CM-P00058626

Ref.TH.1191-CERN

THE SIXTH ORDER RADIATIVE CORRECTION TO THE DIFFERENCE OF THE
ANOMALOUS MAGNETIC MOMENTS OF MUON AND ELECTRON

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A B S T R A C T

We report here the completion of the calculation of the sixth order contribution to the difference of the anomalous magnetic moments of muon and electron. The new theoretical value for the anomalous magnetic moment of the muon is $(1165848 \pm 46) \times 10^{-9}$ in reasonable agreement with experiment.

Ref.TH.1191-CERN

10 July 1970

The difference between the anomalous magnetic moments of the muon and the electron has contributions from the 24 graphs shown in the Figure. The graphs consist of three distinct classes :

Class 1 : Photon-photon scattering, graphs (1-6);

Class 2 : Second order vacuum polarization inserted into fourth order vertex graphs (7-20);

Class 3 : Fourth order vacuum polarization inserted into the second order vertex graph (21-24).

Of these 24 graphs the anomalous magnetic moment contribution from 20 has been calculated so far. Historically, the calculations proceeded as follows :

Kinoshita ¹⁾ (1967), graph 21;
 Lautrup, de Rafael ²⁾ (1968), graphs 22-24;
 Lautrup, de Rafael ³⁾ (1969), graphs 19-20;
 Aldins, Brodsky, Dufner, Kinoshita ⁴⁾ (1969), graphs 1-6;
 Lautrup, Peterman, de Rafael ⁵⁾ (1970), graphs 11-18.

The first three calculations were completely analytic, while the last two were partly analytic, partly numeric. The only graphs not previously calculated are the graphs 7-10 of Class 2. In this paper, we present the results of a calculation of these graphs, and thereby the complete 6th order contribution to $a_\mu - a_e$.

The method generally employed for Class 2 diagrams is to use a spectral decomposition for the inserted vacuum polarization. Thereby we obtain the formula ⁶⁾

$$a = \int_{4m_e^2}^{\infty} \frac{dt}{t} \frac{1}{\pi} \text{Im} \Pi_e(t) A_\mu(t) \quad (1)$$

where $\frac{1}{\pi} \text{Im} \Pi_e(t)$ is the absorption part of the second order vacuum polarization by electrons

$$\frac{1}{\pi} \text{Im} \Pi_e(t) = \frac{\alpha}{\pi} \frac{1}{3} \left(1 + \frac{2m_e^2}{t} \right) \sqrt{1 - \frac{4m_e^2}{t}} \quad (2)$$

and $a_\mu(t)$ is the anomalous magnetic moment obtained from the same graphs by replacing the photon line with electron bubble by a massive photon with a propagator $-i/(k^2-t)$. For a graph or a sum of graphs for which $a_\mu(0)$ exists, we can derive the expression [using the same methods as in Ref. 2]

$$a = \frac{\alpha}{\pi} \left\{ \left(\frac{2}{3} \log \frac{m_\mu}{m_e} - \frac{5}{9} \right) a_\mu(0) + \frac{1}{3} a'_\mu \right\} \quad (3)$$

where

$$a'_\mu = \int_0^{m_\mu^2} \frac{dt}{t} (a_\mu(t) - a_\mu(0)) + \int_{m_\mu^2}^{\infty} \frac{dt}{t} a_\mu(t) \quad (4)$$

This takes care of the logarithmic dependence because a'_μ is independent of the electron mass. For graphs where $a_\mu(0)$ does not exist, it is necessary to refine the method. Often one can separate $a_\mu(t)$ into two parts of which the first does not exist for $t=0$ but is sufficiently simple to be handled directly in (1), and for the second part which exists for $t=0$, one can then use the separation technique outlined above. When, however, $a_\mu(0)$ exists, it must be equal to the contribution from the corresponding fourth order graphs. This is in particular true for the total Class 2 contribution, where we get

$$a_\mu(0) = 2 a^{(4)} \quad (5)$$

where $a^{(4)}$ is the total fourth order contribution ⁷⁾

$$a^{(4)} = \left(\frac{\alpha}{\pi} \right)^2 \left\{ \frac{197}{144} + \frac{\pi^2}{12} - \frac{1}{2} \pi^2 \log 2 + \frac{3}{4} \zeta(3) \right\} \quad (6)$$

