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SIMPLE N - REGGEON VERTEX

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A B S T R A C T

We prove a Veneziano formula for N external Reggeons, no less elegant than that for scalars. It transforms neatly under twists, gauges, propagation and factorization. An operator version suggests a third quantized field theory.

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A Veneziano formula for external Reggeons would be an important ingredient in the Reggeon calculus. The three-Reggeon vertex was obtained by Sciuto ¹⁾ and Caneschi et al. ²⁾. Several authors ^{3),4)} considered N-Reggeons, but their expressions were often over a page long, and always involved the integration variables in a complicated and non-intuitive way. It thus seems remarkable that a simple, explicit Veneziano formula exists for N external Reggeons, which can compete in elegance with the Koba-Nielsen expression ^{5),6)} for scalars.

In Veneziano theory, the state of the i^{th} Reggeon is described by an infinite-dimensional vector $|a^{(i)}\rangle$ with components $a_{n\mu}^{(i)}$, $n = 0, 1, \dots, \infty$; μ being the space-time index. The $n = 0$ components are its momentum ⁷⁾ (units $\alpha' = \frac{1}{2}$)

$$a_{0\mu}^{(i)} = p_i^\mu, \quad (1)$$

and the others label a coherent state basis ⁸⁾ of the harmonic oscillators which generate its excited levels ⁹⁾. (Though they correspond to the harmonic oscillators, the $a_{n\mu}^{(i)}$ here are c-numbers.) All infinite matrices C_{nm} which occur in the theory can be derived from Möbius transformations

$$w = (\alpha z + \beta) / (\gamma z + \delta), \quad (2)$$

by expanding

$$w^n / \sqrt{n} = \sum_{m=0}^{\infty} C_{nm} z^m / \sqrt{m} \quad (3)$$

(with $1/\sqrt{0} \equiv 1$) ^{*}). We will denote the infinite matrix, corresponding to the unique Möbius transformation which takes the complex numbers f, g, h into f', g', h' respectively by

$$\begin{bmatrix} f & g & h \\ f' & g' & h' \end{bmatrix}. \quad (4)$$

^{*}) This is a non-normalizable $\epsilon \rightarrow 0$ limit of the unitary representation D_{ϵ}^+ of the covering group of $SU(1,1)$.

When taking products in this notation, remember that the top row of each bracket acts on the bottom row of the bracket to its right. The order of columns is obviously free. We will need some simple facts about Möbius matrices,

Lemma 1

Define the anharmonic ratio

$$(z_1, z_2, z_3, z_4) = \left(\frac{z_1 - z_2}{z_1 - z_4} \right) \left(\frac{z_3 - z_4}{z_3 - z_2} \right). \quad (5)$$

Then the Möbius transformation (2), corresponding to (4), is

$$(w, f', g', h') = (z, f, g, h). \quad (6)$$

Lemma 2

(a) The inverse matrix is obviously got by switching top and bottom rows in (4),

(b) The transposed matrix is

$$\begin{bmatrix} c & d & e \\ f & g & h \end{bmatrix}^+ = \begin{bmatrix} f^{-1} & g^{-1} & h^{-1} \\ c^{-1} & d^{-1} & e^{-1} \end{bmatrix}. \quad (7)$$

This can be proved by considering first the subgroups corresponding to the generators ^{10), 11)} for which $L_0^+ = L_0$, $L_+^+ = L_-$, and then checking the group property.

