**M - LOOP GENERALIZED VENEZIANO FORMULA**

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**ABSTRACT**

We construct the Veneziano formula with $M$ closed loops, both by Nielsen's representation and by sewing reggeons. The logarithm of its integrand is just the third Abelian integral of a sphere with $M$ handles. Domain variational theory determines all its Regge singularities.
Recently an embryonic pomeron singularity *) was discovered 1), 2) in the generalized Veneziano formula with one closed loop. This makes construction of the M-loop formula urgent. Previous attempts 3) stopped in an impenetrable thicket of harmonic oscillators. We show there is a simple answer. The logarithm of the M-loop integrand is just the third Abelian integral of a sphere with M handles. This is true both for orientable and non-orientable diagrams, they differ only in the co-ordinate system. We give two separate proofs from different hypotheses - one from Nielsen's "almost-physical" representation 4), the other by sewing the N-reggeon vertex 5). The second proof is especially beautiful. Abelian integrals are not tabulated functions for M > 1, but they have been much studied by mathematicians. In particular, domain variational theory 6) determines all Regge singularities of our M-loop formula. We are thus laying the foundation for a dual reggeon calculus.

In Nielsen's representation 4) the duality diagram becomes a two-dimensional conductor, with M holes cut out for the loops, and N electrodes on the boundary for its external lines. Nielsen conjectured that the logarithm of the Veneziano integrand was the heat generated by a steady current. So far this has been verified 4), 1) for M = 0, 1. We start by giving a neat mathematical form to Nielsen's conjecture. Consider the duality diagram as a domain B with boundary b. The harmonic functions on B, which vanish as some point z₀, form a Hilbert space \( \Lambda^2(B) \) with Dirichlet inner product 7)

\[
D_B \{f, g\} = \iint_B (\nabla f \cdot \nabla g) \, d\sigma \, d\nu
\]

\[
= \oint_B f \frac{\partial g}{\partial n} \, ds
\]

(1)

*) This singularity has the right isospin and slope for the pomeron. Also experimentally the pomeron is mainly the shadow of ordinary multi-Regge exchange, and therefore ought to emerge from double scattering graphs in a dual model.
Let $\varphi_m(z)$ be any complete orthonormal basis for $L^2(\mathbb{B})$. Then the harmonic kernel function of $B$ is

$$k_B(z, \omega) = \sum_m \varphi_m(z) \varphi_m(\omega),$$

which converges and is independent of the basis $7)$. $k_B$ is a projection operator for the Hilbert space $L^2(\mathbb{B})$. Therefore

$$D_B^t \{ k_B(z, \omega), k_B(\omega, \omega) \} f = k_B(z, \omega). \quad (3)$$

It also solves the Neumann boundary value problem

$$f(z) = \int_B d\omega_\epsilon \cdot k_B(z, \epsilon) \cdot \frac{\partial f(\epsilon)}{\partial n_\epsilon}, \quad (4)$$

for $f$ harmonic in $B$, since $7)$

$$k_B(z, \omega) = (2\pi)^{-1} \left[ N_B(z, \omega) - G_B(z, \omega) \right], \quad (5)$$

where $N$ and $G$ are the Neumann's and Green's functions of $B$. Now in Nielsen's representation, a current is given on the boundary

$$\frac{\partial f}{\partial n_\epsilon}(\Delta_\epsilon) = (4\pi \alpha')^{\frac{1}{2}} \sum_{i=1}^N p_i \delta(z_i - \delta_\epsilon), \quad (6)$$

where $p_i$ are the external momenta, and $z_i$ the points on the boundary where electrodes are attached (Koba-Nielsen integration variables).

