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M - LOOP GENERALIZED VENEZIANO FORMULA

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A B S T R A C T

We construct the Veneziano formula with M closed loops, both by Nielsen's representation and by sewing reggeons. The logarithm of its integrand is just the third Abelian integral of a sphere with M handles. Domain variational theory determines all its Regge singularities.

Recently an embryonic pomeron singularity ^{*}) was discovered ^{1),2)} in the generalized Veneziano formula with one closed loop. This makes construction of the M-loop formula urgent. Previous attempts ³⁾ stopped in an impenetrable thicket of harmonic oscillators. We show there is a simple answer. The logarithm of the M-loop integrand is just the third Abelian integral of a sphere with M handles. This is true both for orientable and non-orientable diagrams, they differ only in the co-ordinate system. We give two separate proofs from different hypotheses - one from Nielsen's "almost-physical" representation ⁴⁾, the other by sewing the N-reggeon vertex ⁵⁾. The second proof is especially beautiful. Abelian integrals are not tabulated functions for $M > 1$, but they have been much studied by mathematicians. In particular, domain variational theory ⁶⁾ determines all Regge singularities of our M-loop formula. We are thus laying the foundation for a dual reggeon calculus.

In Nielsen's representation ⁴⁾ the duality diagram becomes a two-dimensional conductor, with M holes cut out for the loops, and N electrodes on the boundary for its external lines. Nielsen conjectured that the logarithm of the Veneziano integrand was the heat generated by a steady current. So far this has been verified ^{4),1)} for $M=0, 1$. We start by giving a neat mathematical form to Nielsen's conjecture. Consider the duality diagram as a domain B with boundary b. The harmonic functions on B, which vanish at some point z_0 , form a Hilbert space $\bigwedge^2(B)$ with Dirichlet inner product ⁷⁾

$$\begin{aligned} D_B \{f, g\} &= \iint_B (\text{grad } f \cdot \text{grad } g) \, dx \, dy \\ &= \int_b f \frac{\partial g}{\partial n} \, ds \end{aligned} \tag{1}$$

^{*}) This singularity has the right isospin and slope for the pomeron. Also experimentally the pomeron is mainly the shadow of ordinary multi-Regge exchange, and therefore ought to emerge from double scattering graphs in a dual model.

Let $\varphi_m(z)$ be any complete orthonormal basis for $\Lambda^2(B)$. Then the harmonic kernel function of B is ⁷⁾

$$k_B(z, w) = \sum_m \varphi_m(z) \varphi_m(w), \quad (2)$$

which converges and is independent of the basis ⁷⁾. k_B is a projection operator for the Hilbert space $\Lambda^2(B)$. Therefore

$$D_B^t \{k_B(z, t), k_B(t, w)\} = k_B(z, w). \quad (3)$$

It also solves the Neumann boundary value problem

$$f(z) = \int_b ds_t \cdot k_B(z, t) \cdot \frac{\partial f(t)}{\partial n_t}, \quad (4)$$

for f harmonic in B , since ⁷⁾

$$k_B(z, w) = (2\pi)^{-1} [N_B(z, w) - G_B(z, w)], \quad (5)$$

where N and G are the Neumann's and Green's functions of B . Now in Nielsen's representation, a current is given on the boundary

$$\frac{\partial f}{\partial n_t}(s_t) = (\pi\alpha')^{\frac{1}{2}} \sum_{i=1}^N p_i \delta(z_i - s_t), \quad (6)$$

where p_i are the external momenta, and z_i the points on the boundary where electrodes are attached (Koba-Nielsen integration variables).

