p(a,\mu)$. The supports of $\hat{\psi}(s,t)$ and $\lambda(s)d(s,t)$ are then separated and $\psi(s,t) \rightarrow \lambda(s)d[\hat{\psi}(s,t)]$ is a bounded operator from $\mathcal{L}_c^p(a,\mu)$ into $\mathcal{L}^p(a,\mu)$ if $p \geq 1$, $0 \leq a < p^{-1}$ or $p=1$, $a=1$.

Hence we have obtained

$$\|\lambda(s)d[\psi(s,t) + \psi(t,s)]\| \leq c_1 \|\psi(s,t)\| \quad (5.5)$$

for the spaces $\mathcal{L}_c^p(a,\mu)$ with $p=1$, $a=1$ or $p \geq 1$, $0 \leq a < p^{-1}$ and $0 < \mu < 1$.

These results can be generalized to functions

$$\psi(s,t) \in \tilde{\mathcal{E}}(-a, \mu_1, \mu_2)$$

if the support is restricted as above for $\mathcal{L}_c^p(a,\mu)$. We denote these subspaces by $\tilde{\mathcal{E}}_c(-a, \mu_1, \mu_2)$. The mapping

$$\psi(s,t) \rightarrow \lambda(s)d[\psi(s,t) + \psi(t,s)]$$

is a bounded operator from $\tilde{\mathcal{E}}_c(-a, \mu_1, \mu_2)$ into $\tilde{\mathcal{E}}(-a, \mu_1, \mu_2)$ if $0 \leq a \leq 1$ and $0 < \mu_{1,2} < 1$.

6. APPLICATIONS TO FIXED POINT SOLUTIONS

We first discuss the existence of fixed point solutions. For $\lambda(s)d(s,t) \in \mathcal{L}^1(a,\mu)$, $0 < \mu < \frac{1}{2}$, $\frac{1}{2} + \mu \leq a \leq 1$, the unitarity mapping is a transform from $\mathcal{L}_c^1(a,\mu)$ into $\mathcal{L}_c^1(a,\mu)$ and the

estimates (5.3) and (5.5) allow one to apply the contraction mapping theorem (or the Schauder fixed point principle) as it has been done in Ref. 1). If the norms of $\lambda(s)D(s,t) \in \mathcal{L}^1(a,\mu)$ and $\psi_0(s,t) \in \mathcal{L}_c^1(a,\mu)$ are small enough, the series

$$\psi_0(s,t), \psi_1(s,t), \dots, \psi_{n+1}(s,t) = \Phi[\psi_n(s,t)], \dots$$

converges to a fixed point solution $\psi(s,t) \in \mathcal{L}_c^1(a,\mu)$.

We can extend this result to generalized functions $\lambda(s)D(s,t) \in \tilde{E}(-a, \mu_1, \mu_2)$ and $\psi_0(s,t) \in \tilde{E}(-a, \mu_1, \mu_2)$, $0 < \mu_1 < \frac{1}{4}$, $0 < \mu_2 < 1$. From Section 5.1 we know that the iterated functions $\psi_n(s,t)$, $n=1,2,\dots$, are all elements of $\mathcal{L}^1(a,\mu)$ $\mu = \min(\frac{1}{2} - 2\mu_1, \mu_2)$. The convergence of this series follows as above if $\lambda(s)D(s,t)$ and $\psi_0(s,t)$ are small.

The proof of the existence of fixed point solutions using the spaces $\mathcal{L}^1(a,\mu)$ is easier than Atkinson's method ¹⁾. Compared to Ref. 1) it has also the advantage that functions which decrease only like $(\log t)^{-1-\epsilon}$ are included in $\mathcal{L}^1(1,\mu)$. [This generalization is also possible by a modification of the norm in Ref. 1), see Ref. 6).] But to satisfy the inelastic unitarity bounds

$$\Im m f_\ell(s) \geq |f_\ell(s)|^2, \quad \ell = 0, 2, \dots, s > 16 \quad (6.1)$$

we need some additional work. The calculations of Section 5 imply $\psi(s,t)$ and $\hat{\psi}(t,s) \in \mathcal{L}^q(a-1+\frac{1}{q}, a-1, \mu)$ $1 \leq q < 2$, $0 < \mu < \frac{1}{q} - \frac{1}{2}$. These norm conditions are not sufficient to derive the necessary estimates of the contribution of $\psi(s,t) + \psi(t,s)$ to the partial waves.