Thus we see that the sixth order contribution from Class 2 diagrams can in general be obtained by calculating the fourth order contributions from the corresponding graphs with the vacuum polarization insertion replaced by a massive photon. The parametrization of such graphs in terms of Feynman parameters is quite easy and follows closely the parametrizations one would use for calculating the true fourth order anomalous magnetic moment contributions. Only the denominators are

changed by the introduction of t . Furthermore, it is possible, when $a_\mu(0)$ exists, to check the parametrization by letting $t=0$ and evaluating the integrals analytically or numerically, and compare with the fourth order result. With slight modifications, such a check can even be carried through when $a_\mu(0)$ does not exist.

It is possible to estimate the total Class 2 contributions in the following way⁸⁾. The integrals in Eq. (4) act on a propagator $1/k^2-t$. But as

$$\int_0^{m_\mu^2} \frac{dt}{t} \left\{ \frac{1}{k^2-t} - \frac{1}{k^2} \right\} + \int_{m_\mu^2}^{\infty} \frac{dt}{t} \frac{1}{k^2-t} = \frac{1}{k^2} \log \left(-\frac{k^2}{m_\mu^2} \right)$$

we see that the net effect is to replace the propagator $1/k^2$ in a true fourth order graph by $1/k^2 \log(-k^2/m_\mu^2)$. Now a logarithm is slowly varying and it is probably reasonable to replace it by an average value C . Then we find

$$a'_\mu \approx a_\mu(0) C$$

such that

$$a \approx \frac{\alpha}{\pi} a_\mu(0) \left\{ \frac{2}{3} \log \frac{m_\mu}{m_e} - \frac{5}{9} + \frac{1}{3} C \right\}$$

The constant C can be determined by applying this method to the fourth order contribution to the difference between the anomalous magnetic moments of muon and electron⁹⁾. This leads to Kinoshita's estimate⁸⁾

$$a_{7-20}^{est} = \frac{4}{3} a^{(4)} \frac{\alpha}{\pi} \left(\log \frac{m_\mu}{m_e} - \frac{25}{12} \right)$$

Using the methods as described above, we have obtained

$$a_7 = a_8 = \left(\frac{\alpha}{\pi} \right)^3 \left\{ \left(\frac{1}{9} + \frac{13}{54} \pi^2 - \frac{5}{9} \pi^2 \log 2 + \frac{5}{6} \zeta(3) \right) \log \frac{m_\mu}{m_e} + 1.047 \pm 0.010 \right\}$$

$$a_9 = \left(\frac{\alpha}{\pi} \right)^3 \left\{ \left(\frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36} \right) \log \frac{\lambda}{m_\mu} + \frac{1}{6} \log^2 \frac{m_\mu}{m_e} + \left(-\frac{1}{8} + \frac{\pi^2}{27} \right) \log \frac{m_\mu}{m_e} - 1.058 \pm 0.004 \right\}$$

$$a_{10} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ -\left(\frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36}\right) \log \frac{\lambda}{m_\mu} - \frac{1}{6} \log^2 \frac{m_\mu}{m_e} \right. \\ \left. + \left(\frac{31}{72} + \frac{\pi^2}{27}\right) \log \frac{m_\mu}{m_e} - 1.669 \pm 0.004 \right\}$$

where λ is the photon mass introduced for convergence.