Therefore (4) is the potential inside $B$. The heat generated is its Dirichlet norm $D_B \int_\epsilon f^2 d\omega$, which can be evaluated at once by (3). So Nielsen's conjecture gives the generalized Veneziano integrand in terms of the harmonic kernel function of its duality diagram

$$\exp \left\{ \pi \alpha' \sum_{i \neq j} (p_i \cdot p_j) k_B(z_i, z_j) \right\}. \quad (7)$$

Now we sketch how domain variational theory $6)$ can be used to determine Regge singularities. Consider for simplicity an $M$-loop
orientable diagram B. It is conformally equivalent to the plane with M+1 circular discs removed. Its harmonic kernel function can be got by orthonormalizing inverse power series (plus logarithms) at their centres [7]. For M = 0, 1 this gives the known answers [4, 1]. For M > 1, B is characterized up to conformal equivalence [6] by 3M-3 Riemann moduli \( y_\lambda \). Since \( k_B \) is a conformal invariant, when we sum over all M - loop domains, these \( y_\lambda \) become integration variables, as well as the \( z_i \). Nielsen's representation does not give the integration measure in \( y_\lambda \), but we will determine it below. However [7] already shows that all moving Regge singularities must come from singularities of \( k_B \), either in the \( z_i \) or the \( y_\lambda \). The only singularity in \( z_i, z_j \) is a logarithmic one when \( z_i = z_j \) on the boundary. This gives usual Regge poles. The singularities of \( k_B \) in \( y_\lambda \) are also known [Ref. 6], Ch. 7. They occur only at values where the topology of the duality diagram B changes. There are two cases: a hole shrinking to a point gives the pomeron, two holes touching give Regge cuts [4].

Pomeron singularities always occur [6] in the M-dimensional subspace of \( \Lambda^2(B) \) spanned by the harmonic measures. A harmonic measure \( \omega^B_\mu(z) \) is the function harmonic in \( B \), with \( \omega^B_\mu(z) = 1 \) on the boundary of the \( \mu \)th hole and \( \omega^B_\mu(z) = 0 \) on all other boundaries [7]. There are M+1 boundaries for an orientable M-loop diagram, but only M of the \( \omega^B_\mu(z) \) are linearly independent. They contribute to the integrand [7] a factor

\[
\exp \left\{ -\sum_{\mu, \nu=1}^{M} (P^\mu \cdot P^\nu) Q^B_{\mu \nu} \right\}
\]

where \( P^\mu \) is the total momentum entering through the \( \mu \)th boundary, and \( Q^B_{\mu \nu} \) is the inverse of the period matrix [7]. There is an amusing electrostatic construction for \( Q^B_{\mu \nu} \): if cylindrical conductors are

*) The existence of these singularities is a general theorem, but their physical identification is inferred from the annulus

\[ r \leq |z| \leq 1, \] where \( r \to 0 \) gives pomeron and \( r \to 1 \) Regge cuts.
built on the various boundaries, it relates their surface potentials to their total charges\(^*\). (8) will cancel unless external lines enter through different boundaries. This explains why Regge cuts only occur in such so-called non-planar diagrams. (8) is the factor carrying the pomeron trajectory. For the annulus \( r \leq |z| \leq 1\), \( Q_{11} = -\log r/2\pi\), which illustrates both the logarithmic singularity of \( k_B \) as \( r \to 0\), and how (8) gives Frye and Susskind's \(^1\) pomeron with slope \( \alpha' / 2\).

Domain variational theory shows \([\text{Ref. 6}, p.28]\) that the same singularity will occur in multi-loop diagrams, when the radius \( r \to 0\) of any hole with external lines entering it. It also determines higher powers in \( r\), which give the pomeron residue \(^8\).

Now we express (7) as an Abelian integral. Joining two mirror copies of \( B\) along their boundaries, gives a closed Riemann surface \( R\) called its double. There is a more complicated construction for the double of a non-orientable diagram \([\text{Ref. 6}, p.30]\). The double itself is always orientable. Thus the torus is the double of both annulus and M"obius strip, which explains \(^9\) why they give such similar Veneziano formulae. The third Abelian integral \([\text{Ref. 6}, p.62]\) of a closed Riemann surface \( B\) is a function \( \Omega_{WW_c}(z) \) of \( z\) on \( R\), whose differential is analytic except for two poles, and whose real part is single-valued (= real normalization) and harmonic except for two logarithmic singularities.

\[
\Re \Omega_{WW_c}(z) = \log \left| \frac{z - W_0}{z - W} \right| + \text{harmonic} \quad (9)
\]