But, before we discuss the problem of inelastic unitarity we explore the singularity structure of fixed point solutions of the elastic unitarity integral. The space $\tilde{E}(-a, \mu_1, \mu_2)$, $0 < \mu_1 < \frac{1}{4}$, allows singularities of the type $\delta(t)$ or $\text{Pf} \frac{1}{t}$. The unitarity mapping smoothes these local singularities and they do not show up in the fixed point solution. From Section 5 we infer the following statements for any fixed point solution $\psi(s,t) \in \tilde{E}(-1, \mu_1, \mu)$, $0 < 2\mu_1 + \mu < \frac{1}{2}$ whether it is obtained by iteration or not.

- a. If $\lambda(s)D(s,t) \in \widetilde{E}(-1, \mu_1, \mu)$ then $\psi(s,t)$ is an element of $\mathcal{L}^q(\frac{1}{q}, \mu)$ with $\frac{1}{2} + 2\mu_1 + \mu < q^{-1} \leq 1$;
- b. If $\lambda(s)D(s,t)$ has only $\mathcal{L}^p$ singularities, $1 \leq p \leq \frac{4}{3}$, then $\psi(s,t)$ is locally $\mathcal{L}^r$ integrable for all r with $2p^{-1} - \frac{3}{2} < r^{-1} \leq 2p^{-1} - 1$;
- c. If $\lambda(s)D(s,t) \in \mathcal{L}^q(\frac{1}{q}, \mu)$, $\frac{4}{3} < q \leq 2$, $0 < \mu < \frac{3}{2} - \frac{2}{q}$ then $\psi(s,t)$ is an element of $\mathcal{F}(0, \mu)$.

To obtain δ like singularities for the fixed point solution, if they exist at all, the inhomogeneous term $\lambda(s)D(s,t)$ has to be taken out of a more general function space than $\widetilde{E}(-a, \mu_1, \mu)$, $0 < \mu_1 < \frac{1}{4}$.

At this point we have to remember the inelastic unitarity condition. Using the proofs ^{1), 4), 5)} as an advice on how to obtain the bounds (6.1) the inhomogeneous term $H(s,t)$ has to satisfy (among other conditions):

- a. $D(s,t)$ is a positive measure in t , which depends analytically on s , in the strip $0 < s < 16$;
- b. the inelastic bounds (6.1) apply also to $H(s,t)$ for energies above $s = 17$ [i.e., for those energies with $\lambda(s) \neq 1$].

In the region $4 < s < 16$ and $t > 17$ the property b. restricts the measure $D(s,t)$ to an $\mathcal{L}^p$ function, the values of p depend on s , we can take any p with $1 < p < \sqrt{\frac{16}{s}}$, ⁷⁾. For $s < 9$ we reach values $p \geq \frac{4}{3}$ and the contribution of this part of $D(s,t)$ (i.e., $s < 9$, $t > 17$) to $\psi(s,t)$ is then Hölder continuous in s and t . From the other regions the fixed point solution may however get singular contributions.

But, unfortunately we are not able to prove the existence of such solutions which satisfy the inelastic unitarity bounds and are not Hölder continuous in the elastic strip $4 < s < 16$. To illustrate these difficulties we first consider the case where also $H(s,t)$ is given by an unsubtracted Mandelstam representation ¹⁾

$$H(s,t) = \frac{1}{\pi^2} \int \frac{\omega(s',t')}{(s's)(t'-t)} ds' dt' + \text{crossed terms}$$

$$\omega(s,t) \in \mathcal{L}^1(a, a-1, \mu), \quad \frac{1}{2} + \mu < \alpha < 1.$$

Then we can neglect the cut-off function $\lambda(s)$, i.e., $\lambda(s) = 1$ for $4 < s < \infty$. The proof of the convergence of the iteration $\psi_n(s,t)$ goes through if we start with a smooth $\psi_0(s,t)$. (The fixed point solutions do not depend on this choice.) For the mapping (5.1) we need $D(s,t) \in \mathcal{L}^1(a, \mu)$ which is satisfied. The transform (2.7) is defined for $\psi_n(s,t) + \psi_n(t,s)$ since $\psi_n(s,t)$ and $\hat{\psi}_n(s,t)$ are elements of $\mathcal{L}^1(a, a-1, \mu)$.

A condition like $\omega(s,t) \in \mathcal{L}^1(a, a-1, \mu)$ was necessary to obtain $D(s,t) \in \mathcal{L}^1(a, \mu)$. But then crossing symmetry, $\omega(s,t) = \omega(t,s)$ implies that

$$D(s,t) = \frac{1}{\pi} \int ds' \left[\frac{1}{s's} + \frac{1}{s'-4+s+t} \right] \omega(s',t)$$

is Hölder continuous in s and t outside the support of $\omega(s,t)$. Hence, in this simplest case we always obtain Hölder continuous solutions for $s < 16$.

