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EQUAL TIME SINGULARITIES IN RENORMALIZED QUANTUM FIELD THEORY

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A B S T R A C T

The notion of equal time singularity replacing for the commutator of renormalized local field operators the notion of equal time limit is studied. It is shown that in the Wightman framework the equal time singularity is described by a finite number of one-dimensional tempered distributions.

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1. INTRODUCTION

It was recognized already as early as in 1934 by Dirac ¹⁾ and Heisenberg ²⁾ that local interaction modifies light cone behaviour and that in the presence of such interaction the canonical E.T. ³⁾ limit may not exist. This conclusion became more clear after the introduction of renormalization constants ⁴⁾ and renormalized field operators, satisfying the following formal E.T. canonical commutation relations:

$$[\psi(\vec{x}, t), \psi(\vec{x}', t)] = i Z_3^{-1} \delta(\vec{x} - \vec{x}') \quad (1.1)$$

It follows from the consideration of all non-trivial local interaction Lagrangians ⁵⁾ that at least in perturbation theory the wave renormalization constant Z_3^{-1} is infinite ⁶⁾. It appeared, therefore, that the renormalized field operators are not three-dimensional operator valued distributions, but four-dimensional operator-valued distributions, having the meaning of an operator only if smeared with the test function depending on space as well as time-variables ⁷⁾. The infinite values of wave renormalization constants tell us simply how far unjustified was our effort to define the canonical E.T. limit (1.1). The hope that one can restrict the discussion of physically important field operators to the ones having E.T. limit was revived when the E.T. commutator for the charge densities in the frame of $SU(3) \times SU(3)$ scheme was proposed ⁸⁾. The number of results, derived from this assumption, and their agreement with experiment ⁹⁾ suggests that, indeed, in hadron physics the $SU(3) \times SU(3)$ time component current algebra commutators should be preserved ¹⁰⁾. In order to extend, however, the field of application of such non-perturbative approach, the local E.T. commutators of space-time and space-space components have also been studied. It appeared that in various dynamical models for the E.T. commutators again the infinities are present: in so-called Schwinger terms ¹¹⁾ and in terms which are not of gradient type ¹²⁾. We can conclude, therefore, that even if we restrict the field theoretic approach to the renormalized local currents, at least in perturbation theory we also face the problem of existence of E.T. singularities.

The presence of E.T. singularities was recognized and discussed by several authors ¹³⁾, who considered for one reason or another the timelike E.T. limits for renormalized Green functions. It should be mentioned that the E.T. singularities have also been discussed in papers introducing finite field equations in quantum field theory ¹⁴⁾. In the general case of n point Green function one can define $n(n-1)/2$ different E.T. limits, and the discussion can be only very general. In this paper we shall consider a simple example of the commutator sandwiched between two states $|\alpha\rangle$ and $|\beta\rangle$

$$T^{(n)}(x; p_i) = \langle \alpha | [A^{(n)}(\frac{x}{2}), A(-\frac{x}{2})] | \beta \rangle \quad (1.2)$$

where ¹⁵⁾

$$A^{(n)}(x) = \left(\frac{d}{dx_0} \right)^n A(x) \quad n=0,1,2,\dots \quad (1.3)$$

and A is assumed to be a local real Lorentz scalar ¹⁶⁾. The states $|\alpha\rangle$ and $|\beta\rangle$ are characterized by the four-momenta $p_1, \dots, p_n$, satisfying the mass-shell conditions $p_i^2 = m_i^2$ and normalized in a co-variant way ¹⁷⁾. Further we assume that

$$T^{(n)}(x; p_i) \in S'(R_4) \quad (1.4)$$

in accordance with Wightman scheme ¹⁸⁾. We shall not discuss here the ways of calculating of the distribution (1.2). We would only like to mention that nowadays the standard way through unrenormalized formalism, regularization and renormalization procedure is by no means the only available method ¹⁹⁾.

The usual E.T. limit for the distribution (1.1) exists only if for some neighbourhood of the E.T. value $x_0 = 0$ we are able to express the four-dimensional distribution (1.4) as a family of three-dimensional distributions, depending continuously on x_0 ²⁰⁾.

In such a case the E.T. limit exists in $S'(R_3)$, and using locality together with the temperedness assumption one gets the following most general expression for the E.T. limit:

$$T^{(n)}(\vec{x}, 0; p_i) = \sum_{k=0}^N \sum_{l_1 \dots l_k} \frac{C_{l_1 \dots l_k}^{(n)}(p_i)}{k!} \partial_{l_1} \dots \partial_{l_k} \delta(\vec{x}) \quad (1.5)$$

where $i=1,2,3$ and the coefficients $C_{i_1, \dots, i_k}^{(n)}(p_i)$ are well defined ²¹⁾.

In general the time co-ordinate cannot be treated, however, as a parameter, the restriction of the distribution (1.4) to the hyperplane $x_0=0$ does not exist, and the representation (1.5) does not make any sense. In general, the functional

$$(T^{(n)}(x_0; p_i), \varphi) = \int d^3\vec{x} T^{(n)}(\vec{x}, x_0; p_i) \varphi(\vec{x}) \quad (1.6)$$

describes one-dimensional distribution depending on the test function $\varphi(\vec{x})$. We have ²²⁾

$$(T^{(n)}(x_0; p_i), \varphi) \in S'(R_1) \quad (1.7)$$

We shall discuss in this paper the following generalization of the expression (1.5):

$$T^{(n)}(\vec{x}, x_0; p_i) = \sum_{k=0}^N \sum_{l_1 \dots l_k} \frac{\Gamma_{l_1 \dots l_k}^{(n)}(x_0; p_i)}{k!}.$$

$$\partial_{l_1} \dots \partial_{l_k} \delta(\vec{x}) + R_N^{(n)}(\vec{x}, x_0; p_i)$$

(1.8a)

where

$$\Gamma_{\nu_1 \dots \nu_K}^{(n)}(x_0; p_i) \in S'(R_1) \quad (1.8b)$$

and

$$R_N^{(n)}(\vec{x}, 0; p_i) = 0 \quad (1.8c)$$

It will be shown in Section 2 that the assumption (1.8b) can be proved by using locality. We shall discuss also in Section 2 the explicit formula for the coefficient distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$ in co-ordinate as well as in momentum space. The validity of the relation (1.8c) is shown in Section 3. The main tool in our discussion is the representation of any tempered distribution as a derivative of finite order of some continuous function²³⁾. In the last Section we make some remarks about the classification of distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$.