The existence and uniqueness (up to a constant) of this function is a classical result of the theory of closed Riemann surfaces \(^6\). By (5) and Ref. 6), p.94, the harmonic kernel function of \( E \) can be expressed in terms of the third Abelian integral of its double \( R\). The relation is especially simple for \( z_i, z_j \) on the boundary of \( B\):

\[
k_B(z_i, z_j) = (2\pi)^{-1} \left( \sum_{z_i \to z_j} (z_i) \right) + \Omega_{z_i, z_j} (z_i)^2 \quad (10)
\]

\(^*\) So European experimentalists need not be unemployed without a 300 GeV machine. They can still construct duality diagrams in tinfoil and measure induced charges.
Here \( z_0 \) is an arbitrary point inside \( B \), and \( \tilde{z}_0 \) its conjugate on the other side of the double. Momentum conservation ensures independence from \( z_0 \). For \( B \) the unit disc, we can take the double \( R \) to be the plane plus \( \infty \) (Riemann sphere) with \( \tilde{z} = \bar{z}^{-1} \). In this case \( \text{Re} \Omega \) will be just the first term of (9).

However, Nielsen's representation is not yet proved. Therefore we now establish (7) + (10) by sewing reggeons (i.e., performing unitarity sums on tree graphs with external excited states). This also gives the measure in the integration variables \( y_\lambda \) (which determines the pomeron intercept). The proof is based on the fact (mentioned to by Dr. V. Alessandrin) that the Abelian integrals of a closed Riemann surface correspond to automorphic functions in the plane \( \mathbb{C} \). This generalizes the relation of the one-loop formula to elliptic functions \( \mathbb{C} \).

We need formulae for a Gaussian functional integral over a real Hilbert space

\[
\int \pi^{-\frac{1}{2}} \mathcal{D}f \exp \left[ -\langle f | A | f \rangle + \langle b | f \rangle \right]
= \left[ \text{det} A \right]^{-\frac{1}{2}} \exp \left[ \frac{\langle b | A^{-1} b \rangle}{4} \right],
\]

(11)

and over a complex Hilbert space

\[
\int \pi^{-1} \mathcal{R} \mathcal{E} \mathcal{R} \mathcal{E} \mathcal{R} \exp \left[ -\langle g | A | g \rangle + \langle b | g \rangle + \langle g | c \rangle \right]
= \left[ \text{det} A \right]^{-1} \exp \left[ -\langle b | A^{-1} c \rangle \right].
\]

(12)

These are valid for \( A \) any diagonalizable operator with finite non-zero Fredholm determinant \( \text{det} A \), and can be proved by multiplying one-variable integrals. \( \int \pi^{-\frac{1}{2}} \mathcal{D}f \) means \( \prod_n \int_{-\infty}^{\infty} (df_n / \sqrt{\pi}) \).

We will use notations and results of Ref. 5). The integrand of the \( N \)-reggeon vertex is then

\[
\prod_{1 \leq i < j \leq N} \exp \langle a^{(i)} | \Gamma^{(i)} \Gamma^{(j)} | a^{(j)} \rangle.
\]

(13)
We take $z_i$ always on the unit circle, so $U^{(i)} = [V^{(i)}]^*$. If we then define

$$U^{(i)} V^{(i)} \equiv 0,$$  \hspace{1cm} (14)

we can write (13) in the symmetric form

$$\exp \left[ \frac{1}{2} \sum_{i,j=1}^{N} \langle \alpha^{(i)} | [V^{(i)}]^* V^{(j)} | \alpha^{(j)} \rangle \right]. \hspace{1cm} (15)$$

Now we factorize it by the real functional integral (11) to get

$$\int (2\pi)^{-\frac{1}{2}} \mathcal{D}f \exp \left[ -\frac{1}{2} \langle f | f \rangle + \sum_{i=1}^{N} \langle f | V^{(i)} | \alpha^{(i)} \rangle \right]. \hspace{1cm} (16)$$

Next we sew $M$ pairs of reggeons, and reduce the others to scalars. If the $\mu^{th}$ sewn pairs are $\langle a^{(i)} \rangle$ and $| a^{(j)} \rangle$, then we relabel

$$U_\mu = U^{(i)} = \sqrt{\mu}^*, \hspace{1cm} \langle a_\mu | = \langle a^{(i)} |,$$  \hspace{1cm} (17)

$$\tilde{U}_\mu = U^{(j)} = \sqrt{\mu}^*, \hspace{1cm} | a_\mu \rangle = | a^{(j)} \rangle, \hspace{1cm} \omega_\mu = z_{j-1}.$$

