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CONSTRAINTS ON REGGE TRAJECTORIES FROM FACTORIZATION
OF THE N POINT DUAL AMPLITUDE

II. CONSISTENCY CONDITIONS FOR TRAJECTORIES

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A B S T R A C T

We continue the discussion given previously of the factorization properties of the N point dual amplitude with general Regge trajectories. The most general solution compatible with the Hagedorn temperature **is** found. Consistency for the various couplings of trajectories to the external particles is found to impose selection rules on the total intercept quantum number. The number of external particles has to be even. If the level density is in complete agreement with the Hagedorn level density, the intercept quantum numbers for a given N point function are conserved. Such a model gives meson trajectories which resemble those found in nature. Other models exist where the intercept quantum numbers are partially conserved. In these models the Hagedorn level density is satisfied for most trajectories, but for very low lying trajectories there are deviations. The spectrum of Regge trajectories again resembles that found in nature (for mesons). A concrete example of a finite loop is given in a model with partially conserved trajectory quantum numbers.

1. INTRODUCTION

The factorization properties of the N point dual amplitude with general Regge trajectories have recently been investigated by Mandelstam¹⁾ and by us²⁾ (hereafter referred to as I). Mandelstam's work is done in the framework of a relativistic quark model based on the Veneziano amplitude, whereas we investigated the necessary conditions which many-body Regge trajectories have to satisfy in order that the dual N point amplitude factorizes.

In addition to the constraints arising from the factorization requirement it is also important to investigate various other consistency conditions necessary for the physical meaning of Regge trajectories. The main purpose of this paper is to examine these conditions.

The contents of this paper is the following:

In Section 2 we solve the factorization equation with the subsidiary condition that the level density is consistent with the Hagedorn temperature³⁾. The most general solution to this problem is

$$\alpha_{ik}(0) = b + \tilde{F}_{ik}^0 - c'' \sum_{\alpha=i}^k b''_{\alpha} (B_N'' - \sum_{\alpha=i}^k b''_{\alpha}). \quad (1.1)$$

Here $\alpha_{ik}(0)$ is the intercept of the trajectory in the channel consisting of the particles $i, i+1, \dots, k$, and the "trajectory quantum numbers" b''_{α} are defined by

$$\begin{aligned} & -\alpha_{ik}(0) - \alpha_{i+1,k-1}(0) + \alpha_{i+1,k}(0) + \alpha_{i,k-1}(0) \\ & = -2c'' b_i'' b_k''. \end{aligned} \quad (1.2)$$

B_N'' is the sum of all b''_{α} in a given N point function. The quantity $\tilde{F}_{ik}^0$ is a general solution to the homogeneous equation

obtained by putting $c'' = 0$ in Eq. (1.2). Equation (1.1) is of the same form as the expression for $\alpha_{ik}(0)$ used in I. Various consequences of Eq. (1.1) have been discussed in I and by Lovelace in his recent review ⁴⁾.

In Section 3 it is shown that Eq. (1.1) is only consistent with all requirements on $\alpha_{ik}(0)$ if the total intercept quantum number B_N'' vanishes. The trajectories in the resulting model have a quite rich spectrum. If one allows trajectories with positive intercepts, it is possible to have $\rho - \omega$ degeneracy as well as a single pion trajectory (with the correct values for the intercepts). The model contains an infinite number of different trajectories.

In Section 4 we investigate models where the total trajectory quantum number B_N'' is not in general conserved. From what has been derived in Sections 2 and 3 it follows that such models cannot have a spectrum in exact agreement with the Hagedorn temperature. However, the quark model of Mandelstam ¹⁾ has a spectrum which is very close to the Hagedorn spectrum unless one has a scattering process with a large number of baryons and exotic mesons. We also construct a "mixed model" which has similarities with the model with conserved trajectory quantum numbers (Section 3) and Mandelstam's quark model ¹⁾. In the mixed model B_N'' is in general not zero, but the level density is in most cases in close agreement with Hagedorn's level density. The models in Section 4 correspond to what one can best describe as "partially conserved" quantum numbers. The spectrum of Regge trajectories in these models again resembles that found in nature ^{*)}.

