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CONSTRAINTS ON REGGE TRAJECTORIES FROM FACTORIZATION OF
THE N POINT DUAL AMPLITUDE

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A B S T R A C T

We investigate the factorization properties of the N point Veneziano amplitude with general Regge trajectories. The intercepts turn out to be polynomial functions of the quantum numbers of the constituent particles. From a comparison of the level density with the Hagedorn temperature it is argued that the polynomial is quadratic. A five-dimensional harmonic oscillator formalism is given for the factorization of the N point amplitude with the intercepts as quadratic functions of the constituent particles. Factorizable production amplitudes with satellite terms are investigated in an n dimensional harmonic oscillator formalism, and it is shown that the level density is incompatible with the Hagedorn temperature.

1. - INTRODUCTION

One of the most important problems in connection with the generalized Veneziano model ^{1),2)} concerns the factorization. It has been shown by Fubini, Veneziano ³⁾, Bardakci and Mandelstam ⁴⁾ that the residue at each energy eigenvalue factorizes into a highly degenerate level structure. It was assumed, however, that all trajectories had the same intercepts and slopes. It is obviously desirable to investigate factorization when different trajectories have different intercepts. In the present paper we wish to show that the multi-particle Veneziano model is factorizable if restrictions are imposed on the intercepts. These restrictions are very interesting in the sense that the Regge trajectory has a structure which is related to the particles which build up the trajectory in a multi-particle process. In an intuitive sense one can say that factorization requires the Regge trajectory to remember its origin.

After completion of most of the work reported in this paper the author received a preprint by Mandelstam ⁵⁾, in which it is shown that the generalized Veneziano model factorizes if the intercepts depend quadratically on the total quark number in a given channel. Our results are more general, and do not depend on whether one connects the Veneziano amplitude with the quark model, and we have also investigated the level density at large energies.

In this paper we restrict ourselves to a many-body Veneziano amplitude with spinless external particles. To each channel consisting of particles $i, i+1, i+2, \dots, k$ we associate a leading trajectory of the form

$$\alpha_{ik}(s_{ik}) = a s_{ik} + b_{ik}, \quad (1.1)$$

where $\sqrt{s_{ik}}$ is the total centre of mass energy in the channel ik . It can be shown that universality of the slope a is essential if we are to get finite factorization properties at all. In Section 2 we investigate the restrictions imposed on the intercepts b_{ik} by factorization. It turns out that b_{ik} has a form of the type

$$\begin{aligned}
 b_{ik} = & b + c^{(1)} (b_i^{(1)} + b_{i+1}^{(1)} + \dots + b_k^{(1)}) \\
 & + c^{(2)} (b_i^{(2)} + b_{i+1}^{(2)} + \dots + b_k^{(2)})^2 + \dots \\
 & \dots + c^{(M)} (b_i^{(M)} + b_{i+1}^{(M)} + \dots + b_k^{(M)})^M,
 \end{aligned} \tag{1.2}$$

where $b_{\ell}^{(i)}$ are quantities associated with particle ℓ . Obviously b_{ℓ} must depend on the quantum numbers of particle ℓ , but factorization does not give any hint as to the precise dependence. Equation (1.2) therefore leaves fertile ground for models. It should be mentioned that in addition to the terms in Eq. (1.2) one can have further terms, e.g., of the type

$$c^{(2)'} (b_i^{(2)'} + b_{i+1}^{(2)'} + \dots + b_k^{(2)'}) (b_i^{(2)} + b_{i+1}^{(2)} + \dots + b_k^{(2)}). \tag{1.3}$$

Obviously, Eq. (1.2) is a rather general result, leaving much freedom for b_{ik} . It would therefore be of interest to obtain restrictions on the maximal power M . It turns out that the degree of degeneracy is closely related to M . To see this, in Section 3, we generalize the harmonic oscillator formalism ⁶⁾ to the case when b_{ik} is given by (1.2). As a result we find that the density of degenerate states depend rather crucially on M . If $M = 1$, the degeneracy is the same as if all intercepts were universal. If $M \geq 2$ the degeneracy increases rapidly with M . By comparison with the Hagedorn temperature ⁷⁾ one can see that the best agreement is obtained for $M = 2$. We recall that Hagedorn's temperature is based on experimental distributions of high energy secondary particles in the transverse momentum.