Lemma 3

When the canonical forms of Alessandrini et al. ¹²⁾ act on the Reggeon i , they will just multiply the coherent state $|a^{(i)}\rangle$ by a Möbius matrix [Cf. Eqs. (31)-(33) below]. Transcribing their results, we thus have for the known operators of the theory :

(a) Twisting operator

$$\Omega = \begin{bmatrix} 0 & \infty & 1 \\ 1 & \infty & 0 \end{bmatrix}, \quad \Omega^+ = \begin{bmatrix} \infty & 0 & 1 \\ 1 & 0 & \infty \end{bmatrix}, \quad (8)$$

(b) Gauge transformation

$$G(x) \equiv x^W = \begin{bmatrix} 0 & \infty & 1 \\ 0 & x/(x-1) & 1 \end{bmatrix} = \begin{bmatrix} 0 & (1-x)^{-1} & 1 \\ 0 & \infty & 1 \end{bmatrix}, \quad (9)$$

$$G^\dagger(x) \equiv x^{W^\dagger} = \begin{bmatrix} \infty & (1-x^{-1}) & 1 \\ \infty & 0 & 1 \end{bmatrix} = \begin{bmatrix} \infty & 0 & 1 \\ \infty & (1-x) & 1 \end{bmatrix}, \quad (10)$$

(c) Propagator

$$D(x) \equiv x^{L_0} = \begin{bmatrix} 0 & \infty & 1 \\ 0 & \infty & x \end{bmatrix}, \quad (11)$$

(d) Twisted propagator

$$P(x) \equiv x^{L_0} \Omega (1-x)^W = \begin{bmatrix} 0 & \infty & 1 \\ x & 1 & 0 \end{bmatrix} = \begin{bmatrix} x^{-1} & \infty & 1 \\ \infty & 1 & 0 \end{bmatrix}. \quad (12)$$

The parts of their canonical forms containing the momenta will just be the C_{on} and C_{no} elements of these matrices. (Slight redefinition is needed to get C_{oo} right in Ω .)

The Reggeon contains both physical and spurious states ¹⁰⁾. Physical states are invariant under the gauge transformation (9)

$$x^W |a^{(i)}\rangle = |a^{(i)}\rangle. \quad (13)$$

Since $W^\dagger \neq W$, this gives Reggeon lines a direction.

Now we can write down our N -Reggeon vertex.

THEOREM : Let z_i be the Koba-Nielsen variables ^{5),6)} with $z_{N+i} \equiv z_i$, and define the Möbius matrices

$$U^{(i)} = \begin{bmatrix} z_{i-1} & z_i & z_{i+1} \\ 0 & \infty & 1 \end{bmatrix}, \quad (14)$$

$$V^{(j)} = \begin{bmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{bmatrix}. \quad (15)$$

Then the N -Reggeon vertex is

$$W_N(a) = \int d^N z^{N-3} \varphi_N^0(z) \times \prod_{1 \leq i < j \leq N} \exp \langle a^{(i)} | U^{(i)} V^{(j)} | a^{(j)} \rangle, \quad (16)$$

where $\varphi_N^0(z)$ is the zero momentum limit of the Veneziano integrand for scalar particles ^{5),6)}.

The crucial factorization $U^{(i)} V^{(j)}$ of the Möbius matrix was lost in the algebra previously ^{3),4)}. Before proving, we draw some consequences.

COROLLARY 1 (Möbius invariance). Change all $z_i \rightarrow x_i$ by the same Möbius transformation. Then

$$1 = \begin{bmatrix} x_{i-1} & x_i & x_{i+1} \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} \begin{bmatrix} z_{j-1} & z_j & z_{j+1} \\ x_{j-1} & x_j & x_{j+1} \end{bmatrix}. \quad (17)$$

Inserting (17) between $U^{(i)}$ and $V^{(j)}$ in (16), and multiplying the brackets, turns all z_i into x_i without changing W_N . Thus our N -Reggeon vertex has the same Möbius invariance properties as the scalar Veneziano formula ^{5),6)}. In particular the three fixed z 's can be given arbitrary values. This does not transform the Reggeons.

COROLLARY 2 (Cyclic symmetry). For z 's on the unit circle, $V^{(i)} = [U^{(i)}]^*$ (Hermitian conjugate) by Lemma 2b. This implies that cyclic changes of the product ordering in (16) do not change W_N .