In Section 5 we discuss the situation with respect to loop diagrams, and it turns out that we can give one concrete example of a simple dual loop which is finite by the mechanism previously proposed by us ⁵⁾. The trajectories in this example are constructed

^{*)} with the exception of the quark model in its present form. However, this model works equally well with mesons as external particles (the dual diagrams are then simply a method for constructing trajectories), and in this case the spectrum is reasonable.

according to the mixed model. In the model with conserved trajectory quantum numbers (Section 3) the loops are infinite. The recently proposed renormalization ⁶⁾ of planar loops could then be applied. Since the model contains an infinite number of different parent trajectories, an infinite number of renormalizations is necessary.

It is worth pointing out that we have only been able to obtain a satisfactory factorization scheme (with unequal intercepts) if the number of external particles is even. This might be of interest for the inclusion of various symmetries in the N point dual amplitude.

Finally we mention that in all the factorizable models for Regge trajectories which we have constructed, as well as in Mandelstam's quark model ¹⁾, the infinite number of resonances are classified by an infinite number of quantum numbers (the usual ones as well as an infinite set of trajectory quantum numbers). Present day experimental results do not yield much support for this result, but as pointed out by Lovelace ⁴⁾ it is instructive in this connection to think of the developments in nuclear physics.

2. REGGE TRAJECTORIES FROM FACTORIZATION AND THE HAGEDORN TEMPERATURE

In this Section we shall show that if the N point dual amplitude factorizes in such a way that the level density is in agreement with the Hagedorn temperature, then the trajectories are given by Eq. (1.1).

It was shown in Ref. 1) and in I [see Eqs. (2.1) and (2.2) in I] that the fundamental quantity in the factorization program for the N point Veneziano amplitude is

$$\begin{aligned} \delta(i, k) = & -\alpha_{ik}(0) - \alpha_{i+1, k-1}(0) \\ & + \alpha_{i, k-1}(0) + \alpha_{i+1, k}(0). \end{aligned} \quad (2.1)$$

The indices i and k correspond to the division of the N point diagram shown in Fig. 1, i.e., i belongs to the set of indices $(1, 2, \dots, j)$ and k belongs to $(j+1, j+2, \dots, N)$. In general, the trajectory $\alpha_{ik}(0)$ depends on the quantum numbers (i.e., masses and internal quantum numbers) of the particles in channel ik . We introduce the convention that any index α ($1 \leq \alpha \leq N$) stands for the number of the particle α (with the ordering of external particles in Fig. 1) as well as for the quantum numbers associated with particle number α .

To have factorization properties which are in agreement with the Hagedorn temperature it was shown in I that we must require

$$\delta(i, k) = -2c'' b_i'' b_k'', \quad (2.2)$$

where c'' is a constant. The trajectory quantum numbers b_i'' should only depend on the quantum numbers of particle i . In this Section we shall find the most general solution to Eq. (2.2).

To solve Eq. (2.2) we introduce the quantity

$$B_{ik} = \alpha_{ik}(0) - \alpha_{i+1, k}(0), \quad (2.3)$$

and Eqs. (2.1) and (2.2) can then be written

$$B_{i, k-1} - B_{ik} = -2c'' b_i'' b_k''. \quad (2.4)$$

Equation (2.4) is a difference equation in two variables, with $i \in (1, 2, \dots, j)$ and $k \in (j+1, j+2, \dots, N)$. To solve Eq. (2.4) we put $k = j+2$ and obtain

$$B_{i, j+2} = B_{i, j+1} + 2c'' b_i'' b_{j+2}''. \quad (2.5)$$

Similarly, putting $k = j+3$ we obtain by use of (2.4) and (2.5)

$$\begin{aligned} B_{i,j+3} &= B_{i,j+2} + 2c'' b_i'' b_{j+3}'' \\ &= B_{i,j+1} + 2c'' b_i'' (b_{j+2}'' + b_{j+3}''). \end{aligned} \quad (2.6)$$

Proceeding in this manner we obtain

$$B_{ik} = B_{i,j+1} + 2c'' b_i'' \sum_{\alpha=j+2}^k b_{\alpha}''. \quad (2.7)$$

Here $B_{i,j+1}$ depends on the quantum numbers of particles $i, i+1, \dots, j$. The first term on the right-hand side of (2.7) is a solution to the homogeneous equation corresponding to Eq. (2.4) with $c'' = 0$.