Our main conclusion is the existence of the formula (1.8a) in a well-defined mathematical sense [see (1.8b), (1.8c)], where the relation (1.8c) is understood as valid in $S'(R_3)$ in the sense of Lojasiewicz^{20a)}. The distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$ can be treated as the characterization of interaction in Wightman framework.

Finally, one should note that the E.T. structure for local current operators can be studied experimentally by means of deep inelastic lepton-proton scattering processes²⁴⁾. One can suppose from recent data that the E.T. limits in nature are not so singular as the perturbation theory dictates²⁵⁾. We would like to stress, however, that it seems very improbable that any finite time derivative of local current operator is a three-dimensional operator-value distribution, because in such a case the corresponding spectral functions would have to decrease at infinity faster than inverse of any polynomial²⁶⁾. We conclude, therefore, that if we consider the E.T. limits for sufficiently high time derivatives, the E.T. singularities appear and our considerations should be applied.

2. GENERAL FORMULA FOR EQUAL TIME SINGULARITIES

In this Section we shall discuss the derivation of the formula (1.8a) and the explicit expression for the Fourier transform of the distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$. The validity of the relation (1.8b) will be shown further.

In derivation of the formula (1.5) the main assumption is locality, i.e., the support property of (1.2) in the co-ordinate space

$$\text{supp } T^{(n)}(\vec{x}, x_0; p_i) = \bar{V}^+ \cup \bar{V}^- \quad (2.1)$$

where $\bar{V}^+$ ($\bar{V}^-$) denotes closed future (past) light cone. We therefore see that distribution-valued functional (1.6) depends only on the values of the test functions for $|\vec{x}| \leq x_0$.

Let us introduce the following set of test functions $\Phi_{i_1, \dots, i_K}^{(n)}(\vec{x}; a)$, belonging to $D(R_3)$: ($i_r = 1, 2, 3$; $r = 0, 1, 2, \dots, K$; $K = 0, 1, \dots$)

$$\Phi_{i_1 \dots i_K}(\vec{x}; a) = (-1)^K x_{i_1} \dots x_{i_K} \Phi(\vec{x}; a) \quad (2.2a)$$

where

$$\Phi(\vec{x}; a) = 1 \quad \text{for } |\vec{x}| \leq a \quad (2.2b)$$

$$\Phi(\vec{x}; a) = \exp \left\{ \frac{a^2 - |\vec{x}|^2}{4a^2 - |\vec{x}|^2} \right\} \quad (2.2c)$$

for $a < |\vec{x}| < 2a$

$$\Phi(\vec{x}; a) = 0 \quad \text{for } |\vec{x}| \geq 2a \quad (2.2d)$$

The test function $\Phi(\vec{x}; a)$ is bounded for any $a > 0$ together with its derivatives and the bound contains inverse powers of a ²⁷⁾:

$$\left| \left(\frac{\partial}{\partial x_i} \right)^p \Phi(\vec{x}; a) \right| \leq \frac{A_p}{a^p} \quad (2.2e)$$

Using (2.2a) and (2.2e) one gets also that

$$\left| \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \Phi_{l_1 \dots l_k}(\vec{x}; a) \right| \leq B_p a^{k-p} \quad (2.2f)$$

where $p = p_1 + p_2 + p_3$, because for $|\vec{x}| \leq a$ we have $|\vec{x}_i| \leq a$.

Now we consider the following integral

$$\Gamma_{l_1 \dots l_k}^{(n)}(x_0; p_i) = (-1)^k \int d^3 \vec{x} x_{l_1} \dots x_{l_k} T^{(n)}(\vec{x}, x_0; p_i) \quad (2.3)$$

It is easy to show that if

$$T^{(n)}(x; p_i) \in D'(R_4) \quad (2.4)$$

one gets

$$\Gamma_{i_1 \dots i_k}^{(n)}(x_0; p_i) \in D'(R_1) \quad (2.5)$$

Indeed, taking any given $\chi(x_0) \in D(R_1)$ one can always choose such finite value of the parameter a so that the support of $\chi(x_0)$ is in the interval $-a < x_0 < a$. One can, therefore, always write

$$(\Gamma_{i_1 \dots i_k}(p_i), \chi) = (T^{(n)}(p_i), \chi \otimes \Phi_{i_1 \dots i_k}(a)) \quad (2.6a)$$

where

$$\chi(x_0) \otimes \Phi_{i_1 \dots i_k}(\vec{x}; a) \in D(R_4) \quad (2.6b)$$

We see that it is sufficient only to assume that the relation (2.4) is valid in order to obtain the conclusion that (2.3) is a distribution. Because we have assumed the relation (1.4), we shall seek further restrictions.

In order to show that $\Gamma_{i_1, \dots, i_n}^{(n)} \in S'(R_1)$ one has to prove that the distributions (2.3) are well defined as linear continuous functionals in the class of rapidly decreasing test functions $f(x_0)$, bounded at infinity in the following way

$$|x_0^p \left(\frac{d}{dx_0}\right)^q f(x_0)| \leq C_{pq} \quad (2.7)$$

where p and q can be chosen arbitrarily large. We shall use the Schwartz representation of the tempered distribution $T^{(n)}(\vec{x}, x_0; p_i)$ in terms of derivatives of continuous function, bounded at infinity by polynomials ²³:

$$T^{(n)}(\vec{x}, x_0; p_i) = \left(\frac{\partial}{\partial x_0} \right)^{r+n} \prod_{l=1}^3 \left(\frac{\partial}{\partial x_l} \right)^{s_l} \tau(\vec{x}, x_0; p_i) \quad (2.8)$$

where R , r and s_i are finite,

$$|\tau(\vec{x}, x_0; p_i)| \leq C(p_i) (1+x_0^2)^A \prod_{l=1}^3 (1+x_l^2)^{B_l} \quad (2.9)$$

and τ is continuous. The formula (2.8) provides the following definition of the distribution $T^{(n)}(\vec{x}, x_0; p_i)$:

$$(T^{(n)}, f \otimes \psi) \stackrel{\text{df}}{=} (-1)^{r+s+n} (\tau, \left(\frac{\partial}{\partial x_0} \right)^{r+n} f \otimes \prod_{l=1}^3 \left(\frac{\partial}{\partial x_l} \right)^{s_l} \psi) \quad (2.10)$$

where $s = s_1 + s_2 + s_3$. In order to prove (1.8b) we introduce the following sequence of distributions:

$$\Gamma_{l_1 \dots l_k; \tau}^{(n)}(x_0; p_i) = \Phi(x_0; \frac{\tau}{2}) \Gamma_{l_1 \dots l_k}^{(n)}(x_0; p_i) \quad (2.11)$$

where $\Phi(x_0; \tau/2) \in D(R_1)$ and is defined by the relations (2.2b)-(2.2d). We shall show that for every f the limit

$$\lim_{\tau \rightarrow \infty} (\Gamma_{l_1 \dots l_k; \tau}^{(n)}(p_i), f) = (\Gamma_{l_1 \dots l_k}^{(n)}(p_i), f) \quad (2.12)$$

exists ²⁸⁾. The existence of the limit (2.12) implies that

$$\lim_{\tau, \tau' \rightarrow \infty} |(\Gamma_{l_1 \dots l_k; \tau}^{(n)}(p_i) - \Gamma_{l_1 \dots l_k; \tau'}^{(n)}(p_i), f)| \rightarrow 0 \quad (2.13)$$

