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SOME EXACT RESULTS ON $\pi\pi$ SCATTERING

A. Martin
CERN - Geneva

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1. - INTRODUCTION

This session is in principle devoted to asymptotics. Here what I shall say concerns low energy ! As far as I can see, the only link is that we shall use extensively the fact that for $|t| < 4m_\pi^2$ dispersion relations need at most two subtractions.

The $\pi\pi$ system was the first to which the considerations of unitarity and analyticity were applied by Chew and Mandelstam ¹⁾ because it was recognized that this system has great simplicity features :

- (i) complete crossing symmetry ;
- (ii) stability of the particles and absence of unphysical cuts and anomalous thresholds, due to the small mass of the pion.

For rigorous studies, exactly the same is true ; the $\pi\pi \rightarrow \pi\pi$ scattering amplitude is, undoubtedly, one on which most results are obtained, for the same reasons. First we would like to describe the basic axiomatic results on the $\pi\pi$ amplitude. Some of these results are in fact more general.

2. - BASIC AXIOMATIC RESULTS

We know that the $\pi\pi \rightarrow \pi\pi$ can be described by three functions ¹⁾ :

$$\begin{aligned} & A(s, t, u) \\ & B(s, t, u) \\ & C(s, t, u) , \quad s + t + u = 4m_\pi^2 \end{aligned} \quad (1)$$

B and C are obtained from A by circular permutation ; A is symmetric in t, u, B in s, u, C in s, t, and the isospin combinations in the channel where s is the square of the energy are

$$\begin{aligned} \frac{1}{3}(I=0) + \frac{2}{3}(I=2) &= A+B+C \\ (I=1) &= B-C \\ (I=2) &= B+C \end{aligned} \quad (2)$$

The most remarkable combination is $A+B+C$, the $\pi_0\pi_0 \rightarrow \pi_0\pi_0$ amplitude completely symmetric in s, t, u .

The pure isospin states possess the unitarity property ; they can be expanded in partial waves for $s > 4$ and the expansion is convergent (Lehmann ellipse ²⁾) :

$$F^I(s, t, u) = \sum (2\ell+1) f_\ell^I(s) P_\ell(\cos\theta_s) \quad (3)$$

with $\cos\theta_s = 1 + \frac{2t}{s-4}$

and $\text{Im } f_\ell^I(s) \geq \frac{2k}{\sqrt{s}} |f_\ell^I(s)|^2, \quad s > 4$

and $\text{Im } f_\ell^I(s) = \frac{2k}{\sqrt{s}} |f_\ell^I(s)|^2 \quad 16 > s > 4$

$$\left(k = \sqrt{\frac{s-4}{4}} \right) \quad (4)$$

One of the most important aspects of these unitarity conditions is that they imply the positivity properties which means that the absorptive part of F^I, F_s^I is a function of positive type. This property can be proved directly if one does not like to go through the partial wave expansion. This was done by Glaser ³⁾. Important consequences of this positivity are the following : the absorptive part and all its

derivatives with respect to $\cos\theta$ are positive for $\cos\theta = +1$. Further the modulus of any derivative for $-1 < \cos\theta < +1$ is less than its value for $\cos\theta = 1$.

Of course these positivity properties are valid for a sum of isospin amplitudes with positive coefficients for instance the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ amplitude. They have a rather important consequence which is the following.

We know, from local field theory in any version, including the theory of local observables and the Jaffé fields ⁴⁾ that the amplitude satisfies dispersion relations for $-28 m_\pi^2 < t < 0$, i.e., t "physical" since $t = -2k^2 (1 - \cos\theta)$. We also know that the amplitude is analytic for fixed energy in the Lehmann ellipse ²⁾ with foci ± 1 and extremities $\cos\theta_0 = \left[1 + \frac{64}{k^2 s}\right]^{\frac{1}{2}}$.