In Section 4 we derive some sum rules which are consequences of $M = 2$.

Finally, in Section 5 we discuss the possibility of satellite terms. Using the results obtained by Gross ⁸⁾ we find that even in the simplest case the introduction of satellites implies a huge increase in the level density. For spin J the density behaves like e^J for large values of J , whereas the Hagedorn fit behaves like $e^{\sqrt{J}}$. Hence even the simplest factorizable construction of N point Veneziano amplitudes with satellites implies a very unreasonable hadron spectrum.

2. - CONSTRAINTS ON THE INTERCEPTS FROM FACTORIZATION

The generalization of the Veneziano model ¹⁾ to production amplitudes ²⁾ can easily be carried out for an arbitrary linear trajectory of the form (1.1). Considering the N point function we attempt to factorize the residues at the poles in the variable $s_{1j} = s$ by dividing the external particles $1, \dots, N$ in two groups $1, \dots, j$ and $j+1, \dots, N$. With independent variables chosen according to the multi-peripheral configuration we obtain the following generalization of the expression for the Veneziano integrand used in Refs. ³⁾ and ⁴⁾ (see the Figure)

$$I_N = I_{j+1} I_{N-j} w^{-\alpha_{1j}(s) - 1} \times \prod_{i=2}^j \prod_{k=j+1}^{N-1} \left(1 - w \prod_{s=i}^{j-1} u_{1s} \prod_{s=j+2}^k u_{Ns} \right)^{-2\alpha_i p_k + \delta(i, k)}, \quad (2.1)$$

$$\delta(i, k) = -b_{ik} - b_{i+1, k-1} + b_{i, k-1} + b_{i+1, k}, \quad (2.2)$$

where we have introduced the conventions

$$\prod_{s=j}^{j-1} u_{1s} = 1, \quad \prod_{s=j+2}^{j+1} u_{Ns} = 1, \quad b_{j,j} = b_j, \quad b_{j+1, j+1} = b_{j+1}, \quad b_{j+1, j} = 1. \quad (2.3)$$

In Eq. (2.1) I_{j+1} and I_{N-j} are the Veneziano integrands corresponding to the $(j+1)$ and $(N-j)$ point functions,

$$I_{j+1} = \prod_{p=2}^{j-1} u_{1p}^{-\alpha_{1p}-1} \prod_{p=2}^{j-1} \prod_{q=p}^{j-1} \left(1 - \prod_{s=p}^q u_{1s}\right)^{A(p,q)}, \quad (2.4)$$

with

$$A(p,q) = -\alpha_{pq}(1 - \delta_{pq} - \delta_{p+1,q}) - \alpha_{p-1,q+1} - \delta_{pq} + (1 - \delta_{pq})(\alpha_{p,q+1} + \alpha_{p-1,q}). \quad (2.5)$$

The reason for the conventions (2.3) is that the cases $i = j$, $k = j+1$, $i = j-1$, $k = j+1$ and $i = j$, $k = j+2$ require special attention, as one can see, e.g., from Eqs. (2.4) and (2.5).

Following the procedure in Refs. 3) and 4) we expand in powers of w ,

$$I_N = I_{j+1} I_{N-j} w^{-\alpha_{1j}(s)-1} \times \exp \left\{ \sum_{r=0}^{\infty} \frac{w^r}{r} \left[2\alpha \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+2}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) - \sum_{i=2}^j \prod_{s=i}^{j-1} u_{1s}^r \sum_{k=j+1}^{N-1} \prod_{t=j+2}^k u_{Nt}^r \delta(i,k) \right] \right\}. \quad (2.6)$$

If all intercepts are equal, $\delta(i,k) = 0$ apart from $\delta(j,j+1) = b-1$. We then have the situation discussed in detail in Refs. 3) and 4). If the b 's are not equal, Eq. (2.6) does not in general factorize, as one can easily see by expanding the exponential function in Eq. (2.6) in powers of w . Factorization is only possible if $\delta(i,k)$ itself can be written as a product of two factors, where one factor refers to one part of the diagram and the other to the other part of the diagram.