From Eqs. (2.3) and (2.7) we obtain

$$\alpha_{ik}(0) = \alpha_{i+1,k}(0) + B_{i,j+1} + 2c'' b_i'' \sum_{\alpha=j+2}^k b_{\alpha}''. \quad (2.8)$$

We can solve this recurrence relation by putting $i+1 = j$, $i+1 = j-1$, etc., and in this manner we obtain

$$\begin{aligned} \alpha_{ik}(0) &= \alpha_{jk}(0) + B_{i,j+1} + B_{i+1,j+1} + \dots \\ &\dots + B_{j-1,j+1} + 2c'' \sum_{\beta=i}^{j-1} b_{\beta}'' \sum_{\alpha=j+2}^k b_{\alpha}''. \end{aligned} \quad (2.9)$$

Here $\alpha_{jk}(0)$ is a function of the quantum numbers of the particles $j, j+1, \dots, k$.

The solution (2.9) is not very symmetric. We therefore first extend the summation range by use of

$$2c'' \sum_{\beta=i}^{j-1} b_{\beta}'' \sum_{\alpha=j+2}^k b_{\alpha}'' = 2c'' \sum_{\alpha=i}^j b_{\alpha}'' \sum_{\beta=j+1}^k b_{\beta}'' - 2c'' b_j'' \sum_{\alpha=j+1}^k b_{\alpha}'' - 2c'' b_{j+1}'' \sum_{\beta=i}^j b_{\beta}'' + 2c'' b_j'' b_{j+1}'' \quad (2.10)$$

Then we use the identity

$$\begin{aligned} \sum_{\alpha=j+1}^k b_{\alpha}'' \sum_{\beta=i}^j b_{\beta}'' &= \left[\frac{1}{2} \sum_{\alpha=i}^k b_{\alpha}'' - \frac{1}{2} \sum_{\alpha=i}^j b_{\alpha}'' + \frac{1}{2} \sum_{\alpha=j+1}^k b_{\alpha}'' \right] \\ &\times \left[\sum_{\beta=i}^k b_{\beta}'' - \sum_{\beta=j+1}^k b_{\beta}'' \right] = \frac{1}{2} \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2 - \frac{1}{2} \sum_{\alpha=i}^j b_{\alpha}'' \sum_{\beta=i}^k b_{\beta}'' \\ &+ \frac{1}{2} \sum_{\alpha=i}^j b_{\alpha}'' \sum_{\beta=j+1}^k b_{\beta}'' - \frac{1}{2} \left(\sum_{\alpha=j+1}^k b_{\alpha}'' \right)^2 = \frac{1}{2} \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2 \\ &- \frac{1}{2} \left(\sum_{\alpha=i}^j b_{\alpha}'' \right)^2 - \frac{1}{2} \left(\sum_{\alpha=j+1}^k b_{\alpha}'' \right)^2, \end{aligned} \quad (2.11)$$

and Eq. (2.9) then becomes

$$\alpha_{ik}(0) = G_{i,j+1} + F_{jk} + c'' \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2, \quad (2.12)$$

where F_{jk} and $G_{i,j+1}$ are functions of the quantum numbers of the particles $j, j+1, \dots, k$ and $i, i+1, \dots, j+1$, respectively. It is easily verified that Eq. (2.12) with $c'' = 0$ satisfies the homogeneous equation obtained by putting $c'' = 0$ in Eq. (2.2).

It is clear that the position of the division of the diagram in Fig. 1 at j is arbitrary. For example, we could have $j+1 = k$ or $i = j$. Therefore the functional form of F and G must be the same. We can therefore write Eq. (2.12) as

$$\alpha_{ik}(0) = \tilde{F}_{ik} + c'' \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2, \quad (2.13)$$

where $\tilde{F}_{ik}$ satisfies the "additivity" property

$$\tilde{F}_{ik} = F_{i,j+1} + F_{j,k} . \quad (2.14)$$

Writing Eq. (2.14) with k replaced by $k+1$ we obtain by subtracting the two equations