Choosing $\tau' > \tau$ and using the estimate

$$\begin{aligned} & |(\Gamma_{l_1 \dots l_k; \tau}^{(n)}(p_i) - \Gamma_{l_1 \dots l_k; \tau'}^{(n)}(p_i), f)| \leq \\ & \leq \int_{\tau}^{\tau'} d\xi \left| \frac{d}{d\xi} (\Gamma_{l_1 \dots l_k; \xi}^{(n)}(p_i), f) \right| \end{aligned} \quad (2.14)$$

it is sufficient to show that

$$\left| \frac{d}{d\xi} (\Gamma_{l_1 \dots l_k; \xi}^{(n)}(p_i), f) \right| \leq \frac{C_1[f; p_i]}{\xi^{1+\epsilon}} \quad \epsilon > 0 \quad (2.15)$$

We have, using (2.10)

$$\begin{aligned} & \frac{d}{d\xi} (\Gamma_{l_1 \dots l_k; \xi}^{(n)}(p_i), f) = \\ & = (-1)^{r+p+n} \int_{-\infty}^{+\infty} dx_0 \int d^3 \vec{x} \tau(\vec{x}, x_0; p_i) \\ & \cdot \frac{\partial}{\partial \xi} \left[\left(\frac{\partial}{\partial x_0} \right)^{r+n} \cdot \left\{ \Phi(x_0; \frac{\xi}{2}) f_0(x_0) \right\} \cdot \right. \\ & \cdot \left. \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \Phi_{l_1 \dots l_k}(\vec{x}; \xi) \right] \end{aligned}$$

(2.16)

Because of the support properties

$$\text{supp } \frac{\partial}{\partial \xi} \Phi(x_0; \frac{\xi}{2}) = (\frac{\xi}{2}, \xi) \quad (2.17)$$

and

$$\text{supp } \frac{\partial}{\partial \xi} \Phi_{l_1 \dots l_k}(\vec{x}; \xi) \cup \text{supp } \Phi(x_0; \frac{\xi}{2}) T^{(n)}(\vec{x}, x_0; p_i) = \emptyset \quad (2.18)$$

we have the following inequality

$$\begin{aligned} \left| \frac{d}{d\xi} (\Gamma_{l_1 \dots l_k; \xi}^{(n)}(p_i), f) \right| &\leq C'(p_i) \int_{\frac{\xi}{2}}^{\xi} dx_0 \{ (1+x_0^2)^A \cdot \\ &\cdot \left| \left(\frac{\partial}{\partial x_0} \right)^{r+n} \left[\frac{\partial}{\partial \xi} \Phi(x_0; \frac{\xi}{2}) \cdot f_0(x_0) \right] \right| \} \cdot \\ &\cdot (1+\xi^2)^{B + \frac{k-p+3}{2}} \end{aligned} \quad (2.19)$$

where the estimate

$$\begin{aligned} \left| \int d^3 \vec{x} \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \Phi_{l_1 \dots l_k}(\vec{x}; \xi) \cdot \tau(\vec{x}, x_0; p_i) \right| &\leq \\ &\leq 4\pi C(p_i) B_p \cdot (1+x_0^2)^A (1+\xi^2)^{B + \frac{k-p+3}{2}} \end{aligned} \quad (2.20)$$

based on (2.2f) and (2.9) is used. Because

$$\left| \frac{\partial}{\partial \xi} \left(\frac{\partial}{\partial x_0} \right)^j \Phi(x_0; \frac{\xi}{2}) \right| \leq \frac{A_j}{\xi^{j+1}} \quad (2.21)$$

choosing in (2.7) the value of p equal to $2A+2B+k-p+4+\varepsilon$ one gets the estimate (2.15) with arbitrarily large integer ε ²⁹⁾.

In order to show that the distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$, occurring in (1.8) are given by the formula (2.3) let us expand an arbitrary test function $\varphi(\vec{x})$ around the point $\vec{x}=0$. We have

$$\begin{aligned} \varphi(\vec{x}) = & \sum_{k=0}^N \sum_{l_1 \dots l_k} \frac{x_{l_1} \dots x_{l_k}}{k!} \frac{\partial}{\partial x_{l_1}} \dots \frac{\partial}{\partial x_{l_k}} \varphi(\vec{x}) \Big|_{\vec{x}=0} + \\ & + \varphi_N(\vec{x}) \end{aligned} \quad (2.22)$$

Substituting (2.22) in the functional (1.6) one gets

$$\begin{aligned} (T^{(n)}(x_0; p_i), \varphi) = & \sum_{k=0}^N \sum_{l_1 \dots l_k} \frac{\Gamma_{l_1 \dots l_k}^{(n)}(x_0; p_i)}{k!} \\ & \cdot (-1)^k \partial_{l_1} \dots \partial_{l_k} \varphi(\vec{x}) \Big|_{\vec{x}=0} + (T^{(n)}(x_0; p_i), \varphi_N) \end{aligned} \quad (2.23)$$

where $\Gamma_{i_1, \dots, i_K}^{(n)}$ are given by the formula (2.3). In order to identify the expansion (2.21) with the formula (1.8a) we define the distribution $R_N^{(n)}$, occurring in (1.8a), as follows:

$$(R_N^{(n)}(x_0; p_i), \varphi) \stackrel{df}{=} (T^{(n)}(x_0; p_i), \varphi_N) \quad (2.24)$$

where Ψ_N denotes a remainder, which, in any bounded region of R_3 behaves as the test function $\Psi_N \in S'(R_3)$, satisfying the conditions

$$\prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{\beta_i} \Psi_N(\vec{x}) \Big|_{\vec{x}=0} = 0$$

(2.25)

where $\beta = \beta_1 + \beta_2 + \beta_3 < N$.