COROLLARY 3 (How to twist). The $0, \infty, 1$ in (14), (15) point towards the Reggeons. Therefore transforming $|a^{(i)}\rangle$ by the operators of Lemma 3 leads to very neat results :

- (a) To twist the i^{th} Reggeon, just interchange the side variables z_{i-1} and z_{i+1} in both $U^{(i)}$ and $V^{(i)}$. Twisting all the Reggeons is thus equivalent to anticyclic permutation.
- (b) To apply the gauge transformation $(1-y)^W$ to the i^{th} Reggeon, change 0 to y in $U^{(i)}$, and ω to y^{-1} in $V^{(i)}$.
- (c) To attach a propagator $D(x)$ to the i^{th} Reggeon, change 1 to x in both $U^{(i)}$ and $V^{(i)}$.

COROLLARY 4 (Loops). After using Corollary 3 to attach a propagator or twisted propagator to $\langle a^{(i)} |$, we can identify $|a^{(j)}\rangle = |a^{(i)}\rangle$ for some j , and then sew them by the coherent state trace formula ⁸⁾ [see Eq. (32) below]. The functional integral will be Gaussian with kernel $1-K$, where K is the Möbius matrix $D(x)U^{(i)}V^{(j)}$. It can thus be done by the technique of Alessandrini et al. ¹²⁾, to give a Veneziano formula with one loop and $N-2$ external Reggeons. Similarly we get Gaussian functional integrals for several closed loops.

COROLLARY 5 (Operatorial duality). Introduce a new family of harmonic oscillators $\underline{b}, \underline{b}^\dagger$ with components $b_{n\mu}$, corresponding to the intermediate states between $U^{(i)}$ and $V^{(j)}$ in (16). Let $|0_b\rangle$ be their vacuum. Then we can use these new oscillators to factorize (16) in a standard way ⁹⁾

$$W_N(\underline{a}) = \int d\bar{z}^{N-3} \cdot \varphi_N^0(\bar{z}) \cdot \langle 0_b | \prod_{j=1}^N x$$

$$x : \exp \left\{ -i \langle \underline{a}^{(j)} | U^{(j)} | \underline{b} \rangle + i (\underline{b}^\dagger | V^{(j)} | \underline{a}^{(j)} \rangle \right\} : | 0_b \rangle$$

(18)

Here normal ordering refers to $\underline{b}^\dagger, \underline{b}$, and the different brackets in the exponent mean that $|a^{(i)}\rangle$ is a coherent state while $|\underline{b}\rangle$ remains an operator. Now take z 's on the unit circle, and introduce the infinite component fields

$$\underline{Q}^{(j)} = i \underline{b}^\dagger V^{(j)} - i [\underline{b}^\dagger V^{(j)}]^*.$$

(19)

Then by Corollary 2, (18) becomes

$$W_N(a) = \int dz^{N-3} \cdot \varphi_N^0(z) \times \\ \times \langle 0_b | \prod_{j=1}^N : \exp(Q_{\mu}^{(j)} | a^{(j)}) \rangle : | 0_b \rangle. \quad (20)$$

We see from (15) that $Q_{\mu}^{(j)}$ is a function of z_{j-1} , z_j and z_{j+1} . However, by Lemmas 1, 2b and Eq. (3), its $n = 0$ components are

$$Q_{0\mu}^{(j)} = \sum_{m=0}^{\infty} i [b_{m\mu}^+(z_j)^m - b_{m\mu}(z_j)^{-m}] / \sqrt{m}, \quad (21)$$

and depend on z_j only. For scalar external particles, all $a_{n\mu}^{(i)}$ with $n \neq 0$ vanish, and by (1) and (21), (20) reduces to the operatorial duality formula of Fubini, Veneziano and Gervais ^{7), 11)}.