Therefore (4) is the potential inside B . The heat generated is its Dirichlet norm $D_B \{f, f\}$, which can be evaluated at once by (3). So Nielsen's conjecture gives the generalized Veneziano integrand in terms of the harmonic kernel function of its duality diagram

$$\exp \left\{ \pi\alpha' \sum_{i \neq j} (p_i p_j) k_B(z_i, z_j) \right\}. \quad (7)$$

Now we sketch how domain variational theory ⁶⁾ can be used to determine Regge singularities. Consider for simplicity an M - loop

orientable diagram B. It is conformally equivalent to the plane with $M+1$ circular discs removed. Its harmonic kernel function can be got by orthonormalizing inverse power series (plus logarithms) at their centres ⁷⁾. For $M=0, 1$ this gives the known answers ^{4), 1)}. For $M > 1$, B is characterized up to conformal equivalence ⁶⁾ by $3M-3$ Riemann moduli y_λ . Since k_B is a conformal invariant, when we sum over all M -loop domains, these y_λ become integration variables, as well as the z_i . Nielsen's representation does not give the integration measure in y_λ , but we will determine it below. However (7) already shows that all moving Regge singularities must come from singularities of k_B , either in the z_i or the y_λ . The only singularity in z_i, z_j is a logarithmic one when $z_i = z_j$ on the boundary. This gives usual Regge poles. The singularities of k_B in y_λ are also known [Ref. 6), Ch. 7]. They occur only at values where the topology of the duality diagram B changes. There are two cases: a hole shrinking to a point gives the pomeron, two holes touching give Regge cuts ^{*}).

Pomeron singularities always occur ⁶⁾ in the M -dimensional subspace of $\Lambda^2(B)$ spanned by the harmonic measures. A harmonic measure $\omega_\mu^B(z)$ is the function harmonic in B, with $\omega_\mu^B(z) = 1$ on the boundary of the μ^{th} hole and $\omega_\mu^B(z) = 0$ on all other boundaries ⁷⁾. There are $M+1$ boundaries for an orientable M -loop diagram, but only M of the $\omega_\mu^B(z)$ are linearly independent. They contribute to the integrand (7) a factor

$$\exp \left\{ \pi \alpha' \sum_{\mu, \nu=1}^M (P_\mu P_\nu) \cdot Q_{\mu\nu}^B \right\}, \quad (8)$$

where P_μ is the total momentum entering through the μ^{th} boundary, and $Q_{\mu\nu}^B$ is the inverse of the period matrix ⁷⁾. There is an amusing electrostatic construction for $Q_{\mu\nu}^B$: if cylindrical conductors are

*) The existence of these singularities is a general theorem, but their physical identification is inferred from the annulus $r \leq |z| \leq 1$, where $r \rightarrow 0$ gives pomeron and $r \rightarrow 1$ Regge cuts.

built on the various boundaries, it relates their surface potentials to their total charges ^{*}). (8) will cancel unless external lines enter through different boundaries. This explains why Regge cuts only occur in such so-called non-planar diagrams. (8) is the factor carrying the pomeron trajectory. For the annulus $r \leq |z| \leq 1$, $Q_{11} = -\log r/2\pi$, which illustrates both the logarithmic singularity of k_B as $r \rightarrow 0$, and how (8) gives Frye and Susskind's ¹⁾ pomeron with slope $\alpha'/2$. Domain variational theory shows [Ref. 6), p.285] that the same singularity will occur in multi-loop diagrams, when the radius $r \rightarrow 0$ of any hole with external lines entering it. It also determines higher powers in r , which give the pomeron residue ⁸⁾.

Now we express (7) as an Abelian integral. Joining two mirror copies of B along their boundaries, gives a closed Riemann surface R called its double. There is a more complicated construction for the double of a non-orientable diagram [Ref. 6), p.30]. The double itself is always orientable. Thus the torus is the double of both annulus and Möbius strip, which explains ⁹⁾ why they give such similar Veneziano formulae. The third Abelian integral [Ref. 6), p.69] of a closed Riemann surface B is a function $\Omega_{ww_0}(z)$ of z on R , whose differential is analytic except for two poles, and whose real part is single-valued (= real normalization) and harmonic except for two logarithmic singularities

$$\operatorname{Re} \Omega_{ww_0}(z) = \log \left| \frac{z - w_0}{z - w} \right| + \text{harmonic}. \quad (9)$$