I proved in 1966 that the combination of these "classical" results of field theory with the positivity properties of the absorptive part imply much stronger analyticity properties ⁵⁾, namely the validity of dispersion relations for

$$|t| < 4 m_\pi^2$$

Dispersion relations hold also for fixed u , $|u| < 4$ and fixed s , $|s| < 4$. Such an analyticity domain in two variables is not a natural domain of holomorphy and can be considerably extended. I will not describe here the whole results but just quote the only information which will be used later on, namely that the partial wave amplitudes are analytic in a domain like this at least :

The importance of the fixed, positive t dispersion relations lies in the fact that they imply the Froissart bound ⁶⁾ for the forward amplitude : the fixed t dispersion relations with a finite number of subtractions N , equal, if it is even, to the number of subtractions for $t = 0$, imply that in the mean $F_s(s, t, u)$ is bounded by s^N . Then we have

$$s^N > \sum (2\ell+1) \operatorname{Im} f_\ell(s) P_\ell\left(1 + \frac{2t}{s-4}\right)$$

where the right-hand side is a sum of positive terms, so that each of them is separately bounded by s^N . On the other hand we have

$$|F(s, t=0)| < \sum (2\ell+1) |f_\ell(s)|$$

If you combine this with

$$\operatorname{Im} f_\ell(s) \geq |f_\ell(s)|^2$$

it is an elementary variational problem to show that

$$|F(s, t=0)| < C s (\log s)^2$$

where C depends on N .

At this point you notice that N , the number of subtractions for $t = 0$ (and also for $t < 0$), is less or equal to two. This remains so when one continues dispersion relations to $t = 4$ and allows to sharpen further the Froissart bound by a new application of unitarity. In the end one finds

$$\sigma_t(s) < \frac{\pi}{m_\pi^2} (\log s)^2 \quad (5)$$

This, however, is an asymptotic statement and one could argue that it is useless. In fact this is not true. A striking way of showing this has been invented by Yndurain and Common⁷⁾: they show that one can write

$$\frac{1}{(s-4)^2} \int_4^s (s'-4) \sigma_t(s') ds' < C_1(q_2) \left(\log \frac{s}{\mu} \right)^2 + C_2(q_2) \quad (6)$$

where C_1 and C_2 can be calculated from the D wave scattering length or from some bounds on the amplitude in the unphysical region obtained by Lukaszuk and myself a few years ago and on which we shall come back. This is based on the equation

$$a_2 = c \int \frac{F_s(s', t=4) ds'}{s'^3}$$

where c is a numerical constant.

3. - ABSOLUTE LIMITS TO THE $\pi\pi$ AMPLITUDE

We have seen that

$$\frac{1}{(s-4)^2} \int_4^s (s'-4) \sigma_t(s') ds'$$

can be bounded by a function of

$$\int_4^\infty \frac{F_s(s', t) ds'}{s'^3}$$

with $0 < t < 4$ (for simplicity we work first with the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ amplitude).

Now the question is whether it is not possible to find a strict limit on

$$\int_4^4 \frac{F_s(s'/t) ds'}{s'^3} .$$

I do not want to give a detailed proof of the existence of such a bound, which can be found in the Trieste lecture notes and in my paper with Lukaszuk ⁸⁾. The basic ingredients are crossing symmetry together with the non-linear character of the unitarity condition (4). As an illustration let me indicate that if one takes the normalization of Chew and Mandelstam, the parameter λ , related to the value of the scattering amplitude at $s = t = u = 4/3$ is restricted by

$$- 2.6 < \lambda < 17 \quad (7)$$

One also finds bounds on

$$\int \frac{F_s(s',t) ds'}{s'^3}$$

for $0 < t < 4$. Then it is possible to find a strict numerical bound for

$$\frac{1}{(s-4)^2} \int_4^s (s'-4) \sigma(s') ds'$$

Even if this bound is extremely bad it has the merit of existing and that is a very remarkable fact that micro-causality, crossing symmetry and the existence of a minimum mass allow this.

Perhaps this is even more striking in a scalar theory. You look at the $\sigma\sigma \rightarrow \sigma\sigma$ amplitude (of course my σ 's have to be stable unfortunately). This amplitude has a pole at $t = m_\sigma^2$ since the coupling $\sigma\sigma\sigma$ is not forbidden. (Naturally to make the theory nice you can think that it has also a σ^4 coupling to make the spectrum positive.) Müller ⁹⁾ has been able to calculate, following exactly the same ideas, a bound for the renormalized coupling constant of the trilinear term. I will not tell you what it was, because you would jump into the air, but again it exists.