The formula (1.8a) can be considered also in momentum space. We have the following expression

$$\begin{aligned} \tilde{T}^{(n)}(\vec{q}, q_0; p_i) = & \sum_{k=0}^N \sum_{l_1 \dots l_k} i^k \frac{\tilde{\Gamma}_{l_1 \dots l_k}^{(n)}(q_0; p_i)}{k!} \\ & \cdot q_{l_1} \dots q_{l_k} + \tilde{R}_N^{(n)}(\vec{q}, q_0; p_i) \end{aligned}$$

(2.26)

where

$$\tilde{\Gamma}_{l_1 \dots l_k}^{(n)}(q_0; p_i) = i^k \frac{\partial}{\partial q_{l_1}} \dots \frac{\partial}{\partial q_{l_k}} \tilde{T}^{(n)}(\vec{q}, q_0; p_i) \Big|_{\vec{q}=0}$$

(2.27)

The formula (2.27) should be used, however, with care.

This is caused simply by the fact that the Fourier transform $\tilde{T}^{(n)}(q; p_i)$ is a shorthand notation for the difference of two distributions, well defined as the boundary values of analytic functions. Let us consider, for example, the case of VEV ($|\alpha\rangle = |\beta\rangle = |0\rangle$). The Källen-Lehmann representation

$$\tilde{T}^{(n)}(q) = \epsilon(q_0) q_0^n \vartheta(q_0^2 - \vec{q}^2) \quad (2.28)$$

leads to the following expansion (2.26)

$$\tilde{T}^{(n)}(\vec{q}, q_0) = \sum_{k=0}^N \frac{(-\vec{q}^2)^k}{k!} \tilde{\Gamma}_{2k}^{(n)}(q_0^2) + R_N^{(n)}(\vec{q}^2; q_0) \quad (2.29)$$

where

$$\tilde{\Gamma}_{2k}^{(n)}(q_0^2) = \epsilon(q_0) q_0^n \left(\frac{d}{dq_0^2} \right)^k \vartheta(q_0^2) \quad (2.30)$$

The difficulty mentioned above can be exhibited in a clear way on the standard example of massless free field commutator.

Using the co-ordinate space expansion (1.8a), one gets

$$\begin{aligned} \frac{1}{4\pi} \delta(x_0^2 - \vec{x}^2) &= \Gamma_0^{(0)}(x_0) \delta(\vec{x}) + \\ &+ \frac{1}{2} \Gamma_2^{(0)}(x_0) \Delta \delta(\vec{x}) + \dots \end{aligned} \quad (2.31)$$

where

$$\begin{aligned} \Gamma_0^{(0)}(x_0) &= \frac{1}{4\pi} \int d^3\vec{x} \delta(x_0^2 - \vec{x}^2) = \int_0^\infty r^2 \delta(x_0^2 - r^2) dr = x_0 \\ \Gamma_2^{(0)}(x_0) &= \frac{1}{4\pi} \int d^3\vec{x} \delta(x_0^2 - \vec{x}^2) \vec{x}^2 = x_0^3 \end{aligned} \quad (2.32)$$

etc. On the other hand if we work with the Fourier transform $\epsilon(q_0) \delta(q_0^2 - \vec{q}^2)$, of (2.31), one obtains the expansion

$$\epsilon(q_0) \delta(q_0^2 - \vec{q}^2) = \tilde{\Gamma}_0^{(0)}(q_0) + \tilde{\Gamma}_2^{(0)}(q_0)(-\vec{q}^2) + \dots$$

(2.33)

where, according to the formula (2.30)

$$\tilde{\Gamma}_0^{(0)}(q_0) = \epsilon(q_0) \delta(q_0^2)$$

$$\tilde{\Gamma}_2^{(0)}(q_0) = \epsilon(q_0) \delta'(q_0^2)$$

(2.34)

etc. We see that the distributions (2.34) are not well defined.

Their meaning can be, however, recovered, if we use the formula, giving a unique mathematical meaning to the four-dimensional distribution (2.33)

$$2\pi i \epsilon(q_0) \delta(q_0^2 - \vec{q}^2) = \frac{1}{q_0^2 - \vec{q}^2 + i q_0 0} - \frac{1}{q_0^2 - \vec{q}^2 - i q_0 0}$$

(2.35)

and observe that

$$\left(\frac{\partial}{\partial \vec{q}^2} \right)^k \frac{1}{(q_0^2 - \vec{q}^2 \pm i q_0 0)} \Big|_{\vec{q}=0} = \frac{k!}{(q_0 \pm i 0)^{2k+2}}$$

(2.36)

It can be easily seen that, using the well-defined formulae (2.35)-(2.36) one reproduces the results (2.32)³⁰⁾. We have

$$\epsilon(q_0) \delta(q_0^2) = \delta'(q_0)$$

$$\epsilon(q_0) \delta'(q_0^2) = 3! \delta'''(q_0^2)$$

(2.37)

The relations (2.37) can be treated as definitions of the "distribution-like" objects (2.34).

3. DESCRIPTION OF EQUAL TIME SINGULARITIES BY MEANS OF A FINITE NUMBER OF TERMS

In this Section we shall study the distribution $R_N^{(n)}$, defined by the formula (2.24). Particularly we shall show that [introducing the representation (2.8) and the definition of the distribution at the point given by Lojasiewicz²⁰⁾] there exists for any tempered distribution finite N for which the formula (1.8c) is valid.