If we use a cut-off function $\lambda(s)$ we are not faced with this problem due to crossing symmetry for energies s with $\lambda(s) = 0$. But in that region the inhomogeneous term has to satisfy the inelastic unitarity bounds. So far all such amplitudes which we can write down (see Appendix C), are locally $\mathcal{L}^2$ integrable outside their double spectral region, i.e., $D(s,t)$ is $\mathcal{L}^2$ integrable for $s < 16$. This leads to Hölder continuous fixed point solutions $\psi(s,t)$ if $4 < s < 16$.

We have seen how the elastic unitarity integral can be estimated in function spaces which allow local singularities, and we have proved the existence of fixed point solutions in these spaces. But, if in addition inelastic unitarity bounds are required our present methods only lead to solutions which are Hölder continuous in the elastic strips $4 < s < 16$, $t > 4$ and $4 < t < 16$, $s > 4$. In the region $s > 16$, $t > 16$ local singularities are still possible.

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APPENDIX A

In this Appendix we derive estimates on the unitarity kernel $K(s, t, t_1, t_2)$.

The function (2.10)

$$L(s, t, t_1, t_2) = s(s-4) \varphi(t, t_1, t_2) - 4stt_1t_2 \quad (\text{A.1})$$

with

$$\varphi(t, t_1, t_2) = t^2 + t_1^2 + t_2^2 - 2(tt_1 + tt_2 + t_1t_2) \quad (\text{A.2})$$

may be written as

$$L(s, t, t_1, t_2) = s(s-4) [\alpha(s, t, t_1) - t_2] [\beta(s, t, t_1) - t_2] \quad (\text{A.3})$$

or

$$L(s, t, t_1, t_2) = s(s-4) [t - \alpha(s, t_1, t_2)] [t - \beta(s, t_1, t_2)] \quad (\text{A.4})$$

using the expression (2.11) and

$$\beta(s, t, t_1) = t + t_1 + 2 \frac{tt_1}{s-4} + 2 \sqrt{t+t_1 \left(1 + \frac{t}{s-4}\right) \left(1 + \frac{t_1}{s-4}\right)} \quad (\text{A.5})$$

If $t_1 > 4$ and $t_2 > 4$ we can describe the region where $L(s, t, t_1, t_2)$ is positive in the following ways

$$s > 4, t > 16, 4 < t_1 < \alpha(s, t, 4), 4 < t_2 < \alpha(s, t, t_1)$$

[see (2.9)], or

$$t_1, t_2 > 4, t > t_1 + t_2 + 2 \sqrt{t_1 t_2}, s > 4 + \frac{4tt_1t_2}{\varphi(t, t_1, t_2)}$$

or

$$t_1, t_2 > 4, \quad s > 4, \quad t > \beta(s, t_1, t_2) \quad (> 16).$$

The domain in the variables (s, t, t_1, t_2) characterized by these equivalent conditions is called $\mathcal{D}$. We list some estimates which are valid within $\mathcal{D}$:

$$4 < \alpha(s, t, t_1) < \frac{t}{4t_1} (s-4),$$

$$16 < t_1 t_2 < \frac{1}{4} t (s-4),$$

$$4 \frac{t t_1}{s-4} < \beta(s, t, t_1) - \alpha(s, t, t_1),$$

$$4 \frac{t t_1}{s-4} < \beta(s, t, t_1) < 4 t (1 + \frac{t}{s-4}),$$

$$0 < \varphi(t, t_1, t_2) < t^2$$

(A.6)

These inequalities give sufficient information about the boundaries of the region $\mathcal{D}$. The unitarity kernel $K(s, t, t_1, t_2)$ was defined in (2.9) as

$$K(s, t, t_1, t_2) = \begin{cases} L^{-\frac{1}{2}}(s, t, t_1, t_2) & \text{if } (s, t, t_1, t_2) \in \mathcal{D} \\ 0 & \text{if } (s, t, t_1, t_2) \notin \mathcal{D} \end{cases}$$

For the following we need estimates of the difference $K(s', t', t_1, t_2) - K(s, t, t_1, t_2)$. We notice that $L(s, t, t_1, t_2)$ is a quadratic form in each of its variables, it is therefore sufficient to study $Q(x) = (x-a)(x-b)$ with $a \geq b \geq 0$ and

$$Q_+^\lambda(x) = \begin{cases} Q_+^\lambda(x) & \text{if } x > a \\ 0 & \text{if } x < a \end{cases}$$