To finish this let me say that these results exhibit a basic stability of the theory, because they hold irrespective of the presence of other couplings, other particles (if they are heavy enough !). It seems to me that the only crucial information which has been used is that the $\pi\pi$ (or the $\sigma\sigma$) system has no bound state. The reason why the coupling cannot be too strong is that otherwise you would have a bound state.

An important by-product of these bounds is the existence of lower bounds for the $\pi\pi$ scattering lengths, as obtained first by Vinh Mau and Bonnier ¹⁰⁾ and later by Common ¹¹⁾. For instance one gets

$$a_{\pi_0\pi_0 \rightarrow \pi_0\pi_0} \geq -3 m_\pi^{-1} \quad (8)$$

Similar results have been obtained by Common for definite isospin states. More recently Wit and Common ¹²⁾ have obtained an improved lower bound at the price of an extra assumption which seems to be consistent with experiment, namely that the D wave scattering length is very small. Then they get

$$a_{\pi_0\pi_0 \rightarrow \pi_0\pi_0} \geq -1.1 m_\pi^{-1}$$

4. - POSITIVITY AND LINEAR CONSTRAINTS ON THE PARTIAL WAVES

All of what I have said in Section 3 is certainly important from a fundamental point of view. However, it has little practical importance because of the very large numerical value of the limits obtained. What I want to discuss now might have more practical importance. It is the problem of imposing crossing symmetry on partial wave amplitudes. More exactly we shall discuss first crossing symmetry alone and next crossing symmetry + positivity. To begin the discussion will be limited to $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ and later we shall indicate how to extend this to other isospin combinations.

You all know that historically theoreticians have first looked at $\pi\pi$ partial wave amplitudes. In spite of the work of D. Atkinson, who works with the full amplitude, reported in Section 5, this is still largely true right now. People like partial waves because unitarity is easy to impose, but then crossing symmetry is hard to impose because this requirement involves the full amplitude. A method to impose conditions from crossing symmetry on partial wave amplitude was proposed by Balachandran and Nuyts ¹³⁾ and, in fact, contained all basic ideas. However, this became clear only after Roskies ¹⁴⁾, with the help of Nussinov, and later Morel, Cohen-Tannoudji and Basdevant ¹⁵⁾ understood how simple things were. Well, to make a long story short, take the "Euclidean region" $s > 0$, $t > 0$, $u > 0$, and suppose you have a completely symmetric function $F(s,t,u)$, the $\pi_0 \pi_0 \rightarrow \pi_0 \pi_0$ amplitude. Consider the integral

$$\iint d\sigma (s-u)^{2N+1} F(s,t,u) t^P \quad (9)$$

where $d\sigma$ is the surface element and the integral extends to the triangle $s > 0$, $t > 0$, $u > 0$. P and N are integers. This integral is zero. Now insert the partial wave series in the s channel : the integral becomes

$$0 = \int_0^4 ds \int_{-1}^{+1} (4-s) d\cos\theta \left(s - \frac{4-s}{2}(1+\cos\theta) \right)^{2N+1} \left(\frac{4-s}{2} \right)^P \times (1-\cos\theta)^P \sum (2l+1) f_l(s) P_l(\cos\theta).$$

By orthogonality, in the $\cos\theta$ integration, only a finite number of partial waves will survive and we shall be left with a condition

$$\int_0^4 ds \sum_0^{2N+P+1} C_l^{2N+1, P} f_l(s) = 0 \quad (10)$$

This condition is necessary. However, the complete system of conditions (10), applied to a function which has already $t \rightleftharpoons u$ crossing symmetry built in by taking only l even, is also sufficient, if the f_l 's are known to decrease exponentially for $0 < s \leq 4$ and well behaved near $s = 0$ and $s = 4$. This is because the polynomials $(s-u)^{2N+1} t^P$ form a complete set of functions antisymmetric in $s \rightleftharpoons u$.