The existence and uniqueness (up to a constant) of this function is a classical result of the theory of closed Riemann surfaces ⁶⁾. By (5) and Ref. 6), p.94, the harmonic kernel function of B can be expressed in terms of the third Abelian integral of its double R . The relation is especially simple for z_i, z_j on the boundary of B :

$$k_B(z_i, z_j) = (2\pi)^{-1} \left\{ \Omega_{z_j z_0}(z_i) + \Omega_{z_j \tilde{z}_0}(z_i) \right\}. \quad (10)$$

^{*}) So European experimentalists need not be unemployed without a 300 GeV machine. They can still construct duality diagrams in tinfoil and measure induced charges.

Here z_0 is an arbitrary point inside B , and $\tilde{z}_0$ its conjugate on the other side of the double. Momentum conservation ensures independence from z_0 . For B the unit disc, we can take the double R to be the plane plus ∞ (Riemann sphere) with $\tilde{z} = \bar{z}^{-1}$. In this case $\text{Re } \Omega$ will be just the first term of (9).

However, Nielsen's representation is not yet proved. Therefore we now establish (7)+(10) by sewing reggeons (i.e., performing unitarity sums on tree graphs with external excited states). This also gives the measure in the integration variables y_λ (which determines the pomeron intercept). The proof is based on the fact (mentioned to by Dr. V. Alessandrini) that the Abelian integrals of a closed Riemann surface correspond to automorphic functions in the plane ⁹⁾. This generalizes the relation of the one-loop formula to elliptic functions ¹⁰⁾.

We need formulae for a Gaussian functional integral over a real Hilbert space

$$\begin{aligned} & \int \pi^{-\frac{1}{2}} \delta f \cdot \exp [-\langle f | A | f \rangle + \langle b | f \rangle] \\ &= [\det A]^{-\frac{1}{2}} \exp [\langle b | A^{-1} | b \rangle / 4], \end{aligned} \quad (11)$$

and over a complex Hilbert space

$$\begin{aligned} & \int \pi^{-1} \delta \text{Re } g \cdot \delta \text{Im } g \cdot \exp [-\langle g | A | g \rangle + \langle b | g \rangle + \langle g | c \rangle] \\ &= [\det A]^{-1} \exp [\langle b | A^{-1} | c \rangle]. \end{aligned} \quad (12)$$

These are valid for A any diagonalizable operator with finite non-zero Fredholm determinant $\det A$, and can be proved by multiplying one-variable integrals. $\int \pi^{-\frac{1}{2}} \delta f$ means $\prod_n \int_{-\infty}^{\infty} (df_n / \sqrt{\pi})$.

We will use notations and results of Ref. 5). The integrand of the N -reggeon vertex is then

$$\prod_{1 \leq i < j \leq N} \exp \langle a^{(i)} | U^{(i)} V^{(j)} | a^{(j)} \rangle. \quad (13)$$

We take z_i always on the unit circle, so $U^{(i)} = [V^{(i)}]^*$. If we then define

$$U^{(i)} V^{(i)} \equiv 0, \quad (14)$$

we can write (13) in the symmetric form

$$\exp \left[\frac{1}{2} \sum_{i,j=1}^N \langle a^{(i)} | [V^{(i)}]^* V^{(j)} | a^{(j)} \rangle \right], \quad (15)$$

Now we factorize it by the real functional integral (11) to get

$$\int (2\pi)^{-\frac{1}{2}} \delta f \exp \left[-\frac{1}{2} \langle f | f \rangle + \sum_{i=1}^N \langle f | V^{(i)} | a^{(i)} \rangle \right]. \quad (16)$$