For $Q_+^{-1}(x) - Q_+^{-1}(y)$ we calculate

$$0 < Q_+^{-1}(x) - Q_+^{-1}(y) \leq Q_+^{-1}(x) \left| \frac{(x-y)(x+y)}{Q(y)} \right|^\mu \leq Q_+^{-1-\mu}(x) \left| (x-y) \frac{2x^2}{y} \right|^\mu$$

with $0 \leq \mu \leq 1$, if $a < x < y$, and

$$|Q_+^{-1}(x) - Q_+^{-1}(y)| \leq Q_+^{-1}(y) \leq |(y-x)y|^\mu Q_+^{-1-\mu}(y)$$

with $0 \leq \mu \leq 1$, if $x < y$, $x < a$.

The corresponding results for $Q_+^{-\frac{1}{2}}(x) - Q_+^{-\frac{1}{2}}(y)$ follow immediately from $|A^{\frac{1}{2}} - B^{\frac{1}{2}}| \leq |A - B|^{\frac{1}{2}}$, $A \geq 0$, $B \geq 0$. The unitarity kernel can now be estimated by

$$s' \geq s, t' \geq t, 0 \leq \mu \leq \frac{1}{2},$$

$$\begin{aligned} 0 &\leq K(s, t, t_1, t_2) - K(s', t', t_1, t_2) \leq \\ &\leq |K(s, t, t_1, t_2)|^{1+2\mu} \left\{ |2t^2 s^2 \frac{s'-s}{s'}|^\mu + |2s(s-4)t^2 \frac{t'-t}{t'}|^\mu \right\} \\ &\text{if } (s, t, t_1, t_2) \in \mathcal{D}, \end{aligned}$$

$$\begin{aligned} |K(s, t, t_1, t_2) - K(s', t', t_1, t_2)| &\leq |K(s', t', t_1, t_2)|^{1+2\mu} |s(s-4)t'(t'-t)|^\mu \\ &\text{if } (s, t, t_1, t_2) \notin \mathcal{D}, (s', t', t_1, t_2) \in \mathcal{D}, \end{aligned}$$

$$\begin{aligned} |K(s, t, t_1, t_2) - K(s', t, t_1, t_2)| &\leq |K(s', t, t_1, t_2)|^{1+2\mu} |t^2 s'(s'-s)|^\mu \\ &\text{if } (s, t, t_1, t_2) \notin \mathcal{D}, (s', t, t_1, t_2) \in \mathcal{D}. \end{aligned}$$

(A.7)

These results are also contained in Atkinson's paper ¹⁾. But we use somewhat simpler techniques and need (A.5)-(A.7) anyhow for further reference.

For different values of t_1 (or t_2) we derive in the same way

$$0 < K(s, t, t_1', t_2) - K(s, t, t_1, t_2) \leq \\ \leq |K(s, t, t_1', t_2)|^{1+2\mu} |4st(s+t)(t_1' - t_1)|^\mu$$

$$\text{if } (s, t, t_1', t_2) \in \mathcal{D}, t_1' > t_1,$$

$$|K(s, t, t_1', t_2) - K(s, t, t_1, t_2)| \leq \\ \leq |K(s, t, t_1, t_2)|^{1+2\mu} |4st(s+t)(t_1' - t_1)|^\mu$$

$$\text{if } (s, t, t_1', t_2) \notin \mathcal{D}, (s, t, t_1, t_2) \in \mathcal{D},$$

$$0 \leq \mu \leq \frac{1}{2}.$$

(A.8)

For application in Section 4 we calculate integrals over $K(s, t, t_1, t_2)$.

Let q be a real number, $1 \leq q < 2$, then

$$\int |t_2^a K(s, t, t_1, t_2)|^q dt_2$$

can be evaluated using the representation (A.3) for $L(s, t, t_1, t_2)$ and the bounds (A.6). Estimates for s or t integrations are most easily derived from (A.1) or (A.4). We write the results as follows:

$$\int |(t_1 t_2)^a K(s, t, t_1, t_2)|^q dt_{1,2} \leq C \cdot s^{-\frac{q}{2}} (s-4)^{aq+1-\frac{q}{2}} t^{aq+1-q} t_{2,1}^{-1},$$

$$a \geq \frac{1}{2} - \frac{1}{q},$$

$$\int |s^{-a} K(s, t, t_1, t_2)|^q ds \leq C \cdot t^{aq-1} (t_1 t_2)^{1-aq-q}, \quad a \geq \frac{1}{q} - \frac{1}{2},$$

$$\int |t^{-a} K(s, t, t_1, t_2)|^q dt \leq C \cdot s^{-\frac{q}{2}} (s-4)^{aq-1+\frac{q}{2}} (t_1 t_2)^{1-aq-q},$$

$$a \geq \frac{1}{q} - \frac{1}{2},$$

$$1 \leq q < 2.$$

(A.9)