Firstly we shall discuss the definition of the distribution $T(x_0)$ at the point $x_0=0$. Introducing the following sequence of test functions which approach Dirac delta with support in $(\epsilon/2, -\epsilon/2)$

$$\chi_\epsilon(x_0) = \frac{1}{\epsilon} \chi\left(\frac{x_0}{\epsilon}\right) \quad \epsilon > 0 \quad (3.1)$$

where

$$a) \quad \chi \in D(R_1)$$

$$b) \quad \chi \geq 0$$

$$c) \quad \int dx_0 \chi(x_0) = 1$$

$$d) \quad \text{supp } \chi \subset \left(\frac{1}{2}, -\frac{1}{2}\right)$$

we define the value $T(0)$ of distribution $T(x_0)$ at $x_0=0$ as follows ²⁰⁾:

$$T(0) = \lim_{\epsilon \rightarrow 0} (T, \chi_\epsilon) \quad (3.2)$$

Let us consider now the space of test functions, representing remainders in the Taylor expansion (2.22). From the relations (2.25) we see that the leading term around the point $\vec{x}=0$ contains at least N products of the variables x_1, x_2 and x_3 . For small values of $\vec{x}$, satisfying the relation $|\vec{x}| \leq x_0$, one can make the following estimate:

$$\left| \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{\beta_i} \varphi_N(\vec{x}) \right| \leq H_N |x_0|^{N-\beta} \quad (3.3)$$

Let us write now the expression (3.2) for the distribution $R_N^{(n)}(\vec{x}, x_0; p_i)$, smeared with arbitrary function $\varphi \in S(R_3)$. From (2.24) and (2.8) it follows that

$$(R_N^{(n)}(p_i), \chi_\epsilon \otimes \varphi) = (-1)^{r+p+n} \int_{-\frac{\epsilon}{2}}^{+\frac{\epsilon}{2}} dx_0 \int d^3\vec{x}. \quad (3.4)$$

$$\cdot \tau(\vec{x}, x_0; p_i) \left(\frac{\partial}{\partial x_0} \right)^{r+n} \chi_\epsilon(x_0) \cdot \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \varphi_N(\vec{x})$$

Using the support properties of the distribution (2.8) one can replace $\varphi_N(\vec{x})$ by the following test function, belonging to $D(R_3)$:

$$\Phi_N(\vec{x}; \epsilon) = \varphi_N(\vec{x}) \Phi(\vec{x}; \epsilon) \quad (3.5)$$

where $\Phi(\vec{x}; \varepsilon)$ is defined by (2.2b)-(2.2d). The estimate for upper bound on the derivatives of Φ_N is given on the basis of the relations (3.3) and (2.2e):

$$\left| \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \Phi_N(\vec{x}; \varepsilon) \right| \leq D_N \varepsilon^{N-p} \quad (3.6)$$

One can write for $\varepsilon < 1$

$$\begin{aligned} |(R_N^{(n)}(p_i), \varphi \otimes \chi_\varepsilon)| &= |(T^{(n)}(p_i), \Phi_N(\varepsilon) \otimes \chi_\varepsilon)| \leq K(p_i) \cdot \int d^3 \vec{x} \left| \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} \right)^{p_i} \Phi_N(\vec{x}; \varepsilon) \right| \cdot \\ &\cdot \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} dx_0 \left| \left(\frac{d}{dx_0} \right)^{r+n} \chi_\varepsilon(x_0) \right| \end{aligned} \quad (3.7)$$

where K denotes the upper bound of the value of continuous function $\tau(\vec{x}, x_0)$ in the compact region $|\vec{x}|^2 + |x_0|^2 \leq 1$. Using (3.1) one gets

$$dx_0 \left| \left(\frac{d}{dx_0} \right)^r \chi_\varepsilon(x_0) \right| = \frac{d\xi}{\varepsilon^r} \left| \left(\frac{d}{d\xi} \right)^r \chi(\xi) \right|_{\xi = \frac{x_0}{\varepsilon}} \quad (3.8)$$

From (3.8) and (3.6) one gets

$$\begin{aligned} |(R_N^{(n)}(p_i), \varphi \otimes \chi_\varepsilon)| &\leq D'_N(p_i) \varepsilon^{N-p-r-n+3} \\ &\cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi \left| \left(\frac{d}{d\xi} \right)^{r+n} \chi(\xi) \right| \end{aligned} \quad (3.9)$$

where $D'_N = 4\pi K D_N$ is a finite number for any choice of N and the test function φ . We therefore see that it is sufficient to assume that

$$N = p + r - 2 \quad (3.10)$$

in order to get the result

$$\begin{aligned} \lim_{x_0 \rightarrow 0} (R_N^{(n)}(x_0; p_i), \varphi) &\stackrel{df}{=} \\ &\stackrel{df}{=} \lim_{\epsilon \rightarrow 0} (R_N^{(n)}(p_i), \varphi \otimes \chi_\epsilon) = 0 \end{aligned} \quad (3.11)$$

It follows from (3.10) that the number N is determined by the order of tempered distribution. Similarly like in the proof of the relation (1.8b), the basic property which has been used, is the form (2.1) of the support in the co-ordinate space.

Let us discuss finally the meaning of the limit (3.11). It is easy to see that in the case when $(R_N^{(n)}(x_0), \varphi)$ is a function, continuous at $x_0 = 0$ the Lojasiewicz definition is equivalent to the ordinary limit. One can state, therefore, that if $(R_N^{(n)}(x_0; p_i), \varphi)$ is a function, the definition (3.10) means that this function vanishes at $x_0 = 0$. It may happen, however, for example in the case of rapid oscillations around the point $x_0 = 0$, that the ordinary limit does not exist. In such a case the definition (3.10), anyway, may give a vanishing result. A good example is provided by the function $\frac{1}{|x_0|^\alpha} \sin \frac{1}{|x_0|^\beta}$, which does not have a definite limit if x_0 goes to zero, Ref. 31), but if we use the formula (3.2), one gets for arbitrary α and $\beta > 0$ the value zero.

4. DISCUSSION

Having established the existence of the expansion (1.8a) in the Wightman formulation of quantum field theory, we ask now the next question: how can one characterize the distributions (1.8b). We shall distinguish the three following types of one-dimensional distributions $\Gamma_{i_1, \dots, i_K}^{(n)}(x_0; p_i)$:

- a) Distributions described by a bounded function for all $x_0 \neq 0$.

Such distributions occur in the theories having only E.T. singularities. The Fourier transforms of these distributions are not integrable at infinity, and do not exhibit oscillations which shift the position of the singularity from the origin ³²⁾. All choices of spectral functions, occurring in perturbation theory, for example in the case of VEV having asymptotic form

$$\varrho(q^2) \xrightarrow{q^2 \rightarrow \infty} A(q^2)^\alpha \ln^m q^2 \quad (4.1)$$

lead to the sequence of distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$, containing only E.T. singularities.