Next we sew M pairs of reggeons, and reduce the others to scalars. If the μ^{th} sewn pairs are $\langle a^{(i)} |$ and $| a^{(j)} \rangle$, then we relabel

$$\begin{aligned} U_\mu &= U^{(i)} = V_\mu^*, \quad \langle a_\mu | = \langle a^{(i)} |, \\ \tilde{U}_\mu &= U^{(j)} = \tilde{V}_\mu^*, \quad | a_\mu \rangle = | a^{(j)} \rangle, \quad w_\mu = z_{j-1}. \end{aligned} \quad (17)$$

For an external scalar

$$\langle a^{(i)} | U^{(i)} = (2\alpha')^{\frac{1}{2}} p_i \langle z_i | = [V^{(i)} | a^{(i)} \rangle]^*, \quad (18)$$

where p_i is its momentum and

$$\langle z_i |_n = (z_i)^{-n} / \sqrt{n}, \quad {}_n | z_i \rangle = (z_i)^n / \sqrt{n}. \quad (19)$$

Before sewing, we must attach the propagator Δ_μ to $\langle a_\mu |$, where Δ_μ is either ⁵⁾ the twisted propagator $\Delta_\mu = P(x_\mu)$, or the untwisted one $\Delta_\mu = D(x_\mu)$. (16) thus becomes

$$\int (2\pi)^{-\frac{1}{2}} \delta f \cdot \exp \left\{ -\frac{1}{2} \langle f | f \rangle + \sum_i (2\alpha')^{\frac{1}{2}} \rho_i \langle z_i | f \rangle + \right. \\ \left. + \sum_{\mu=1}^M \left[\langle a_\mu | \Delta_\mu U_\mu | f \rangle + \langle f | \tilde{V}_\mu | a_\mu \rangle \right] \right\} \quad (20)$$

(because $|f\rangle$ is real, $\langle b | f \rangle = \langle f | b \rangle$). We now sew the reggeons by functionally integrating each $|a_\mu\rangle$ with the measure ¹¹⁾

$$\int \pi^{-1} \cdot \delta \operatorname{Re} a_\mu \cdot \delta \operatorname{Im} a_\mu \cdot \exp [-\langle a_\mu | a_\mu \rangle]. \quad (21)$$

However, we leave the loop momenta $a_\mu^0 = (2\alpha')^{\frac{1}{2}} k_\mu$ unintegrated for the moment. By (12) the integrals (20)-(21) give

$$\int (2\pi)^{-\frac{1}{2}} \delta f \cdot \exp \left\{ -\frac{1}{2} \langle f | f \rangle + \sum_{\mu=1}^M \langle f | \operatorname{Re} S_\mu | f \rangle + \right. \\ \left. + \sum_i (2\alpha')^{\frac{1}{2}} \rho_i \langle z_i | f \rangle + \sum_{\mu=1}^M (2\alpha')^{\frac{1}{2}} k_\mu \langle w_\mu | [S_\mu - 1] | f \rangle \right\}, \quad (22)$$

where

$$S_\mu \equiv \tilde{V}_\mu \Delta_\mu U_\mu. \quad (23)$$

Since $|f\rangle$ is real, only $\operatorname{Re} S_\mu = \frac{1}{2}(S_\mu + S_\mu^*)$ will contribute to the second term. S_μ corresponds to a Möbius (homographic) transformation ⁵⁾. It can be shown that

$$S_\mu^* = V_\mu \Delta_\mu \tilde{U}_\mu = S_\mu^{-1}. \quad (24)$$

S_μ is unitary in the $n \geq 1$ subspace of $_n |a_\mu\rangle$, which is the D_1^+ representation ¹²⁾ of $O(2,1)$. Thus $\operatorname{Re} S_\mu = \frac{1}{2}(S_\mu + S_\mu^{-1})$. Lastly, we do the δf integral by (11) to get the M -loop integrand

$$\det \left[1 - \sum_{\mu=1}^M (S_\mu + S_\mu^{-1}) \right]^{-2} \times \\ \times \exp \left\{ \alpha' \sum_{i \neq j} (\rho_i \rho_j) \langle z_i | \left[1 - \sum_{\mu=1}^M (S_\mu + S_\mu^{-1}) \right]^{-1} | z_j \rangle \right\}, \quad (25)$$

where we have temporarily omitted the k_μ terms for brevity. The power -2 of the Fredholm determinant comes from the four space-time dimensions ¹¹⁾.