For an external scalar

$$\langle a^{(i)} | U^{(i)} = (2\alpha')^{\frac{1}{2}} p_i z_i \hspace{1cm} p_i <z_i | = [V^{(i)} | a^{(i)} \rangle]^* \hspace{1cm} (18)$$

where $p_i$ is its momentum and

$$<z_i | n = (z_i)^{-n}/\sqrt{n}, \hspace{1cm} n | z_i \rangle = (z_i)^{n}/\sqrt{n}. \hspace{1cm} (19)$$

Before sewing, we must attach the propagator $\Delta_\mu$ to $\langle a_\mu |$, where $\Delta_\mu$ is either 5) the twisted propagator $\Delta_\mu = p(x_\mu)$, or the untwisted one $\Delta_\mu = D(x_\mu)$. (16) thus becomes
\[ \int (2\pi)^{-\frac{1}{2}} \delta f \cdot \exp \left\{ -\frac{1}{2} \langle f|f \rangle + \sum_{i} (2\alpha')^{\frac{1}{2}} \rho_i \langle \tilde{z}_i|f \rangle + \sum_{\mu=1}^{M} \left[ \langle a_{\mu}|\Delta_{\mu} U_{\mu}|f \rangle + \langle f|\tilde{\nu}_{\mu}|a_{\mu} \rangle \right] \right\}^{2} \]  

(because \( |f\rangle \) is real, \( \langle b|f\rangle = \langle f|b\rangle \)). We now sew the reggeons by functionally integrating each \( |a_{\mu}\rangle \) with the measure \( \int_{\pi}^{1} \delta \text{Re} a_{\mu} \cdot \delta \text{Im} a_{\mu} \cdot \exp \left[ -\langle a_{\mu}|a_{\mu} \rangle \right] \). (21)

However, we leave the loop momenta \( a_{\mu} = (2\alpha')^{\frac{1}{2}} k_{\mu} \) unintegrated for the moment. By (12) the integrals (20)-(21) give

\[ \int (2\pi)^{-\frac{1}{2}} \delta f \cdot \exp \left\{ -\frac{1}{2} \langle f|f \rangle + \sum_{\mu=1}^{M} \langle f|\text{Re} S_{\mu}|f \rangle + \sum_{i} (2\alpha')^{\frac{1}{2}} \rho_i \langle \tilde{z}_i|f \rangle + \sum_{\mu=1}^{M} (2\alpha')^{\frac{1}{2}} k_{\mu} \langle \omega_{\mu}|[S_{\mu}^{-1}]-1|f \rangle \right\}^{2} \]

where

\[ S_{\mu} \equiv \tilde{\nu}_{\mu} \Delta_{\mu} U_{\mu} \]  

Since \( |f\rangle \) is real, only \( \text{Re} S_{\mu} = \frac{1}{2}(S_{\mu} + S_{\mu}^{-1}) \) will contribute to the second term. \( S_{\mu} \) corresponds to a Möbius (homographic) transformation.

It can be shown that

\[ S_{\mu}^{*} = \nu_{\mu} \Delta_{\mu} U_{\mu} = S_{\mu}^{-1} \]  

\( S_{\mu} \) is unitary in the \( n \geq 1 \) subspace of \( n|a_{\mu}\rangle \), which is the \( D_1^+ \) representation of \( O(2,1) \). Thus \( \text{Re} S_{\mu} = \frac{1}{2}(S_{\mu} + S_{\mu}^{-1}) \). Lastly, we do the \( \delta f \) integral by (11) to get the \( M \)-loop integrand

\[ \det \left[ 1 - \sum_{\mu=1}^{M} (S_{\mu} + S_{\mu}^{-1}) \right]^{-\frac{1}{2}} \times \]

\[ \exp \left\{ \alpha' \sum_{i \neq j} (\rho_i \rho_j) \langle \tilde{z}_i|1 - \sum_{\mu=1}^{M} (S_{\mu} + S_{\mu}^{-1}) \right\}^{-1}|\tilde{z}_j\rangle \]  

(25)
where we have temporarily omitted the $k_\mu$ terms for brevity. The power $-2$ of the Fredholm determinant comes from the four space-time dimensions $^{11)}$.