It is convenient to introduce the q dependent kernel

$$\mathcal{R}(s, t, t_1, t_2) = (st)^{-a+1-\frac{2}{q}} (t_1 t_2)^a K(s, t, t_1, t_2)$$

$$1 \leq q < 2, \quad a \geq \frac{1}{2} - \frac{1}{q}$$

(A.10)

then (A.9) can be formulated in a more symmetric way

$$\int |R(s, t, t_1, t_2)|^q dt_{1,2} \leq C (st t_{2,1})^{-1},$$

$$\int |R(s, t, t_1, t_2)|^q ds \leq C (t t_1 t_2)^{-1},$$

$$\int |R(s, t, t_1, t_2)|^q dt \leq C (st_1 t_2)^{-1}$$

(A.11)

where the right-hand side of the first and the third inequality can be multiplied by a threshold factor

$$\left(\frac{s-4}{s}\right)^{aq - \frac{q}{2} + 1}.$$

The unitarity mapping (4.1) is then transformed to

$$B(s, t) = (st)^{a-1+\frac{2}{q}} \int R(s, t, t_1, t_2) t_1^{-a} f(t_1) t_2^{-a} g(t_2) dt_1 dt_2$$

(A.12)

If we substitute the kernel $K(s, t, t_1, t_2)$ in (4.1) by

$$(st)^{2\mu} |K(s, t, t_1, t_2)|^{1+2\mu}, \quad 0 < \mu < \frac{1}{q} - \frac{1}{2},$$

$$G(s, t) = (st)^{2\mu} \int |K(s, t, t_1, t_2)|^{1+2\mu} f(t_1) g(t_2) dt_1 dt_2$$

(A.13)

the function $G(s, t)$ can also be written in the form (A.12),

$$G(s, t) = (st)^{a-1+\frac{2}{q}} \int \tilde{R}(s, t, t_1, t_2) t_1^{-a} f(t_1) t_2^{-a} g(t_2) dt_1 dt_2,$$

$$a \geq \frac{1}{2} - \frac{1}{q} + \mu,$$

with

$$\tilde{R}(s, t, t_1, t_2) = (st)^{-a+1-\frac{2}{q}+2\mu} |K(s, t, t_1, t_2)|^{1+2\mu} (t_1 t_2)^a,$$

$$0 \leq \mu < \frac{1}{q} - \frac{1}{2}, \quad a \geq \frac{1}{2} - \frac{1}{q} + \mu.$$

This kernel $\tilde{R}$ satisfies again the estimates (A.11) (with another constant C and without the threshold factor). So the problem is reduced to an integral transform of the type (A.12).

If we calculate the differences $B(s', t) - B(s, t)$ or $B(s, t') - B(s, t)$ we are led to integrals like (A.13). Using (A.7) we obtain the bounds

$$|B(s', t) - B(s, t)| \leq |2 \frac{s' - s}{s}|^\mu \cdot (M(s, t) + M(s', t))$$

where $M(s, t)$ is given by the modulus of (A.13).

Finally, we would like to mention that $f(t)$ and $g(t)$ can be generalized functions. If $f(t)$ is a linear functional on the space of Hölder continuous functions with index μ then we have to handle integral kernels like $|t_1 - v|^{-\mu} |K(s, t, t_1, t_2) - K(s, t, v, t_2)|$. In (A.8) this function is estimated by $|8s^2 t^2|^\mu \cdot \max_{x=t_1, v} |K(s, t, x, t_2)|^{1+2\mu}$ what again leads to (A.13) and hence to (A.12). We have only to take care of the range of the different Hölder indices. If $f(t)$ and $g(t)$ are elements of $E'(-a, \mu_1)$ (see Section 3) then

$$|B(s', t) - B(s, t)| \cdot |s' - s|^{-\mu}$$

has a representation (A.12) only if $0 < \mu + 2\mu_1 < \frac{1}{q} - \frac{1}{2}$, $a \geq \frac{1}{2} - \frac{1}{q} + \mu + 2\mu_1$.