Now we recall the constraint (14). For external scalars, this gives $i \neq j$ in (25). For the sewn reggeons it means by (23) and (24) that all terms containing adjacent factors S_μ and S_μ^{-1} are to be zeroed when expanding the matrix inverse in (25). To count the remaining terms, we introduce the infinite automorphism group of Möbius transformations T_α generated by the S_μ . Each element T_α is an ordered product of positive and negative powers of the various S_μ . If we drop all terms with adjacent S_μ and S_μ^{-1} , the inverse in (25) generates all elements of the automorphism group once each. Thus

$$\begin{aligned} \langle z_i | [1 - \sum_{\mu=1}^M (S_\mu + S_\mu^{-1})]^{-1} | z_j \rangle \\ = \sum_{\alpha} \langle z_i | T_\alpha | z_j \rangle. \end{aligned} \quad (26)$$

But this is just a Poincaré θ -series ¹³⁾ ! In fact, Burnside ¹⁴⁾ showed that the third Abelian integral with complex normalization [Ref. 6), p.75] is the θ -series

$$\omega_{ab}(z) = \sum_{\alpha} \log \left\{ \left(\frac{T_\alpha(z) - b}{T_\alpha(z) - a} \right) \left(\frac{T_\alpha(c) - a}{T_\alpha(c) - b} \right) \right\}. \quad (27)$$

So, writing $\approx$ for equality up to terms cancelled by momentum conservation, we have by (19)

$$\begin{aligned} \sum_{\alpha} \langle z_i | T_\alpha | z_j \rangle &= - \sum_{\alpha} \log |z_i - T_\alpha(z_j)| \\ &\approx \text{Re } \omega_{z_i z_0}(z_j). \end{aligned} \quad (28)$$

Now, (thanks to V. Alessandrini) we can integrate the loop momenta k_μ . The terms omitted from the exponent of (25) are

$$\begin{aligned} & \sum_i \sum_{\mu=1}^M 2\alpha'(k_\mu k_i) \sum_{\alpha} \langle w_\mu | [S_\mu - 1] T_\alpha | z_i \rangle + \\ & + \sum_{\mu, \nu=1}^M \alpha'(k_\mu k_\nu) \sum_{\alpha} \langle w_\mu | [S_\mu - 1] T_\alpha [S_\nu^{-1} - 1] | w_\nu \rangle \end{aligned} \quad (29)$$

$$\approx - \sum_i \sum_{\mu=1}^M 2\alpha'(k_\mu k_i) \operatorname{Re} \varphi_\mu(z_i) + \sum_{\mu, \nu=1}^M \alpha'(k_\mu k_\nu) \operatorname{Re} A_{\mu\nu},$$

where, as shown by Burnside ¹⁴⁾, $\varphi_\mu(z_i)$ and $A_{\mu\nu}$ are the first Abelian integrals and their period matrix, and are independent of w_μ , w_ν . Integrating k_μ thus adds to the exponent of (25), exactly the term needed to convert the third Abelian integral (28) from complex ω to real Ω normalization, in accord with (10). Our final formula for the M-loop integrand is then

$$\begin{aligned} & \det(\alpha' \operatorname{Re} A)^{-2} \cdot [\det(\sum_{\alpha} T_{\alpha})]^2 \times \\ & \times \exp \left\{ \alpha' \sum_{i \neq j} (k_i k_j) \operatorname{Re} \Omega_{z_i z_0}(z_j) \right\}. \end{aligned} \quad (30)$$

It can be shown that the $Q_{\mu\nu}^B$ of (8) is

$$Q_{\mu\nu}^B = \pi \operatorname{Re} [A^{-1}]_{\mu\nu}, \quad (31)$$

so Regge singularities will depend also on $\operatorname{Im} A_{\mu\nu}$, and thus on the ordering of the sewn reggeons. The determinants in (30) give the integration measure missing from Nielsen's representation. (For no loops, the automorphism group is just $T_{\alpha} = 1$, and (28) gives the tree formula).