If you look at the simplest conditions you see that they are such that except for the S wave ($l=0$) you will never be able to get a constraint on the l^{th} wave alone. Constraints involve the l^{th} wave and all the lower waves. Let us write the lowest constraints :

$$l=0 \rightarrow N=0, P=0$$

$$\int_0^4 ds (4-s)(3s-4) f_0(s) ds = 0 \quad (11)$$

$$l=0 + l=2 \rightarrow \text{we can take} \quad \begin{array}{ll} N=0 & P=1 \\ N=0 & P=2 \\ N=1 & P=0 \end{array} \quad (12)$$

we have therefore three equations linking f_0 and f_2 . Of course these equations apply to the unphysical interval $0 < s < 4$. They are therefore difficult to confront with "experimental" $\pi\pi$ phase shifts

but can be confronted with models. For instance the Brown-Gobel model ¹⁴⁾ did not pass the test. The Padé models of the Saclay group ¹⁵⁾ passed the test.

Now you may wonder whether there is anything else to do since the system is complete. Well this is an illusion, because in practice you have only the lowest partial waves at your disposal and you want to concentrate as much information as possible on these partial waves. If you want to do that you have to use another ingredient which is positivity. It is obvious that this plays a role because for instance there are combinations of the $\ell=0$, $\ell=2$ constraints which contain an integral over the D wave with a positive weight function. Now from our "first principles" quoted in Section 2, we know that for $0 < s < 4$ we have dispersion relations with at most two subtractions. Therefore the Froissart - Gribov representation is valid for $\ell \geq 2$.

$$f_{\ell}(s) = \frac{4}{\pi(4-s)} \int_4^{\infty} Q_{\ell}\left(\frac{2t}{4-s} - 1\right) F_t(s,t) dt \quad (13)$$

(where $u \rightleftharpoons t$ crossing is taken care of). F_t is the absorptive part in the t channel for $0 < s < 4$. In this interval the partial wave expansion

$$F_t = \sum (2\ell+1) \operatorname{Im} f_{\ell}(t) P_{\ell}(\cos \theta_t)$$

converges, and is a sum of positive terms from unitarity. Therefore, $f_{\ell}(s) > 0$ for $\ell > 0$, $0 < s < 4$ and hence that Roskies' equation involving $\ell=0$ and $\ell=2$ with the coefficient of $\ell=2$ positive can be turned into an inequality involving the $\ell=0$ wave only.

There are essentially two approaches to accumulate conditions (all of which are inequalities). One followed by Roskies ¹⁶⁾ and independently Piguet and Wanders ¹⁷⁾ consists in looking for conditions on

moments of the S wave for $0 < s < 4$. The other, followed by Auberson, Brander, Mahoux and myself consists in looking for inequalities involving values of $f_0(s)$ at various points in the interval $0 < s < 4$.

Let me show the principle of this for the $\pi^0 \eta^0$ case ¹⁸⁾. You write

$$F(s, t, u) = \sum_0^{L-2} (2\ell+1) f_\ell(s) P_\ell(\cos \theta_s) + R_L(s, \cos \theta_s) \quad (14)$$

where R_L is the rest of the series. R_L has a simple expression :

$$R_L = \frac{2L}{\pi} \int_{z_0}^{\infty} dz \frac{[z Q_{L-1}(z) P_L(\cos \theta) - \cos \theta P_{L-1}(\cos \theta) Q_L(z)] F_t(s, t')}{z^2 - (\cos \theta)^2} \quad (15)$$

where F_t is the absorptive part in the t channel. It must be understood that $z = (2t'/4-s)-1$. We already see that if we can show that the bracket in front of F_t has a constant sign, we get the sign of $R_L(s, \cos \theta_s)$. It is also allowed to write a partial wave expansion in the t channel

$$F = \sum_0^{L'-2} (2\ell+1) f_\ell(t) P_\ell(\cos \theta_t) + R_{L'}(t, \cos \theta_t)$$

These two expressions should agree. It is now clear that a sufficient condition to obtain an inequality involving only a finite number of partial waves is to find values of s and t such that

$$R_L(s, \cos \theta_s) R_{L'}(t, \cos \theta_t) < 0 \quad (16)$$

Then one of the two partial sums is less than F , the other larger than F . It is possible to study this question because F_t in (15) is positive definite.