APPENDIX B

In this Appendix we derive a theorem on integral transforms of the type (4.1). We consider complex valued Lebesgue measurable functions defined on the real axis. The function spaces $\mathcal{L}^p[\sigma]$, $1 \leq p \leq \infty$, are defined as usual ⁸⁾ with a measure $d\sigma(x)$ and the norms

$$\|f\|_{p,\sigma} = \left[\int |f(x)|^p d\sigma(x) \right]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_{\infty,\sigma} = \operatorname{ess. sup}_x |f(x)|$$

In the case of the Lebesgue measure $d\sigma(x) = dx$ we drop the index σ .

Theorem Given the functions $K(x_1, x_2, x_3)$, $f(x)$ and $g(x)$ with the following restrictions

a) the integral kernel $K(x_1, x_2, x_3)$ is estimated by

$$\left[\int |K(x_1, x_2, x_3)|^q dx_i \right]^{\frac{1}{q}} \leq \varphi(x_j) \varphi(x_k)$$

for $i=1,2,3$, $(i,j,k) = \operatorname{perm}(1,2,3)$ and a value of q , $1 \leq q < \infty$;

b) the functions $f(x)$ and $g(x)$ satisfy

$$f(x) [\varphi(x)]^{1-q+\frac{q}{v}} \in \mathcal{L}^v, \quad g(x) [\varphi(x)]^{1-q+\frac{q}{w}} \in \mathcal{L}^w$$

for some real numbers $v \geq 1$ and $w \geq 1$, $v^{-1} + w^{-1} \geq 2-q^{-1}$,

then the bilinear form

$$F(x_3) = \int K(x_1, x_2, x_3) f(x_1) g(x_2) dx_1 dx_2$$

is defined for almost all values of x_3 and we obtain

$$F(x) [\varphi(x)]^{-1+\frac{q}{\tau}} \in \mathcal{L}^\tau, \quad \tau^{-1} = v^{-1} + w^{-1} + q^{-1} - 2$$

and

$$\|F \cdot \varphi^{-1+\frac{q}{\tau}}\|_\tau \leq \|f \cdot \varphi^{1-q+\frac{q}{v}}\|_v \cdot \|g \cdot \varphi^{1-q+\frac{q}{w}}\|_w.$$

Proof

The Hölder inequality for the x_1 or the x_2 integration gives

$$|F(x_3)| \leq \begin{cases} \|f\|_p \cdot \|g \cdot \varphi\|_1 \cdot \varphi(x_3) \\ \|f \cdot \varphi\|_1 \cdot \|g\|_p \cdot \varphi(x_3) \end{cases} \quad \text{with } p^{-1} + q^{-1} = 1$$

(B.1)

In addition we calculate

$$\left[\int |F(x_3)|^q dx_3 \right]^{\frac{1}{q}}$$

If $q=1$ this can be done by an interchange of the order of integration,

$$\int |F(x)| dx \leq \|f \cdot \varphi\|_1 \|g \cdot \varphi\|_1.$$

If $1 < q < \infty$ we multiply $F(x)$ by a function $h(x) \in \mathcal{L}^p$, the dual space of $\mathcal{L}^q$, and the Hölder inequality yields

$$\int |F(x) h(x)| dx \leq \|h\|_p \cdot \|f \varphi\|_1 \|g \varphi\|_1.$$

Hence we obtain in both cases

$$\|F\|_q \leq \|f \varphi\|_1 \|g \varphi\|_1, \quad 1 \leq q < \infty$$

(B.2)

Now we define the functions

$$\tilde{f}(x) = \begin{cases} [\varphi(x)]^{1-q} f(x) & \text{if } \varphi(x) > 0 \\ 0 & \text{if } \varphi(x) = 0 \end{cases}$$

and

$$\tilde{F}(x) = \begin{cases} [\varphi(x)]^{-1} F(x) & \text{if } \varphi(x) > 0 \\ 0 & \text{if } \varphi(x) = 0 \end{cases}$$

and introduce the spaces $\mathcal{L}^r[\sigma]$, $1 \leq r \leq \infty$, with the measure $d\sigma(x) = [\varphi(x)]^q dx$. The results (B.1) and (B.2) can then be written as

$$\|\bar{F}\|_{\infty, \sigma} \leq \begin{cases} \|\tilde{f}\|_{p, \sigma} & \|\tilde{g}\|_{1, \sigma} \\ \|\tilde{f}\|_{1, \sigma} & \|\tilde{g}\|_{p, \sigma} \end{cases}$$

and

$$\|\bar{F}\|_{q, \sigma} \leq \|\tilde{f}\|_{1, \sigma} \|\tilde{g}\|_{1, \sigma}$$

The Riesz convexity theorem ⁸⁾ allows to generalize these inequalities to

$$\|\bar{F}\|_{r, \sigma} \leq \|\tilde{f}\|_{v, \sigma} \|\tilde{g}\|_{w, \sigma}$$

with $v, w \geq 1$ and $r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geq 0$. But this is exactly our theorem if we use the original functions f , g and F .