As the simplest case, let us consider $\delta(i,k) = 0$ (with the exception of a few values of i and k). From Eq. (2.2) we obtain

$$b_{ik} - b_{i,k-1} = b_{i+1,k} - b_{i+1,k-1}. \quad (2.7)$$

The solution of this equation is obviously

$$b_{ik} - b_{i,k-1} = f(k), \quad (2.8)$$

where $f(k)$ is an arbitrary function of k , i.e., f depends in an arbitrary way of the quantum numbers of the particle k . Putting $f(k) = b'_k$, Eq. (2.8) has the solution

$$b_{ik} = b + b'_i + b'_{i+1} + \dots + b'_k \quad (2.9)$$

where b is a universal constant. It is easily seen that the only cases where $\delta(i,k)$ does not vanish with the solution (2.9) are the following :

$$\delta(j, j+1) = -b'_j - b'_{j+1} - b + b_j + b_{j+1} - 1,$$

$$\delta(j-1, j+1) = b - b_j + b'_j,$$

$$\delta(j, j+2) = b - b_{j+1} + b'_{j+1}. \quad (2.10)$$

Each non-zero δ increases the degree of degeneracy. However, from a bootstrap point of view we are led to consider the external particles as lying on a Regge trajectory, and we then have

$$b_j = b + b'_j, \quad b_{j+1} = b + b'_{j+1}, \quad (2.11)$$

so that the last two δ 's in Eq. (2.10) vanish. Hence the only non-vanishing δ is

$$\delta(j, j+1) = b - 1, \quad (2.12)$$

and the degeneracy is now the same as in the case when all trajectories have universal intercepts. Equation (2.9) shows that the intercepts correspond to additive quantum numbers.

Equation (2.9) only represents the simplest solution for the intercepts. It is natural to guess that a polynomial function also leads to factorization. Let us write

$$b_{ik} = b + c^{(1)} B_{ik}^{(1)} + c^{(2)} [B_{ik}^{(2)}]^2 + \dots + c^{(M)} [B_{ik}^{(M)}]^M, \\ B_{ik}^{(P)} = b_i^{(P)} + b_{i+1}^{(P)} + \dots + b_k^{(P)}. \quad (2.13)$$

The quantities $b_\ell^{(p)}$ can of course depend in different ways on the quantum numbers of particle ℓ for different p 's. The c 's are universal constants. For the sake of a simple exposition, we begin by considering the case $M = 2$. Since we already know that most δ 's vanish for $M = 1$, we easily find

$$\delta(i, k) = -2c^{(2)} b_i^{(2)} b_k^{(2)}, \\ \delta(j, j+1) = b - 1 - 2c^{(2)} b_j^{(2)} b_{j+1}^{(2)}, \quad (2.14)$$

and hence Eq. (2.6) becomes

$$\begin{aligned}
 I_N = & I_{j+1} I_{N-j} w^{-\alpha_{1j}(s)-1} (1-w)^{b-1} \\
 & \times \exp \left\{ 2 \sum_{r=0}^{\infty} \frac{w^r}{r} \left[a \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) \right. \right. \\
 & \left. \left. + c^{(2)} \left(\sum_{i=2}^j b_i^{(2)} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b_k^{(2)} \prod_{t=j+2}^k u_{Nt}^r \right) \right] \right\}. \quad (2.15)
 \end{aligned}$$

By expanding in powers of w we see that the coefficient of w^n is a finite sum of the form

$$\begin{aligned}
 & \sum_{\ell_r} \prod_{r=1}^n \frac{1}{\ell_r! r^{\ell_r}} \left\{ a \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) \right. \\
 & \left. + c^{(2)} \left(\sum_{i=2}^j b_i^{(2)} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b_k^{(2)} \prod_{t=j+2}^k u_{Nt}^r \right) \right\}^{\ell_r}, \quad (2.16)
 \end{aligned}$$

where the integer ℓ 's satisfy the partition equation

$$\sum_{r=1}^{\infty} r \ell_r = n = \alpha_{1j}(s_n). \quad (2.17)$$

The number of terms is evidently finite and does not depend on j and N . However, because of the additional terms proportional to powers of $c^{(2)}$ the degree of degeneracy is higher than in the case $c^{(2)} = 0$. In the next Section the level of degeneracy will be discussed in the asymptotic limit $n \rightarrow \infty$.