In this way you get for instance

$$f_0(0) > f_0(3.155) \quad (17)$$

$$f_0(3.205) > f_0(0.2134) > f_0(2.9863) \quad (18)$$

or as another example

$$\begin{aligned} 1.494 f_2(0.537) - 1.623 f_2(2.363) \\ < f_0(0.537) - f_0(2.363) < \\ 1.510 f_2(0.537) - 1.622 f_2(2.363). \end{aligned} \quad (19)$$

One can get many such inequalities. An improvement possible is that it is sufficient to have $R_L(s, \cos\theta_s) - R_L(t, \cos\theta_t)$ of a definite sign. This leads to tighter inequalities. For instance (17) is replaced by

$$f_0(0) > f_0(3.189) \quad (20)$$

Before leaving the $\pi\pi_0 \rightarrow \pi\pi_0$ let me indicate the important inequality ¹⁹⁾

$$\frac{df_0}{ds} > 0 \quad \text{for } s > 1.7 \quad (21)$$

$$\frac{df_0}{ds} < 0 \quad \text{for } s < 1.12 \quad (22)$$

(this last result has been obtained recently by Auberson ²⁰⁾) and

$$\frac{d^2 f_0}{ds^2} > 0 \quad \text{in between (by Common ²¹⁾)} \quad (23)$$

which guarantees the uniqueness of the minimum.

All these considerations can be extended to the cases with isospin ²²⁾.

Now are these inequalities useful ? Honestly I do not know. It seems that they restrict very severely unitary amplitudes. This is what has been experienced by Wanders ^{23),24)} and his collaborators, by Vinh Mau, Bonnier and Gauron ²⁵⁾, and by Le Guillou, Morel and Navelet ²⁶⁾.

I shall give some details on the latter work with which I am more acquainted since Le Guillou and Navelet are working at CERN. They have a seven-parameter formula for the S wave. They feed in the experimental information about the P wave absorptive part (the **P** peak !), construct the left-hand cut by exact crossing relations and then require unitarity. Unitarity of the S waves is built in. Then they use the five Nuyts-Balachandran-Roskies constraints to reduce the number of parameters to two. Finally they use the positivity - crossing inequalities (including a new one by Auberson ²⁰⁾) on the derivative of the S wave at $s = 0$) and find that the allowed domain for the remaining parameters almost reduces to a point. Figure 1 shows the fit they are using for the P wave. Figure 2 shows what they get for the S waves. The low energy region looks very much like what one gets from current algebra, even though this was not put in. The question of δ_0^0 going through 90° is unclear, within the freedom allowed on the remaining two parameters. The results are nevertheless very encouraging. It will be interesting to confront them with those of the other groups who are using somewhat different parametrizations.

5. - CONSTRUCTION OF EXACTLY UNITARY AMPLITUDE

By this I mean that one wants to fulfil conditions (4) exactly. You all remember the work of Atkinson ²⁷⁾ presented by Wightman at the Vienna Conference. I shall only mention one development : a deficiency of the first example of crossing symmetric and unitary amplitude produced was that it was going to zero at high energies, which corresponds to $\sigma_{\text{tot}} \lesssim s^{-1}$. Now, after some exploratory work by myself ²⁸⁾ and by Kupsch ²⁹⁾, Atkinson has finally succeeded to prove the possibility of the iterative construction of an amplitude which has a total cross-section behaving like $(\log s)^{-2}$.

Unfortunately it will be very difficult to construct an example with constant total cross-section at high energies. This is because one encounters all the difficulties which have been described by Gribov ³⁰⁾ and Gribov and Pomeranchuk ³¹⁾ : first of all a fixed diffraction peak is forbidden, but then Regge poles alone have the famous $\ell = -1$ singularity and have to be accompanied by Regge cuts. These difficulties appear automatically in Atkinson's work since exact elastic unitarity is guaranteed !

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FIG.1

FIG 2