APPENDIX C

We want to construct amplitudes which satisfy Mandelstam analyticity and the inelastic unitarity bounds

$$\text{Im } f_\ell(s) \geq |f_\ell(s)|^2, \quad \ell = 0, 1, 2, \dots, \quad s > s_0. \quad (\text{C.1})$$

Moreover, these functions should show strong local singularities outside the physical region.

So far this problem has been considered only for smooth amplitudes ^{1),4),5)}. But, the methods of Ref. 5), Section 2, which allow a polynomial increase at infinity can be modified to obtain also local singularities. [Thereby we correct a mistake in Ref. 5) Section 2.]

Since we are interested in the local behaviour we first study non-crossing symmetric expressions.

We define a function

$$F_1(s, t) = \frac{A}{\cos \pi \gamma(t)} (s_1 - s)^{-\gamma(t)}, \quad s_1 > s_0 + 1, \quad (\text{C.2})$$

with

$$\gamma(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\text{Im } \gamma(t')}{t' - t} dt', \quad \text{Im } \gamma(t) \geq 0,$$

and $0 < \gamma(t) < \frac{1}{2}$ for $t < t_0$, which is holomorphic below the threshold $t = t_0 = s_0 > 0$. To study the partial waves of $F_1(s, t)$ in the neighbourhood of $s = s_1$, $|s - s_1| < 1$, we use the property that a power series in $x = t + \frac{s - 4}{2}$ with positive coefficients has also positive partial waves.

The function $\gamma(t)$ has a positive power series in x since $\text{Im} \gamma(t) \geq 0$. From this fact we derive the following statements

$$\begin{aligned} \text{a. } |s-s_1|^{-\gamma(t)} &= \exp [\gamma(t) \log |s-s_1|] = \\ &= \sum (n!)^{-1} [\gamma(t) \log |s-s_1|]^n, \quad |s-s_1| < 1, \end{aligned}$$

shows positive coefficients as power series in x ;

$$\text{b. } \frac{1}{\cos \pi \gamma(t)} = \sum c'_v [\gamma(t)]^v, \quad c'_v \geq 0,$$

has a positive series in x ;

$$\text{c. } \tan \pi \gamma(t) = \sum c''_v [\gamma(t)]^v, \quad c''_v \geq 0,$$

has a positive series in x ,

$$\tan \pi \gamma(t) = \sum c_n x^n, \quad c_n \geq 0.$$

All these series converge for values of t between $-t_0 - s + 4 < t < t_0$.

In the interval $s_1 - 1 < s < s_1$ we obtain from a. and b.

$$F_1(s, t) = \frac{A}{\cos \pi \gamma(t)} (s_1 - s)^{-\gamma(t)} = \sum a_n(s) x^n \quad \text{with } a_n \geq 0$$

for a positive A .

If $s_1 < s < s_1 + 1$ the function $F_1(s, t)$ develops an imaginary part ($t < t_0$)

$$\phi(s, t) = \text{Im } F_1(s, t) = A \tan \pi \gamma(t) \cdot (s - s_1)^{-\gamma(t)},$$

$$\psi(s, t) = \text{Re } F_1(s, t) = A (s - s_1)^{-\gamma(t)}$$

From a. and c. we infer

$$\psi(s, t) = \sum a_n(s) x^n \quad \text{with} \quad a_n(s) \geq 0 \quad \text{and}$$

$$\phi(s, t) = \sum b_n(s) x^n \quad \text{with} \quad b_n(s) \geq 0.$$

Moreover we know

$$b_n = c_0 a_n + c_1 a_{n-1} + \dots + c_n a_0 \geq c_0 a_n,$$

$$c_0 = \operatorname{tg} \pi \gamma \left(-\frac{s-4}{2}\right).$$

The actual amplitude is now defined by

$$F(s, t) = \sigma(s) F_1(s, t)$$

(C.3)

with

$$\sigma(s) = -(s_0 - s)^{-\delta-1} (s_1 - s)^\delta = \begin{cases} e^{i\pi\delta} |\sigma(s)| & \text{if } s_0 < s < s_1 \\ |\sigma(s)| & \text{if } s_1 < s \end{cases}$$