Now we recall the constraint (14). For external scalars, this gives $ij$ in (25). For the sewn reggeons it means by (23) and (24) that all terms containing adjacent factors $S_\mu$ and $S^{-1}_\mu$ are to be zeroed when expanding the matrix inverse in (25). To count the remaining terms, we introduce the infinite automorphism group of M"ubius transformations $T_\alpha$ generated by the $S_\mu$. Each element $T_\alpha$ is an ordered product of positive and negative powers of the various $S_\mu$. If we drop all terms with adjacent $S_\mu$ and $S^{-1}_\mu$, the inverse in (25) generates all elements of the automorphism group once each. Thus

$$<z_i| \left[1 - \sum_{\mu=1}^M (S_\mu + S^{-1}_\mu) \right]^{-1} |z_j>$$

$$= \sum_{\alpha} <z_i| T_\alpha |z_j> .$$

(26)

But this is just a Poincaré $\theta$-series $^{13)}$! In fact, Burnside $^{14)}$ showed that the third Abelian integral with complex normalization [Ref. 6, p.75] is the $\theta$-series

$$\omega_{ab}(z) = \sum_\alpha \log \left\{ \left( \frac{T_\alpha(z) - b}{T_\alpha(z) - a} \right) \left( \frac{T_\alpha(c) - a}{T_\alpha(c) - b} \right) \right\} .$$

(27)

So, writing $\approx$ for equality up to terms cancelled by momentum conservation, we have by (19)

$$\sum_\alpha <z_i| T_\alpha |z_j> = - \sum_\alpha \log |z_i - T_\alpha(z_j)|$$

$$\approx \Re \omega_{z_i z_0}(z_j).$$

(28)

Now, (thanks to V. Alessandrini) we can integrate the loop momenta $k_\mu$. The terms omitted from the exponent of (25) are
\[
\sum_{i} \sum_{\mu=1}^{M} 2\alpha' (k_{\mu} k_{i}) \sum_{\alpha} \langle \omega_{\mu} | [S_{\mu} - 1] T_{\alpha} | z_{i} \rangle + \\
+ \sum_{\mu, \nu=1}^{M} \alpha' (k_{\mu} k_{\nu}) \sum_{\alpha} \langle \omega_{\mu} | [S_{\mu} - 1] T_{\alpha} [S_{\nu}^{-1} - 1] | \omega_{\nu} \rangle 
\]

\[
\approx - \sum_{i} \sum_{\mu=1}^{M} 2\alpha' (k_{\mu} k_{i}) \text{Re } \Phi_{\mu}(z_{i}) + \sum_{\mu, \nu=1}^{M} \alpha' (k_{\mu} k_{\nu}) \text{Re } A_{\mu\nu},
\]

where, as shown by Burnside \(^{14}\), \(\Phi_{\mu}(z)\) and \(A_{\mu\nu}\) are the first Abelian integrals and their period matrix, and are independent of \(w_{\mu}, w_{\nu}\). Integrating \(k_{\mu}\) thus adds to the exponent of (25), exactly the term needed to convert the third Abelian integral (28) from complex \(\omega\) to real \(\Omega\) normalization, in accord with (10). Our final formula for the \(M\)-loop integrand is then

\[
\text{det} (\alpha' \text{Re } A)^{-2} \cdot \left[ \text{det} \left( \sum_{\alpha} T_{\alpha} \right) \right]^{-2} \times \\
x \exp \left\{ \alpha' \sum_{i \neq j} (k_{i} k_{j}) \text{Re } \Omega_{z_{i} z_{0}} (z_{j}) \right\}.
\]

It can be shown that the \(Q_{\mu\nu}^{B}\) of (8) is

\[
Q_{\mu\nu}^{B} = \pi \text{Re } [A^{-1}]_{\mu\nu},
\]

so Regge singularities will depend also on \(\text{Im } A_{\mu\nu}\), and thus on the ordering of the sewn reggeons. The determinants in (30) give the integration measure missing from Nielsen's representation. (For no loops, the automorphism group is just \(\alpha = 1\), and (28) gives the tree formula).