COROLLARY 6 (Point change). Changing one $z_i \rightarrow y$ is equivalent, apart from its effect on $\varphi_N^0(z)$, to the following transformation of the Reggeons

$$|a^{(i-1)}\rangle \rightarrow (y, z_{i-1}, z_i, z_{i-2})^{L_0} |a^{(i-1)}\rangle, \quad (22)$$

$$|a^{(i)}\rangle \rightarrow (y, z_{i-1}, z_i, z_{i+1})^{-W^+} |a^{(i)}\rangle, \quad (23)$$

$$|a^{(i+1)}\rangle \rightarrow (y, z_{i+2}, z_i, z_{i+1})^{-W} |a^{(i+1)}\rangle. \quad (24)$$

The proof uses Lemmas 1 and 3. This generalizes a known ^{13), 14)} invariance of the three-Reggeon vertex, since W_3 is independent of z_1 .

PROOF OF THEOREM

This N -Reggeon vertex was originally derived from Collopy's expression ⁴⁾, by writing out all the Chan variables, transforming to Koba-Nielsen variables, and doing many pages of algebra. However, there is a nicer direct proof. First we check (16) for three Reggeons. Then we show that the N -Reggeon vertex splits under factorization into two smaller copies of itself.

For $N = 3$, we have

$$U^{(1)} V^{(2)} = \begin{bmatrix} z_3 & z_1 & z_2 \\ 0 & \infty & 1 \end{bmatrix} \begin{bmatrix} \infty & 0 & 1 \\ z_1 & z_2 & z_3 \end{bmatrix} \\ = \begin{bmatrix} 0 & \infty & 1 \\ 1 & \infty & 0 \end{bmatrix} = \Omega, \quad (25)$$

by (8). Similarly $U^{(2)} V^{(3)} = \Omega$, $U^{(1)} V^{(3)} = \Omega^\dagger$. Putting the $a^{(i)}$ in cyclic order, all Möbius matrices are equal to that in the twisting operator, and we get the three-Reggeon vertex of Caneschi et al. ²⁾. (25) also checks independence from z_1, z_2, z_3 .

Now consider factorization. For easy comparison with the scalar case, we use Shapiro's notation ¹⁵⁾. Thus we have a big vertex with lines $(1, 2, \dots, j, j+1, \dots, N)$, and want to factorize it into two little vertices $(1, 2, \dots, j, E)$ and $(F, j+1, \dots, N)$. The new lines E and F will be connected by the propagator. We take z_1, z_j, z_{j+1} to be fixed for the big vertex, z_1, z_j, z_E and z_F, z_{j+1}, z_N to be fixed for the little ones. The new variable z_N of the big vertex will be related to the integration variable x of the propagator.

The constants z_E, z_F must occur only in the little vertices. An important difference from the scalar case is that $U^{(i)}, V^{(i)}$ for $i = E \pm 1, F \pm 1$, also depend on them. So to get these correct for the big vertex, we must change

$$z_E \rightarrow z_N \quad \text{in} \quad U^{(1)}, \quad V^{(1)}, \quad (26)$$

$$z_E \rightarrow z_{j+1} \quad \text{in} \quad U^{(j)}, \quad V^{(j)}, \quad (27)$$

$$z_F \rightarrow z_j \quad \text{in} \quad U^{(j+1)}, \quad V^{(j+1)}, \quad (28)$$

$$z_F \rightarrow z_1 \quad \text{in} \quad U^{(N)}, \quad V^{(N)}. \quad (29)$$

As in Corollary 6, (26) and (28) can be done by gauge transformations (24) of the Reggeons, which are legal by (13), but (27) and (29) would require an attached propagator (22). Therefore we must choose

$$z_E = z_{j+1}, \quad z_F = z_1, \quad (30)$$

which defines the semi-peripheral (s.p.) configuration ^{6),15)}. The freedom to factorize in different configurations by varying z_E, z_F ¹⁵⁾ does not exist for Reggeons. Presumably other configurations need an explicit physical state projection operator.