- b) Even in the case of VEV of the commutator function, with additionally imposed positivity condition on the spectral function $\varrho(q^2)$, one cannot exclude the existence of the singularities on any hyperboloid $x^2 = a^2$ ³³⁾. In general, the set of distributions (1.8b) can exhibit a singularity in the parameter x_0 for any value $x_0 = a$. We shall now distinguish the class of distributions, described by bounded function for all x_0 except (finite or infinite) number of discrete points $x_0 = a_1, x_0 = a_2 \dots$ ³⁴⁾. Such theories will contain unequal time singularities and the Fourier transforms (2.27) contain the non-integrable oscillating terms, leading to the shift of the singularities to unequal times $x_0 \neq 0$ ³⁵⁾. There were recently introduced "a priori" models of high energy behaviour with unequal time singularities by Brandt and Sucher ³⁶⁾. Sometimes, occurrence of unequal time singularities is called a "Brandt-Sucher disease".

c) The third form of distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$, occurring in QFT, is caused by the presence of infra-red divergences. The local non-integrability of the Fourier transforms (2.27) implies correspondingly the behaviour of the objects (1.8b) for $x_0 \rightarrow \infty$, which is not bounded by $A, |x_0|^{\frac{3}{2} + \epsilon}$. Such assumption meets, however, serious difficulties in renormalized theory, because the large distance behaviour is fixed by the asymptotic condition, and restricted severely by the positiveness assumption of the underlying Hilbert space. If we assume, as it is usually done, that the massless particles are not present in the theory, the third type of distributions never appear in the expansion (1.8a).

These three classes of distributions (1.8b) occur in present models of QFT. It is possible, however, to invent the examples which are less familiar: for example, one can assume that the function $\tau(\vec{x}, x_0)$ in some part of support (2.1) is continuous but does not have a derivative at all. In such a case, one would have a continuous distribution of singularities. We do not know of any model in QFT with such a structure of singularities, but one can mention that such models open new possibilities of introducing parameters not having the interpretation of elementary interval, like in the case of unequal time singularities. Another possibility is the presence of infinite asymptotic oscillations (of type $\sin a q_0^\alpha$, with $\alpha > 1$), of the Fourier transform (2.27). For such oscillations the statement that the asymptotic behaviour of the Fourier transform determines the local structure of the function is not true.

Let us consider now some particular class of distributions (1.8b) containing only E.T. singularities. If the Fourier transform (2.20) for some sequence of indices $i_1, \dots, i_K$ behaves as follows:

$$\tilde{\Gamma}_{i_1, \dots, i_K}^{(n)}(q_0; p_i) \sim_{|q_0| \rightarrow \infty} q_0^k \quad k=0, 1, 2, \dots$$

(4.2)

one gets for the distribution (1.8b)

$$\Gamma_{i_1 \dots i_K}^{(n)}(x_0; p_i) \underset{x_0 \approx 0}{\sim} \delta^{(K)}(x_0) \quad (4.3)$$

We see that the particular asymptotic behaviour (4.2) implies the existence of discrete E.T. singularities. If these singularities are present in the theory, it is not possible to detect the value of the E.T. singularity by considering the behaviour of $\Gamma_{i_1, \dots, i_K}^{(n)}$ around $x_0 = 0$. On the other hand the asymptotic behaviour (4.2) is the only one which cannot be detected in such a way. All other types of distributions (2.3) have extended support and may be represented for all values $x_0 \neq 0$ by singular functions³⁷⁾. Finally, we would like to mention that often the discussion of E.T. behaviour is performed under the assumption that only the symmetric part of the distribution (1.8b) is taken into account. Formally we have

$$\begin{aligned} T^{(n)}(\vec{x}, x_0; p_i) &\rightarrow T_{\text{symm}}^{(n)}(\vec{x}, x_0; p_i) = \\ &= \frac{1}{2} \{ T^{(n)}(\vec{x}, x_0; p_i) + T^{(n)}(\vec{x}, -x_0; p_i) \} \end{aligned} \quad (4.4)$$

Such truncated definition simplifies for the example the discussion of the problem of E.T. limit of any field operator with itself, and gives always a vanishing result:

$$[A(\vec{x}, x_0), A(\vec{x}', x_0)]_{\text{symm}} = 0 \quad (4.5)$$

It is easy to see, however, that the limit (4.5), in the case of presence of sufficiently strong non-canonical singularities³⁸⁾ [for example, proportional to $\delta'(x^2)$ in VEV of the commutator] is not well defined. It is easy to see also that the formula (4.4)

cannot be differentiated (in the sense of distribution theory), and it is not possible to use the relation

$$\Gamma_{l_1 \dots l_k}^{(n+m)}(x_0; p_i) = \left(\frac{d}{dx_0} \right)^m \Gamma_{l_1 \dots l_k}^{(n)}(x_0; p_i) \quad (4.6)$$

The last argument (and the strongest one for the physicist) against the definition (4.4) is provided by examples where the anti-symmetric part of the commutator function with respect to the change $x_0 \rightarrow -x_0$ has an important meaning³⁹⁾.

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REFERENCES AND FOOTNOTES

- 1) P.A.M. Dirac, Proc.Phi.Soc. (Cambridge) 30, 150 (1934).
- 2) W. Heisenberg, Z.Physik 90, 209; 92, 692 (1934).
- 3) E.T. = equal time.
- 4) See for example:
F.J. Dyson, Phys.Rev. 75, 486; 75, 1736 (1949).
- 5) We consider in this paper only four-dimensional theory.
- 6) The statement that the wave renormalization constant is divergent was also supported by arguments which do not use explicitly the perturbation expansion. See for example:
G. Källén, Kgl.Danske Videnskab.Selskab, Mat.-fys.Medd. 27, No. 12 (1953);
G. Källén, Acta Phys.Austr. Suppl. 2, 133 (1965);
R.T. Powers, Commun.Math.Phys. 4, 145 (1967);
K. Sinha and G.C. Emch, Bull.Am.Phys.Soc. 14, 86 (1969).
- 7) See for example:
A.S. Wightman and L. Garding, Arkiv Fysik 28, 129 (1964).
- 8) See for example:
M. Gell-Mann, Physics 1, 63 (1964).
- 9) See for example:
S.L. Adler and R.F. Dashen "Current algebra", W.A. Benjamin, New York (1968).
- 10) A more cautious statement would be that the E.T. local algebra for charge densities is satisfied if sandwiched between the same one-particle states (forward virtual Compton scattering). Such an assumption allows the presence of additional terms, proportional to the powers of Laplacean, in the local commutator of charge densities. In such a case it seems that the conclusion about non-Regge behaviour (fixed pole in j plane) for weak amplitudes [see for example, J.B. Bronzan, J.S. Gerstein, B.W. Lee and F.E. Low, Phys.Rev.Letters 18, 32 (1967)] need not be necessarily valid.