To find the Riemann surface \(R\) to which this Abelian integral belongs, we write the Möbius transformation \(S_{\mu}\) in normal form

\[
S_{\mu}(z) = \frac{\alpha_{\mu} z + \beta_{\mu}}{\delta_{\mu} z + \gamma_{\mu}}, \quad \alpha_{\mu} \delta_{\mu} - \beta_{\mu} \gamma_{\mu} = 1,
\]
and construct the isometric circles [Ref. 13], p.25 of \( S^\mu \) and \( S^{-1}_\mu \):

\[
\begin{align*}
\xi^\mu : |\gamma^\mu z + \delta^\mu | &= 1, \\
\check{\xi}^\mu : |\gamma^\mu z - \alpha^\mu | &= 1, \quad \mu = 1, \ldots, M.
\end{align*}
\] (33)

\( S^\mu \) transforms \( c^\mu \) into \( \check{c}^\mu \). The plane outside all these \( 2M \) circles is a fundamental region for the automorphism group [Ref. 13], p.45. Therefore the Riemann surface \( R \) is got by removing their insides, and sewing each \( c^\mu \) to \( \check{c}^\mu \), by identifying each point on \( c^\mu \) with its image under \( S^\mu \). Considering the plane plus \( \infty \) as the Riemann sphere, this obviously gives a sphere with \( M \) handles.

By (23) and (17), the generators when written out in the notation of Ref. 5) are

\[
S^\mu = \begin{bmatrix}
\infty & 0 & 1 \\
\overline{z_{j-1}} & \overline{z}_j & \overline{z}_{j+1} \\
\end{bmatrix}
\begin{bmatrix}
0 & \infty & 1 \\
\overline{x}_j & 1 & 0 \\
0 & \infty & 1 \\
\end{bmatrix}
\begin{bmatrix}
\overline{z}_{i-1} & \overline{z}_i & \overline{z}_{i+1} \\
\end{bmatrix}
\] (34)

for \( \Delta^\mu = P(x^\mu) \) a twisted propagator. Reading from right to left, we have: unit circle anticlockwise \( \to \) real axis leftwards \( \to \) real axis rightwards \( \to \) unit circle anticlockwise. Therefore \( S^\mu \) containing twisted propagators transform the interior of the unit circle into itself. Similarly \( S^\mu \) containing untwisted propagators \( \Delta^\mu = D(x^\mu) \), transform the interior of the unit circle into its exterior. Orientable diagrams can be constructed by sewing the symmetric \( N \)-reggeon vertex with twisted propagators \( 2 \), so their automorphism group is Fuchsian [Ref. 13], p.66, and Burnside's first criterion \( \endnote{14} \) immediately proves convergence of the \( \Theta \)-series, provided all circles \( c^\mu, \check{c}^\mu \) are external to each other. For non-orientable diagrams, the untwisted propagators give loxodromic generators \( S^\mu \) [Ref. 13], p.21 and we have a Kleinian group. However, examination of Burnside's first convergence proof \( \endnote{14} \) shows that it only requires the circumference of the unit circle to be invariant under the whole group, and therefore applies to non-orientable diagrams also.

These sewn circles \( c^\mu, \check{c}^\mu \) will all intersect the invariant unit circle [Ref. 13], pp.28,67. Therefore the unit circle becomes a sum of closed curves on the Riemann surface \( R \), running round all its
handles. When we cut $R$ along this transformed unit circle, we get the original duality diagram $B$, with the Koba-Nielsen variables $z_i$ on its boundary. $R$ is the orientable double of $B$ [Ref. 6], p. 22.

The Fredholm determinant, $\text{det}(\sum \alpha T_\alpha)$ in (30), does not look to me like any known function, but its convergence can be investigated by $\theta$-series techniques 13). Loop divergences come from "parabolic" points where circles $\alpha'$, $\tilde{\alpha}$ touch.

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*) After this paper was finished, Dr. L.P. Yu (Berkeley) wrote me that he had discovered something like Eq. (25), but apparently not its solution by Abelian integrals.
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