To find the Riemann surface R to which this Abelian integral belongs, we write the Möbius transformation S_μ in normal form

$$S_\mu(z) = \frac{\alpha_\mu z + \beta_\mu}{\gamma_\mu z + \delta_\mu}, \quad \alpha_\mu \delta_\mu - \beta_\mu \gamma_\mu = 1, \quad (32)$$

and construct the isometric circles [Ref. 13), p.25] of S_μ and S_μ^{-1} :

$$\begin{aligned} c_\mu &: |\gamma_\mu z + \delta_\mu| = 1, \\ \tilde{c}_\mu &: |\gamma_\mu z - \alpha_\mu| = 1, \quad \mu = 1, \dots, M. \end{aligned} \quad (33)$$

S_μ transforms c_μ into $\tilde{c}_\mu$. The plane outside all these $2M$ circles is a fundamental region for the automorphism group [Ref. 13), p.45]. Therefore the Riemann surface R is got by removing their insides, and sewing each c_μ to $\tilde{c}_\mu$, by identifying each point on c_μ with its image under S_μ . Considering the plane plus ∞ as the Riemann sphere, this obviously gives a sphere with M handles.

By (23) and (17), the generators when written out in the notation of Ref. 5) are

$$S_\mu = \begin{bmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{bmatrix} \begin{bmatrix} 0 & \infty & 1 \\ x_\mu & 1 & 0 \end{bmatrix} \begin{bmatrix} z_{i-1} & z_i & z_{i+1} \\ 0 & \infty & 1 \end{bmatrix} \quad (34)$$

for $\Delta_\mu = P(x_\mu)$ a twisted propagator. Reading from right to left, we have: unit circle anticlockwise $\rightarrow$ real axis leftwards $\rightarrow$ real axis rightwards $\rightarrow$ unit circle anticlockwise. Therefore S_μ containing twisted propagators transform the interior of the unit circle into itself. Similarly S_μ containing untwisted propagators $\Delta_\mu = D(x_\mu)$, transform the interior of the unit circle into its exterior. Orientable diagrams can be constructed by sewing the symmetric N -reggeon vertex with twisted propagators ²⁾, so their automorphism group is Fuchsian [Ref. 13), p.66], and Burnside's first criterion ¹⁴⁾ immediately proves convergence of the θ -series, provided all circles $c_\mu, \tilde{c}_\mu$ are external to each other. For non-orientable diagrams, the untwisted propagators give loxodromic generators S_μ [Ref. 13), p.21] and we have a Kleinian group. However, examination of Burnside's first convergence proof ¹⁴⁾ shows that it only requires the circumference of the unit circle to be invariant under the whole group, and therefore applies to non-orientable diagrams also.

These sewn circles $c_\mu, \tilde{c}_\mu$ will all intersect the invariant unit circle [Ref. 13), pp.28,67]. Therefore the unit circle becomes a sum of closed curves on the Riemann surface R , running round all its

handles. When we cut R along this transformed unit circle, we get the original duality diagram B , with the Koba-Nielsen variables z_i on its boundary. R is the orientable double of B [Ref. 6), p.29].

The Fredholm determinant, $\det(\sum_{\alpha} T_{\alpha})$ in (30), does not look to me like any known function, but its convergence can be investigated by Θ -series techniques ¹³⁾. Loop divergences come from "parabolic" points where circles $c_{\mu}, \tilde{c}_{\nu}$ touch.

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*) After this paper was finished, Dr. L.P. Yu (Berkeley) wrote me that he had discovered something like Eq. (25), but apparently not its solution by Abelian integrals.

R E F E R E N C E S

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