11) See for example:

P. Pocsik, Nuovo Cimento 43A, 541 (1966);

S. Okubo, Nuovo Cimento 44A, 1015 (1966).

Particularly in QED the divergence of Schwinger terms follows from the divergence of the wave renormalization constant Z_3^{-1} , because

$$Z_3^{-1} = \int_0^\infty \frac{d\kappa^2 \rho(\kappa^2)}{\kappa^2} = \infty \Rightarrow \int_0^\infty d\kappa^2 \kappa^2 \rho(\kappa^2) = \infty$$

(coefficient of the VEV
of Schwinger term)

on the basis of positivity of spectral function and ultra cause of the divergence of Z_3^{-1} .

12) See for example:

A.I. Vainstein and B. Ioffe, JETP Letter 6, 917 (1967);

S.L. Adler and Wu-Ki-Ting, Phys.Rev.Letters 22, 978 (1969);

R. Jackiw and G. Preparata, Phys.Rev.Letters 22, 975 (1969);

S.L. Adler and Wu-Ki-Ting, Institute of Advanced Study preprint (Oct 1969);

G. Altarelli and H.R. Rubinstein, New York University preprint (May 1969).

It is clear from these papers that calculating the E.T. commutator for space-space components it is not possible to include all infinities in the multiplicative renormalization constants.

13) See for example:

J. Valatin, Proc.Roy.Soc. A, 222, 93, 228; 225, 535; 226, 254 (1954);

K. Wilson, "On products of quantum field operators at short distances", Cornell University preprint (1964) unpublished;

J. Lukierski, Nuovo Cimento 49A, 312 (1967);

J. Langerholc, DESY preprint 67/26;

W.S. Hellmann and R. Roman, Nuovo Cimento 52A, 1341 (1967);

B. Schroer and P. Stichel, Commun.Math.Phys. 8, 327 (1968);

R.A. Brandt, Phys.Rev. 166, 1795 (1968);

V. Völkel and U. Völkel, Pittsburgh University preprint (1968);

- W.W. Sokolov, Jad.Fizika (Sov.Nucl.Phys.) 8, 559 (1968);
F. Kaschluchn, E. Wieczoreh and W. Zollner, Nuclear Phys. B8,
84 (1968);
N.J. Usukina, Jad.Fizika (Sov.Nucl.Phys.) 8, 1253 (1968);
E. Pfaffelhuber, University of Munich preprint (Oct 1968);
J. Lukierski, Nuovo Cimento Letters 1, 259 (1969);
F. Ottoson, J.Math.Phys. 10, 1525 (1969);
F. Krause, Bonn University thesis, (Nov. 1969).

The author is indebted to Dr. Kupsch for pointing out after the presentation of this paper at a CERN seminar the last reference. In order to avoid overlap with the methods presented in Ref. 13) (F. Krause), some parts of our original version, using momentum space methods, have been removed.

14) See for example:

- W. Zimmermann, Commun.Math.Phys. 6, 161 (1967); 10, 325 (1968);
R.A. Brandt, Ann.Phys. 44, 221 (1967); 52, 122 (1969).

In these papers the infinite renormalization constants and counterterms in local field equations are interpreted as a result of limiting procedure with respect to the regularizing parameter, represented by some space-like vector, introducing non-locality. We see, therefore, that one can get the E.T. singularity as a singular limit of the sequence of canonical theories. In this paper we shall deal directly with the renormalized solutions.

- 15) The space derivatives of $A(x)$ do not modify the singularities in the x_0 variable and shall not be discussed here.
- 16) The commutator function (1.2) of tensor fields can be easily reduced to the scalar case. The presence of additional tensor indices introduces only kinematic factors, composed out of the momenta p_i and the four-derivative operator. All the dynamical singularities are present only in the scalar structure functions, which are the distributions of type (1.2). For example VEV of the conserved current operator is described by one scalar distribution, and for non-conserved currents we have

$$\langle 0 | [j_\mu(\frac{x}{2}), j_\nu(-\frac{x}{2})] | 0 \rangle = (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) F_1(x) + \partial_\mu \partial_\nu F_2(x)$$

If we discuss the forward virtual Compton scattering we need in general five structure functions for vector current, and in the case of non-forward virtual Compton scattering tensor distributions.

In the case of half-integer spin the procedure is analogous, but somewhat more complicated because of the occurrence of γ_μ -matrices in the kinematic factors.

Finally, the field operator A in (1.1) is chosen to be Hermitian for the sake of simplicity.

- 17) For example, for one-particle states, we assume $\langle p | p' \rangle = (2\omega)^{\frac{1}{2}}(2\omega')^{\frac{1}{2}} \delta(\vec{p} - \vec{p}')$. We shall not discuss in this paper a possible distribution-like behaviour with respect to the external momenta, and all p_i are treated in further formulae as harmless parameters [see, however, footnote 21)].

- 18) A.S. Wightman, Phys.Rev. 101, 860 (1956).

Because the formula (1.2) requires the existence of asymptotic states, we are dealing here with Wightman field, satisfying additional technical assumptions allowing the definitions of asymptotic fields and free asymptotic states. See for example:

R. Jost, "The general theory of quantized fields", Am.Math.Soc., Providence, R.I., (1965); Sect. VI.

- 19) The methods leading to renormalized results without any infinities at intermediate steps combine analyticity properties (dispersion relations) and unitarity. See for example

G. Källén and A. Sabry, Kgl.Danske Videnskab, Selskab, Mat.-fys.Medd 29, No. 17 (1955)

where the fourth order vacuum polarization diagram is calculated by means of the dispersion technique.

- 20) A more "sophisticated" definition of E.T. limit, due to Łojasiewicz
- S. Łojasiewicz, *Studia Math.* 16, 1 (1957); 17, 1 (1958) is discussed and applied in Section 3.
- 21) The coefficients $c_{i_1, \dots, i_K}^{(n)}(p_i)$ are not, in general, functions, because of the contributions from disconnected parts, proportional to $\delta(p_i - p_j)$.
- 22) Because $S(R_3) \times S(R_1)$ is dense in $S(R_4)$, in order to know the distribution (1.2) it is sufficient to determine the one-dimensional distributions (1.7) for all choices of the test functions $\varphi \in S(R_3)$.
- 23) L. Schwartz, *Medd.Lunds.Mat.Sem.Suppl.* 192 (1952), theorem VI.
- 24) See for example:
J. Bjorken, "Current algebra at small distances", Varenna lecture, July 1967.
- 25) See for example:
F.J. Gilman, "High energy electroproduction", invited talk at the 1969 Internal symposium on electron and photon interactions at high energies, Liverpool, England (Sep. 1969).
- 26) Particularly, the expansions of high energy behaviour in terms of E.T. commutators of time derivatives of the field operator cannot be used to an arbitrary order.
- 27) We have for example for $2a \geq |\vec{x}| \geq a$

$$\frac{\partial}{\partial x_i} \Phi(\vec{x}; a) = - \frac{2x_i}{4a^2 - |\vec{x}|^2} \left(1 - \frac{a^2 - |\vec{x}|^2}{4a^2 - |\vec{x}|^2} \right) \Phi(\vec{x}; a)$$