$$\tilde{F}_{i,k+1} - \tilde{F}_{ik} = F_{j,k+1} - F_{j,k} \quad (2.15)$$

The right-hand side of this equation is independent of the internal quantum numbers of the particles $i, i+1, \dots, j-1$. Taking $j = k-1$ the right-hand side is only a function of the quantum numbers of the particles $k-1, k$, and $k+1$, i.e.,

$$\tilde{F}_{i,k+1} = \tilde{F}_{ik} + f(k+1, k, k-1), \quad (2.16)$$

where f is a function of the quantum numbers of the particles $k+1, k$, and $k-1$. Equation (2.16) is a recursion relation which is easily solved,

$$\begin{aligned} \tilde{F}_{ik} &= f(k, k-1, k-2) + f(k-1, k-2, k-3) + \dots \\ &\dots + f(i+2, i+1, i) + \tilde{F}_{i,i+1}. \end{aligned} \quad (2.17)$$

Finally, we shall apply the consistency requirement that the trajectory α_{ik} given by Eq. (2.13) is the same whether it is constructed from the particles $i, i+1, \dots, k$ or the particles $k+1, k+2, \dots, N, 1, 2, \dots, i-1$. This leads to the consistency requirement

$$\begin{aligned}
 & f(k, k-1, k-2) + \dots + f(i+2, i+1, i) + \tilde{F}_{i, i+1} \\
 & = f(k+1, k+2, k+3) + \dots + f(N-1, N, 1) + f(N, 1, 2) \\
 & + f(1, 2, 3) + \dots + f(i-3, i-2, i-1) + \tilde{F}_{i-2, i-1} \\
 & + c'' B_N''^2 - 2c'' \sum_{\alpha=i}^k b_{\alpha}'',
 \end{aligned}
 \tag{2.18}$$

where

$$B_N'' = \sum_{\alpha=1}^N b_{\alpha}''. \tag{2.19}$$

To solve Eq. (2.18) we note that the complete solution of Eq. (2.18) is the sum of the most general solution to the homogeneous equation [obtained by putting $c'' = 0$ in (2.18)] and a special solution to the inhomogeneous equation. To find the solution to the inhomogeneous equation we write down (2.18) for the channel $i, i+1, \dots, k, k+1$, and subtract the two relations. In this way we obtain

$$f(k+1, k, k-1) = -f(k+1, k+2, k+3) - 2c'' B_N'' b_{k+1}''. \tag{2.20}$$

This equation shows that $f(x, y, z)$ only depends on x . Hence we obtain

$$\begin{aligned}
 f(k+1, k, k-1) &= f(k+1, k+2, k+3) \equiv f(k+1) \\
 &= -c'' B_N'' b_{k+1}''.
 \end{aligned}
 \tag{2.21}$$

Hence the complete solution to Eq. (2.18) is [use Eqs. (2.17) and (2.21)]

$$\tilde{F}_{ik} = -c'' B_N'' \sum_{\alpha=i}^k b_{\alpha}'' + \tilde{F}_{ik}^0 + b, \quad (2.22)$$

where b is a constant and where $\tilde{F}_{ik}^0$ is the most general solution to the homogeneous Eq. (2.18) [obtained by putting $c'' = 0$ in Eq. (2.18)]. Hence $\tilde{F}_{ik}^0$ must satisfy the three requirements [note that Eq. (2.25) is a consequence of Eq. (2.24)],

$$\tilde{F}_{ik}^0 = \tilde{F}_{k+1, k+2, \dots, N; 1, 2, \dots, i-1}^0 \quad (\text{all } i \text{ and } k), \quad (2.23)$$

$$\tilde{F}_{ik}^0 = \tilde{F}_{ij}^0 + \tilde{F}_{j+1, k}^0, \quad (2.24)$$

$$-\tilde{F}_{ik}^0 - \tilde{F}_{i+1, k-1}^0 + \tilde{F}_{i, k-1}^0 + \tilde{F}_{i+1, k}^0 = 0. \quad (2.25)$$

The requirement (2.24) is a consequence of the additivity property (2.14) as well as Eq. (2.16). Constructing the trajectory from Eqs. (2.13) and (2.22), we obtain

$$\alpha_{ik}(0) = b + \tilde{F}_{ik}^0 - c'' \sum_{\alpha=i}^k b_{\alpha}'' (B_N'' - \sum_{\alpha=i}^k b_{\alpha}''). \quad (2.26)$$

The solution (2.26) is the most general solution of the factorization problem which is consistent with the Hagedorn fit. For a fixed number of external particles Eq. (2.26) satisfies all possible consistency relations. However, we have not investigated

what happens when N varies. Therefore Eq. (2.24) is only a necessary condition for factorization with a Hagedorn level density. In the next Section we shall investigate the effect of varying the number of external particles.