Observe that the coefficient of $\log(m_\mu/m_e)$ is $4/3$ times the corresponding fourth order contribution for $2a_7 = 2a_8 = a_7 + a_8$, while this is only true for the sum $a_9 + a_{10}$. The uncertainties on the constant terms stem from the integration subroutine used [see Ref. 5)]. They are 90% confidence limits.

Thus we have

$$a_{7-10} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(\frac{19}{36} + \frac{5}{9} \pi^2 - \frac{10}{9} \pi^2 \log 2 + \frac{5}{3} \zeta(3)\right) \log \frac{m_\mu}{m_e} \right. \\ \left. - 0.633 \pm 0.012 \right\}$$

The remaining Class 2 contributions are 5)

$$a_{11-18} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(-\frac{28}{9} + \frac{4}{9} \pi^2 \log 2 - \frac{2}{3} \zeta(3)\right) \log \frac{m_\mu}{m_e} \right. \\ \left. + 0.676 \pm 0.014 \right\}$$

and 3)

$$a_{19-20} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(\frac{119}{27} - \frac{4}{9} \pi^2\right) \log \frac{m_\mu}{m_e} + \frac{\pi^2}{27} - \frac{61}{162} \right\}$$

Hence the total Class 2 contribution is

$$a_{7-20} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(\frac{197}{108} + \frac{\pi^2}{9} - \frac{2}{3} \pi^2 \log 2 + \zeta(3)\right) \log \frac{m_\mu}{m_e} \right. \\ \left. + 0.032 \pm 0.018 \right\}$$

As the coefficient of the logarithmic term is -0.438 , we see that Kinoshita's estimate is not verified for the total Class 2 diagrams. (It would require a constant term of $+0.8$.) The reason for the deviation seems to lie in graphs 17 and 18 [see Ref. 5]. These are the only graphs that individually violate Kinoshita's conjecture that constant terms ought to be of the opposite sign of the coefficient of the logarithm and, needless to say, we have performed several double checks on those graphs ¹⁴⁾.

We can now quote the value for the full 6th order difference. The individual numerical contributions are

$$\begin{aligned} \text{Class 1 } ^{4)} \quad & a_{1-6} = (18.4 \pm 1.1) \left(\frac{\alpha}{\pi}\right)^2 \\ \text{Class 2} \quad & a_{7-20} = (-2.30 \pm 0.02) \left(\frac{\alpha}{\pi}\right)^3 \\ \text{Class 3 } ^{1), 2)} \quad & a_{21-24} = 4.2414 \left(\frac{\alpha}{\pi}\right)^3 \end{aligned}$$

such that the total contribution is

$$(a_{\mu} - a_e)^{(6)} = (20.3 \pm 1.1) \left(\frac{\alpha}{\pi}\right)^3$$

We observe that the sum of the coefficients of the contributions from Class 2 and 3 is 1.94. The previous value was 2.82, using Kinoshita's estimate.

Using the most recent experimental value for a_e ¹⁰⁾ and the calculated strong ¹¹⁾ and weak ¹²⁾ interaction corrections, we obtain the theoretical prediction for the muon anomalous magnetic moment (see the table)

$$a_{\mu}^{th} = (1165848 \pm 46) \times 10^{-9}$$

This should be compared with the experimental value ¹³⁾

$$a_{\mu}^{exp} = (1166160 \pm 310) \times 10^{-9}$$

The agreement is reasonable.

Finally, it should be stressed that independent checks on these calculations are necessary.

The author wants to thank Dr. E. de Rafael for a long collaboration on this subject, and Professor A. Peterman for many discussions, and also for checking the analytic part of graphs 9 and 10.

TABLE : for the theoretical value for the anomalous magnetic moment of the muon

	$\times 10^{-9}$
a_e	1159644 ± 7
4 th order $a_{\mu} - a_e$	5904
6 th order $a_{\mu} - a_e$	255 ± 14
strong interactions	65 ± 5
weak interactions	-20 ± 20
a_{μ}	1165848 ± 46

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 -0.1778 ± 0.0003 .

FIG. 1