Now it is a simple matter to proceed to the case when M is arbitrary. Here one obtains

$$\begin{aligned}
 I_N &= I_{j+1} I_{N-j} w^{-\alpha_{ij}(s)-1} (1-w)^{b-1} \\
 &\times \exp \left\{ \sum_{r=0}^{\infty} \frac{w^r}{r} \left[2a \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) \right. \right. \\
 &+ \sum_{n=2}^M c^{(n)} \sum_{S=2}^n \sum_{t=1}^{S-1} \sum_{u=0}^{n-S} (-1)^S \binom{n}{S} \binom{S}{t} \binom{n-S}{u} \\
 &\times \left(\sum_{i=2}^j (b_i^{(n)})^{S-t} (b_{ij}^{(n)})^{n-S-u} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} (b_k^{(n)})^t (b_{j+1,k}^{(n)})^u \right. \\
 &\left. \left. \times \prod_{t=j+2}^k u_{Nt}^r \right) \right] \Big\}
 \end{aligned} \tag{2.18}$$

with

$$b_{\alpha\beta}^{(n)} = b_{\alpha}^{(n)} + b_{\alpha+1}^{(n)} + \dots + b_{\beta}^{(n)}. \tag{2.19}$$

Again we can expand the exponential in powers of w , and we see that the coefficient of w^n does indeed factorize with a finite number of levels. However, the number of levels at a given n increases rapidly with M , as we shall see in the next Section. It should also be mentioned that the large number of additional terms (for $M > 2$) makes it likely that more ghosts enter. On the other hand, the terms proportional to powers of the c 's do not contribute to the leading trajectory. Hence, no matter how large M is, the states on the leading trajectory are still non-degenerate.

We wish to end this Section by discussing the more general case where $\delta(i,k)$ cannot be written in the separable form. Consider the case when b_{ik} is an analytic function of the form

$$b_{ik} = f(b_i + b_{i+1} + \dots + b_k). \quad (2.20)$$

This case can be obtained from the previous solution by performing the limit $M \rightarrow \infty$. It is easily seen by this argument that the amplitude is factorizable. However, the number of levels at a given energy is infinite. Hence, solutions of the type (2.19) are not acceptable physically. The same remark applies if f is a rational function (not a polynomial) of its variables.

In addition to the polynomial solution (2.13) one might think of a multiplicative solution of the type

$$b_{ik} = b + c b'_i b'_{i+1} \dots b'_k. \quad (2.21)$$

However, solutions of this type are already contained in the polynomial solution (2.13), since the product in (2.21) can always be written as a suitable combination of polynomials, à la

$$2 b'_1 b'_2 = (b'_1 + b'_2)^2 - (b'^2_1 + b'^2_2).$$

In the Appendix we have discussed the special case (2.21) in more detail.

3. - HARMONIC OSCILLATOR FORMALISM AND THE LEVEL DENSITY

We have seen that the factorization allows a very large class of Regge intercepts, and it would be interesting if one can say which values of the maximum power M in the solution (2.13) for b_{ik} are reasonable from a physical point of view. One feature which is characteristic

for the factorization is the rapid increase with M of the number of states at a given energy $\sqrt{s_n}$. To investigate this effect more qualitatively we shall extend the harmonic oscillator formalism recently proposed by Fubini, Gordon and Veneziano⁶⁾ to the case when b_{ik} is not universal. This treatment also makes much more transparent the real mechanism in the solutions obtained for b_{ik} .