3. SELECTION RULES FOR THE TOTAL TRAJECTORY QUANTUM NUMBER B_N'' : CONSERVED TRAJECTORY QUANTUM NUMBERS

The main problem in connection with the actual construction of trajectories from the factorization requirement (2.2) is to obtain selection rules for the total trajectory quantum number B_N'' . To see how this problem arises let us consider the simple examples shown in Fig. 2. In Fig. 2a it is shown how the two-body trajectory α_2 is constructed (with identical external particles). The trajectory α_2 can easily be constructed from Eq. (2.26) and becomes

$$\alpha_2(0) = b + \tilde{F}_2^0 - c'' \sum_{\alpha=1}^2 b_{\alpha}'' \sum_{\alpha=3}^4 b_{\alpha}'' . \quad (3.1)$$

Now, factorization requires that the same trajectory couples in Fig. 2b. Thus it is necessary that

$$\alpha_2(0) = b + \tilde{F}_2^0 - c'' \sum_{\alpha=1}^2 b_{\alpha}'' \sum_{\alpha=3}^6 b_{\alpha}'' , \quad (3.2)$$

where we have used the property (2.23) for $\tilde{F}_{ik}^0$. In general, the two equations (3.1) and (3.2) do not lead to the same trajectory. Therefore it is necessary to impose selection rules on B_N'' .

The clue to the understanding of these rules is the following important symmetry [this symmetry has also been noticed by

Lovelace ⁴⁾7. In I it was assumed that the external particles with masses m_l lie on Regge trajectories,

$$\alpha_l(0) = -am_l^2 = b + \tilde{F}_l^0 + c'' b_l''^2, \quad (3.3)$$

where a is the universal slope of the Regge trajectories. Thus, with given external masses m_l the trajectory quantum number is only determined up to a sign, and we can write

$$b_l'' = \epsilon_l b_l^0, \quad \epsilon_l = \pm 1, \quad b_l^0 > 0. \quad (3.4)$$

It is obvious that the assignment of ϵ 's to a multiparticle amplitude gives a large degree of freedom. Equations (3.1) and (3.2) can be made identical, e.g., by the assignments $\epsilon_5 = +1$, $\epsilon_6 = -1$, and similar rules can be applied for more complicated examples. Extending the arguments leading from (3.1) to (3.2) to an arbitrary N point amplitude, it is easily seen that the selection rule for B_N'' requires

$$\sum_{\alpha=i}^k b_{\alpha}'' = \pm (B_N'' - \sum_{\alpha=i}^k b_{\alpha}''). \quad (3.5)$$

This guarantees that the trajectories in Figs. 1a, 1b, and 1c are all identical. As to the sign of the right-hand side of (3.5) it is seen that the one-body trajectory (3.3) requires a negative sign. A positive sign would give

$$\sum_{\alpha=i}^k b_{\alpha}'' = \frac{1}{2} B_N'', \quad (3.6)$$

which implies that the sum of b_{α}'' for any channel is a constant for a given N . However, varying N it is easily seen that this

leads to trajectories depending on N , which is not acceptable.
Hence, Eq. (3.5) yields

$$B_N'' = 0. \quad (3.7)$$

Equation (2.26) therefore becomes

$$\alpha_{ik}(0) = b + \tilde{F}_{ik}^0 + c'' \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2 \quad (3.8)$$

where

$$\sum_{\alpha=i}^k b_{\alpha}'' = - \left(\sum_{\alpha=k+1}^N b_{\alpha}'' + \sum_{\alpha=1}^{i-1} b_{\alpha}'' \right), \quad (3.9)$$

as one can see from Eq. (3.5) with the negative sign on the right-hand side.