and we choose a δ , $\gamma(0) \leq \delta < \frac{1}{2}$. (We may also take a bounded function

$$\sigma(s) = - \int ds' \rho(s') (s' - s)^{-\delta-1} (s_1 - s)^\delta$$

The coefficients of $F(s, t) = \sum d_n(s) x^n$ are easily determined. For the interval $s_1 - 1 < s < s_1$ we obtain

$$F(s, t) = e^{i\pi\delta} |F(s, t)|, \quad d_n(s) = \sigma(s) a_n(s),$$

$$\operatorname{Im} d_n(s) = \operatorname{tg} \pi \delta \cdot \operatorname{Re} d_n(s) \geq 0$$

and for $s_1 < s < s_1 + 1$

$$F(s, t) = |\sigma(s)| F_1(s, t), \quad d_n(s) = |\sigma(s)| (a_n(s) + i b_n(s)),$$

$$\operatorname{Im} d_n(s) \geq \operatorname{tg} \pi \gamma \left(-\frac{s-4}{2}\right) \cdot \operatorname{Re} d_n(s) \geq 0.$$

Therefore, an estimate $\text{Im } d_n(s) \geq c |\text{Re } d_n(s)|$ with $c > 0$ is valid in the whole interval $s_1 - 1 < s < s_1 + 1$. The same inequality applies to the partial waves

$$F(s, t) = 2 \left(\frac{s}{s-4} \right)^{\frac{1}{2}} \sum (2\ell+1) f_\ell(s) P_\ell(z),$$

$$\text{Im } f_\ell(s) \geq c |\text{Re } f_\ell(s)|.$$

On the other hand $F(s, t)$ is bounded in the physical region $t \leq 0$, $s_1 - 1 < s < s_1 + 1$, and by the choice of the parameter A we obtain

$$|f_\ell(s)| \leq \frac{1}{2} [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 |F(s, t)| dt \leq \frac{1}{2} \min(1, c^{-1}).$$

This implies the final result

$$\text{Im } f_\ell(s) \geq |f_\ell(s)|^2, \quad \ell = 0, 1, 2, \dots, \quad s_1 - 1 < s < s_1 + 1.$$

(C.4)

The type of singularity of $F(s, t)$ is determined by the expression $(s_1 - s)^{\delta - \gamma(t)}$. Since $\gamma(t) > 0$, $t < t_0$, the positive $\delta \geq \gamma(0)$ is necessary to get a bounded amplitude in the physical region.

The strong restriction on this singularity is given by $\gamma(t) < \frac{1}{2}$ for $t < t_0$. A value of $\gamma(t) = \frac{1}{2}$ introduces a pole in $F(s, t)$ which is not allowed below $t = t_0$. We do not see how to avoid such a restriction by another ansatz without violation of (C.4).

An amplitude which satisfies crossing symmetry and the inelastic bounds for all s above threshold can be obtained as in Ref. 5); one has to symmetrize $F(s, t)$ and to add a symmetric function $G(s, t, u)$ which is given by a Mandelstam representation with smooth spectral functions

$$A(s, t, u) = F(s, t) + F(t, s) + F(u, t) + \dots + G(s, t, u).$$

We do not go into the details to estimate the partial waves of the whole expression. The local singularities of our amplitudes are in any case too weak to persist after the elastic iteration. So we

cannot use them in Section 6 for a proof of the existence of unitary amplitudes with singular double spectral function in the elastic strip.

Finally, we want to correct a mistake in Ref. 5). The arguments of Section 2.1.2 in Ref. 5) to find a function $\beta(t)$ such that $\beta(t) \cdot \cos \pi \gamma(t)$ has negative partial waves are not correct. One can solve this problem in a simple way, which we have used for the above amplitude (C.1), by an ansatz

$$\beta(t) = - \frac{c}{\cos \pi \gamma(t)}$$

or more generally

$$\beta(t) = - f(t) \cos^{-1} \pi \gamma(t) ,$$

the function $f(t)$ has to be chosen with positive partial waves. But this solution is only possible if $\gamma(t)$ does not admit half integer values for $t < t_0$. This restriction and condition (2.6) of Ref. 5) imply the bounds $-2n < \alpha(t) < -2n+1$ for all $t < t_0$ and some integer n . In the physical region the asymptotic behaviour of the whole amplitude is then dominated by the additional term $G(s,t,u)$. But in the double spectral region $\text{Re} \alpha(t)$ can reach large positive values and the Regge term increases polynomially.

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