In order to keep the number of indices at a reasonable level we shall first discuss the case $M = 2$. It will then be evident how to treat the general case. In the case $M = 2$ we found the factorizable expression (2.15)

$$\begin{aligned}
 I_N = & I_{j+1} I_{N-j} w^{-\alpha_{1j}(s)-1} (1-w)^{b-1} \\
 & \times \exp \left\{ 2 \sum_{r=0}^{\infty} \frac{w^r}{r} \left[a \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) \right. \right. \\
 & \left. \left. + c^{(2)} \left(\sum_{i=2}^j b_i^{(2)} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b_k^{(2)} \prod_{t=j+2}^k u_{Nt}^r \right) \right] \right\}. \quad (3.1)
 \end{aligned}$$

To write this expression in a simple form we introduce the set of five-dimensional vectors

$$\begin{aligned}
 P^{(r)} = & \left(\sqrt{2}a \sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r, \sqrt{2}c^{(2)} \sum_{i=2}^j b_i^{(2)} \prod_{s=i}^{j-1} u_{1s}^r \right) \\
 Q^{(r)} = & \left(\sqrt{2}a \sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r, \sqrt{2}c^{(2)} \sum_{k=j+1}^{N-1} b_k^{(2)} \prod_{t=j+2}^k u_{Nt}^r \right) \quad (3.2)
 \end{aligned}$$

and defining a metric $+1$ for the fifth component of the five-vector we can rewrite Eq. (3.2) as

$$I_N = I_{j+1} I_{N-j} w^{-\alpha_{ij}(s)-1} (1-w)^{b-1} \times \exp \left[\sum_{r=0}^{\infty} \frac{w^r}{r} P^{(r)} Q^{(r)} \right]. \quad (3.3)$$

For each component $P_{\mu}^{(r)}$ we introduce a harmonic oscillator operator $a_{\mu}^{(r)}$, so $a_{\mu}^{(r)}$ is five-dimensional. We then have the commutation relations

$$[a_{\mu}^{(r)}, a_{\nu}^{\dagger(r')}] = \delta_{rr'} g_{\mu\nu}, \quad (3.4)$$

where $g_{55} = +1$ and $g_{\mu\nu}$ is diagonal. It is then easy to show by the same arguments as in Ref. ⁶⁾ that the amplitude is given by

$$V_N = \sum_{\lambda} \langle 0 | \mathcal{G}(Q, p, a) | \lambda \rangle \langle \lambda | \mathcal{D}(R, s) | \lambda \rangle \langle \lambda | \mathcal{G}(p, p, a^{\dagger}) | 0 \rangle, \quad (3.5)$$

with

$$\mathcal{G}(p, p, a^{\dagger}) = \int \prod_{i=2}^{j-1} du_{1i} I_{j+1} \exp \left[\sum_r P^{(r)} a^{\dagger(r)} / \sqrt{r} \right], \quad (3.6)$$

$$\mathcal{D}(R, s) = \int_0^1 dw w^{-\alpha_{ij}(s)-1+R} (1-w)^{b-1}, \quad (3.7)$$

$$R = \sum_n n a^{\dagger(n)} a^{(n)}. \quad (3.8)$$

Here $|\lambda\rangle$ denotes a complete set of states of the type

$$|\lambda, \mu\rangle = |\lambda_1, \lambda_2, \dots, \lambda_i, \dots\rangle = \prod_{i=1}^{\infty} \frac{[a^{\dagger(i)}]^{\lambda_i}}{\sqrt{\lambda_i!}} |0\rangle. \quad (3.9)$$

Formally, Eqs. (3.5)-(3.9) are the same as in Ref. ⁶⁾. The difference is that the creation operators $a^{+(i)}$ in (3.9) are five-dimensional, and hence the number of states is larger than in Ref. ⁶⁾, where the a 's are four-dimensional. This difference shows up when we calculate the multiplicity $d(n)$ of the level $\alpha_{1j}(s_n) = n$ in the asymptotic limit of large values of n ,

$$d^{M=2}(n) \sim e^{a\sqrt{5n}} \sim e^{6.2\sqrt{n}}, \quad a = \frac{2\pi}{\sqrt{6}}. \quad (3.10)$$

The factor $\sqrt{5}$ in the exponent arises from the fact that $a_{\mu}^{(r)}$ are five-dimensional vectors. With $M = 1$ or $M = 0$ the a 's are four-dimensional, and instead of $\sqrt{5}$ we would obtain $\sqrt{4}$.