We now join E and F by the twisted propagator (12), which projects onto physical states when integrated ^{12),14),16),4)}. All factors containing $a^{(E)}$ or $a^{(F)}$ are then

$$\prod_{k=1}^j \exp \langle a^{(k)} | U^{(k)} V^{(E)} | a^{(E)} \rangle \times \quad (31)$$

$$\times \exp \langle a^{(E)} | P(x) | a^{(F)} \rangle \cdot \prod_{l=j+1}^N \exp \langle a^{(F)} | U^{(F)} V^{(l)} | a^{(l)} \rangle$$

The factors omitted here are those obviously common to big and little vertices, and also the scalar measure. Now we must take the trace over the coherent states $a^{(E)}$ by a functional integral with measure ⁸⁾

$$\int \pi^{-1} \cdot \delta \operatorname{Re} a^{(E)} \cdot \delta \operatorname{Im} a^{(E)} \cdot \exp [-\langle a^{(E)} | a^{(E)} \rangle], \quad (32)$$

and similarly for $a^{(F)}$. The functional integrals can be done immediately, and (31) becomes

$$\prod_{k=1}^j \prod_{l=j+1}^N \exp \langle a^{(k)} | U^{(k)} V^{(E)} P(x) U^{(F)} V^{(l)} | a^{(l)} \rangle \quad (33)$$

These will be the correct factors of the big vertex (16), if we can prove

$$V^{(E)} P(x) U^{(F)} = 1. \quad (34)$$

By writing out the Möbius matrices (12), (14), (15) and using Lemma 2a, (34) becomes

$$\begin{bmatrix} z_j & z_E & z_1 \\ \infty & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \infty & 1 \\ z_N & z_F & z_{j+1} \end{bmatrix} = \begin{bmatrix} 0 & \infty & 1 \\ x & 1 & 0 \end{bmatrix}. \quad (35)$$

Direct calculation using Lemma 1 shows that the conditions for the Möbius transformations defined by the two sides of (35) to be the same, are (30) and

$$x = (z_N, z_{j+1}, z_1, z_j). \quad (36)$$

Equation (30) again require the s.p. configuration, while (36) relates the new variable z_N of the big vertex to the integration variable x of the propagator. (Just as in the scalar case ¹⁵⁾, we can use Möbius invariance to transform z_N to rest inside the right-hand little vertex.) For $0 \leq x \leq 1$, (36) gives $z_{j+1} \leq z_N \leq z_1$, which is the correct integration range, since these are the nearest fixed points on either side. The measures in the integration variables z_i and x will be the same as for the scalar case ^{6), 15)}, where they are already known to factorize correctly. Thus we have proved (16) by induction. Q.E.D.

We end with some remarks on Corollary 5. The infinite set of harmonic oscillators in Veneziano theory form a second quantized field theory ¹⁷⁾, but this only describes the internal quantum numbers ⁹⁾. Their Hilbert space becomes an infinite-dimensional "momentum space" for the physical fields. Dual field theory is thus third quantized. The infinite-dimensional vectors $\underline{a}^{(i)}$ representing the Reggeons, are just the external momenta in this analogy. In fact their $n=0$ components are the physical momenta by (1), so we have unified external and internal degrees of freedom. The Reggeons are the eigenstates of the free Hamiltonian. (This is why they correspond to zero width resonances.) The usual harmonic oscillator representation ⁹⁾ which diagonalizes them, is therefore the momentum representation. By contrast $\underline{b}$ of Corollary 5 has nothing to do with excited states. It is an infinite-dimensional position vector. The exponential in (18) is the Fourier transform from momentum to position representation ^{*}). The N -Reggeon vertex, which is diagonal in this position representation $|b\rangle$ by (18), is just

^{*}) The analogy is even better if we factorize (16) by a functional integral over a c -number $\underline{b}$, as in (31)-(33) reversed, instead of an harmonic oscillator vacuum expectation value.

the interaction of the third quantized field theory. The fact that it contains N momentum vectors $\underline{a}^{(i)}$ but only one position vector $\underline{b}$, means that this dual field theory is local ! It turns out ¹⁸⁾ that amplitudes with M loops can also be factorized in the position representation, but then $M+1$ infinite-dimensional position vectors $\underline{b}^{(r)}$ are required to form a basis.

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