If $a^2 \leq |\vec{x}|^2 \leq 2a^2$ we have

$$\left| \frac{\partial}{\partial x_i} \Phi(\vec{x}; a) \right| \leq \frac{3}{2a} |\Phi(\vec{x}; a)| \leq \frac{3}{2} a$$

and for $2a^2 < |\vec{x}|^2 \leq 4a^2$, because the maximum of the function $y^n e^{-y}$ is at $y = n$, we get

$$\left| \frac{\partial}{\partial x_i} \Phi(\vec{x}; a) \right| \leq \frac{2}{a} (e^{-1} + 2e^{-2})$$

The discussion of bounds for higher derivatives is analogous.

- 28) From the existence of the limit (2.12) for all $f \in S(R_1)$ it follows that

$$a) \quad \Gamma_{l_1 \dots l_K; \tau}^{(n)} \xrightarrow[\tau \rightarrow \infty]{\text{strong}} \Gamma_{l_1 \dots l_K}^{(n)}$$

$$b) \quad \Gamma_{l_1 \dots l_K}^{(n)} \subset S'(R_1)$$

See for example Ref. 18) (R. Jost), p. 8.

- 29) One can also prove the relation (1.8b) directly, by constructing a test function equal in the domain (2.1) to $(-1)^k x_{i_1}, \dots, x_{i_K} \times f$, where $f \in S(R_1)$, and belonging to $S(R_4)$. The behaviour of such test function for large values of x_0 can be given by the following product

$$\Phi_{l_1 \dots l_K}(\vec{x}; x_0) f(x_0) \tag{A}$$

It is easy to show that for large values of x_0 the function

$$\frac{\Phi_{l_1 \dots l_K}(\vec{x}; x_0)}{(1+x_0^2)^K}$$

and all its derivatives are bounded. Because the estimate (2.7) is also valid for the function $(1+x_0^2)f(x_0^2)$, we see that

$$|x_0^p \left(\frac{\partial}{\partial x_0} \right)^2 \{ \Phi_{l_1 \dots l_K}(\vec{x}; x_0) f(x_0) \}| \leq C'_{p2} \tag{B}$$

i.e., the test function (A) is a rapidly decreasing function if $x_0 \rightarrow \infty$.

30) I.M. Gelfand and G.E. Shilov, Generalized functions, vol. 1, Moscow 1958 (in Russian).

31) One can get any value k for the limit

$$\lim_{x_{0n} \rightarrow 0} \frac{1}{|x_{0n}|^\alpha} \sin \frac{1}{|x_{0n}|^\beta}$$

by suitable choice of the sequence $x_{01}, x_{02}, \dots$

32) The leading oscillations of the function $\tilde{g}(q_0)$ at infinity are defined by means of the limit $q_0 \rightarrow \infty$ for the function

$$\tilde{\omega}(q_0) = \frac{1}{i} \frac{d}{dq_0} \ln \tilde{g}(q_0)$$

(A)

If $\tilde{g}(q_0)$ is not integrable for large q_0 , and

$$\lim_{q_0 \rightarrow \infty} \tilde{\omega}(q_0) = 0$$

(B)

the leading singularity of the Fourier transform $g(x_0)$ is at the point $x_0 = 0$. In the case when $\tilde{g}(x_0)$ is a distribution, one should replace

$$\tilde{g}(q_0) \rightarrow \int dq'_0 \Phi(q_0 - q'_0; 1) \tilde{g}(q'_0)$$

in the formula (A).

The condition that all singularities of the Fourier transform $g(x_0)$ are at the point $x_0 = 0$ means that all asymptotically non-integrable terms satisfy the relation (B). Having only the notion of a non-integrable function $\tilde{g}(q_0)$ it is not easy to write down necessary and sufficient conditions for the absence of singularities for its Fourier transform at $x_0 = 0$.

33) The assumption of positive definite spectral function in Källén-Lehmann representation implies only that the strongest singularity should be on the light cone. If we introduce, for example, on the light cone the singularity proportional to $\delta'(x^2)$ the singularities proportional to $\delta(x^2 - a^2)$ are allowed. The problem of unequal time singularities will be studied by the author in a separate publication.

34) The parameters a_i depend generally on the four vectors p_i .

35) The position of leading oscillating terms are defined by means of the relation [see footnote 32), formula (A)]

$$\lim_{q_0 \rightarrow \infty} \tilde{\omega}(q_0) = a$$

The parameter a describes the position $x_0 = a$ of unequal time singularity.

36) See:

R.A. Brandt and J. Sucher, Phys.Rev. Letters 20, 1131 (1968);

R.A. Brandt and J. Sucher, Phys.Rev. 177, 2218 (1968).

37) For example the distributions $\Gamma_{i_1, \dots, i_K}^{(n)}$ corresponding to the spectral function (4.1) have for the choice of $\alpha \neq -\frac{1}{2}, 0, \frac{1}{2}, \dots$ the following behaviour around $x_0 = 0$:

$$n \text{ odd: } |x_0|^{-\beta} \ln^r |x_0| \quad (A)$$

$$n \text{ even: } \epsilon(x_0) |x_0|^{-\beta} \ln^r |x_0| \quad (B)$$

where β is a real number and r is an integer. The distributions (A) and (B) can be treated as a linear combination of the distributions

$$|x_0|_+^{-\beta} \ln^r |x_0|_+ \quad |x_0|_-^{-\beta} \ln^r |x_0|_-$$

[for details, see Ref. 30)].

- 38) By non-canonical singularities we call the ones leading to the infinite value of the wave renormalization constant Z_3^{-1} [see (1.1)]. On the other hand, as non-canonical terms we treat the ones which modify the canonical value $Z_3^{-1} = 1$.
- 39) See for example the local E.T. commutators for energy-momentum tensor density
P.A.M. Dirac, Rev.Modern Phys. 34, 592 (1962);
J. Schwinger, Phys.Rev. 130, 406 (1963).