In the more general case when M is arbitrary Eqs. (3.3) to (3.9) are still valid, but the vectors P , Q , a and a^+ now have a higher dimension than five, and correspondingly the density $d^M(n)$ increases more rapidly with n . Considering the general case we can now easily formulate the general condition that the N point function is factorizable. The condition is that

$$\delta(i,k) = \sum_{s=5}^{s_{\max}} p_{i,s} p_{k,s}, \quad (3.11)$$

where $s_{\max}$ is the dimension of the vector $P^{(r)}$. The polynomial solution (2.13) guarantees that Eq. (3.11) is satisfied, as one can see from Eq. (2.18). The density then behaves like

$$d^{s_{\max}}(n) \sim e^{a\sqrt{s_{\max}n}}. \quad (3.12)$$

It is interesting to compare the level densities obtained for various values of M with the Hagedorn temperature ⁷⁾. By fitting the experimental data Hagedorn found

$$d^H(n) \sim e^{6\sqrt{n}}. \quad (3.13)$$

If we calculate the density for $M = 3$ (where the dimension is 9) we obtain

$$d^{M=3}(n) \sim e^{7.7\sqrt{n}} \quad (3.14)$$

which is not in good agreement with (3.13). For $M > 3$ the discrepancy increases even more. For $M = 0$ or $M = 1$ one obtains ⁶⁾

$$d^{M=0 \text{ or } M=1}(n) \sim e^{5.1\sqrt{n}} \quad (3.15)$$

and it therefore seems as if the experimental hadron distribution is best fitted by taking the Regge intercepts to be quadratic functions of the b 's.

The n dimensional vector formalism shows an intimate connection between internal degrees of freedom and the level structure. The more internal degrees of freedom one has, the more does the level density increase, which is intuitively acceptable.

4. - SOME PHENOMENOLOGICAL CONSEQUENCES OF THE STRUCTURE OF THE INTERCEPTS

We have found that factorization of the N point Veneziano amplitude is consistent with the Hagedorn temperature if

$$b_{ik} = b + c' (b_i' + b_{i+1}' + \dots + b_k') \\ + c'' (b_i'' + b_{i+1}'' + \dots + b_k'')^2. \quad (4.1)$$

Although Eq. (5.1) does not impose strong constraints on the Regge trajectories, one can obtain certain sum rules for the intercept of leading Regge trajectories. It is easily seen that for reactions involving four or more particles, we have the term structure

$$-\alpha_{ik}(0) - \alpha_{i+1,k-1}(0) + \alpha_{i,k-1}(0) + \alpha_{i+1,k}(0) \quad (4.2) \\ = -2c'' b_i'' b_k''. \quad (4.2)$$

Taking the particles i and k to be the same in different reactions, the right-hand side of Eq. (5.2) is independent of the particles $i+1, i+2, \dots, k-1$. By taking different quantum numbers for these particles one can obtain an infinite number of sum rules. As an example we mention

$$-\alpha_{4\pi}(0) - \alpha_{2\pi}(0) + 2\alpha_{3\pi}(0) \quad (4.3) \\ = -\alpha_{3\pi K}(0) - \alpha_{K\pi}(0) + 2\alpha_{2\pi K}(0).$$

Here "isospin complications" have been ignored. These sum rules involve intercept of trajectories where the first resonance is rather heavy, and therefore not yet found experimentally.

5. - SATELLITES AND THE LEVEL DENSITY

Although factorization of the production amplitude is possible with the polynomial solution for the intercepts, many problems remain, From the point of view of principles one of the most important is concerned with the uniqueness of the representation of the N point function. It has

been shown by Gross ⁸⁾ that one can construct an N point function with satellites that factorizes along the leading as well as the non-leading trajectories. Our method where we introduce n dimensional vectors is very convenient for finding the level density. In the simplest case discussed by Gross, the vector product $P^{(r)} Q^{(r)}$ in Eq. (3.3) is replaced by (see Ref. ⁸⁾ for details)

$$P^{(r)} Q^{(r)} + \left(\frac{d}{dw} \right)^r F_N(w, u) \Big|_{w=0}, \quad (5.1)$$

where $(d/dz)^r F_N|_{z=0}$ is a sum of $2n+2$ factorizable terms ⁸⁾, i.e.,

$$P^{(r)} Q^{(r)} + \left(\frac{d}{dw} \right)^r F_N \Big|_{w=0} = \bar{P}^{(r)} \bar{Q}^{(r)} \quad (5.2)$$

where the dimension of the vectors $\bar{P}$ and $\bar{Q}$ is $2n+7$ (if $M=2$) or $2n+6$ (if $M=0$ or 1). Equation (3.12) therefore gives

$$d^{\text{satellites}}(n) \sim e^{\frac{2\pi}{\sqrt{3}} n}, \quad (5.3)$$

which is violently incompatible with the Hagedorn fit (3.13). On the basis of these considerations one would therefore be inclined to exclude satellite terms.

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APPENDIX

In this Appendix we discuss the multiplicative type of intercept,

$$b_{ik} = b + c b'_{ik}, \quad b'_{ik} = b'_i b'_{i+1} \dots b'_k. \quad (A.1)$$

We obtain

$$\begin{aligned} \delta(i,k) = & b'_{i+1,j} (-b'_i b'_k + b'_i + b'_k - 1 + \delta_{k,i+1}) \\ & \times b'_{j+1,k-1} + (b-1) \delta_{k,i+2}, \quad k \geq j+1, \quad i \leq j, \end{aligned} \quad (A.2)$$

with the convention

$$b'_{j+1,j} = 1. \quad (A.3)$$

It follows that $\delta(i,k)$ consists of four terms, each of which can be "factorized" in two factors coming from the two halves of the diagram in the Figure. Proceeding as in Section 2, we see that the coefficient of the factorized form

$$\begin{aligned} & \prod_{l_r} \left\{ \left(\sum_{i=2}^j p_i \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} p_k \prod_{t=j+2}^k u_{Nt}^r \right) \right. \\ & + \left(\sum_{i=2}^j b'_{i,j} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b'_{j+1,k} \prod_{t=j+2}^k u_{Nt}^r \right) \\ & + \left(\sum_{i=2}^j b'_{i+1,j} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b'_{j+1,k-1} \prod_{t=j+2}^k u_{Nt}^r \right) \end{aligned}$$

$$\begin{aligned}
 & + \left(\sum_{i=2}^j b'_{i,j} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b'_{j+1,k-1} \prod_{t=j+2}^k u_{Nt}^r \right) \\
 & + \left(\sum_{i=2}^j b'_{i+1,j} \prod_{s=i}^{j-1} u_{1s}^r \right) \left(\sum_{k=j+1}^{N-1} b'_{j+1,k} \prod_{t=j+2}^k u_{Nt}^r \right) \}^{\ell_r} \quad (A.4)
 \end{aligned}$$

where the ℓ 's satisfy the usual partition equation. Introducing the harmonic oscillator formalism of Section 4 we have to associate eight-dimensional vector with the two halves of the diagram. As in Eq. (3.10) it then follows that

$$d(n) \sim e^{a\sqrt{8n}} \sim e^{7.3\sqrt{n}} \quad (A.5)$$

which is not in good agreement with the Hagedorn temperature [see Eq. (3.13)]. The degree of degeneracy exhibited in Eq. (A.5) is very close to the degree of degeneracy obtained with the cubic solution ($M=3$) for the intercepts [see Eq. (3.14)]. In fact, the degree of degeneracy corresponds to $M=3$ with $c^{(2)} = 0$.

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FIG. 1

The N point amplitude