In the remaining part of this Section we shall discuss the simplest model where all external particles are identical. We can therefore write Eq. (3.4) as

$$b_{\ell}'' = \epsilon_{\ell} b^0, \quad \epsilon_{\ell} = \pm 1, \quad b^0 > 0, \quad (3.10)$$

and we write

$$B_{ik}''^2 \equiv \left(\sum_{\alpha=i}^k b_{\alpha}'' \right)^2 = (b^0)^2 N_{ik}^2, \quad (3.11)$$

where

$$N_{ik} = \sqrt{\left(\sum_{\alpha=i}^k \epsilon_{\alpha} \right)^2}. \quad (3.12)$$

N_{ik} is thus always positive, and N_{ik} is the same whether it is constructed from channel $i, i+1, \dots, k$ or the equivalent channel $k+1, k+2, \dots, N, 1, 2, \dots, i-1$, as is seen from Eqs. (3.9) and (3.12). The quantity N_{ik} furthermore satisfies Eqs. (2.24) and (2.25), as one can see by noting that if

$$\sum_{\alpha=i}^k \epsilon_{\alpha} > 0,$$

Eqs. (2.24) and (2.25) follow trivially from (3.12), and if

$$\sum_{\alpha=i}^k \epsilon_{\alpha} < 0, \quad N_{ik} = - \sum_{\alpha=i}^k \epsilon_{\alpha},$$

so again Eqs. (2.24) and (2.25) follow. Assuming that no other quantity exists which has the properties (2.23), (2.24), and (2.25) we can therefore write Eq. (3.8) in the form

$$\alpha_{ik}(0) = b + c_1 N_{ik} + c_2 N_{ik}^2, \quad (3.13)$$

where c_1 and c_2 are constants.

The selection rule (3.7) is easily seen to imply that the total number of particles N must be even. Consider, e.g., the process 2 particles $\rightarrow$ 3 particles, where the particles are numbered as $(12) \rightarrow (345)$. Taking $\epsilon_1 + \epsilon_2 = 0$ (which is possible) we must require $\epsilon_3 + \epsilon_4 + \epsilon_5 = 0$. However, the last equation has no solution with $\epsilon_{\alpha} = \pm 1$, $\alpha = 3, 4, 5$. Similar arguments apply to more complicated cases. Therefore it follows that factorization of N point amplitudes with identical external particles is only possible if N is even ^{*)}, provided one assumes that not all N body

*) The assumption of the validity of the Hagedorn fit is not important for this conclusion. If the level density increases faster than the Hagedorn fit, Eq. (3.13) has to be replaced by a polynomial of N_{ik} of finite degree.

trajectories are identical (if there is only one trajectory the N point amplitude of course factorizes). The result implies that the external particles can be e.g. pions or scalar baryons.

Let us now consider an N point amplitude with N arbitrary and let us discuss the resulting spectrum for the Regge trajectories. The one-body trajectory is given by

$$\alpha_1(0) = b + c_1 + c_2. \quad (3.14)$$

We can construct two two-body trajectories by assigning the ϵ 's such that $N_2 = 0$ or $N_2 = 2$. We therefore obtain the two two-body trajectories

$$\alpha_0(0) = b, \quad \alpha_2(0) = b + 2c_1 + 4c_2. \quad (3.15)$$

Similarly the three-body trajectory can be constructed in two ways, $N_3 = 1$ or $N_3 = 3$, and we obtain

$$\alpha_3(0) = b + 3c_1 + 9c_2, \quad (3.16)$$

and the other three-body trajectory is equal to the one-body trajectory (3.14). Four external particles can couple to $\alpha_2(0)$ (with $N_4 = 2$), $\alpha_0(0)$ (with $N_4 = 0$), or to

$$\alpha_4(0) = b + 4c_1 + 16c_2. \quad (3.17)$$

In general, if we have n external particles, they can couple to $\alpha_0, \alpha_2, \alpha_4, \dots, \alpha_n$, if n is even, where

$$\alpha_n(0) = b + nc_1 + n^2c_2. \quad (3.18)$$

If n is odd, the n particles can couple to $\alpha_1, \alpha_3, \dots, \alpha_n$. Thus the spectrum consists of a few leading trajectories and an infinite number of secondary trajectories. The secondary trajectories are of course in general not identical to daughter trajectories.

It is interesting to note that the spectrum exhibited in Eq. (3.18) can be brought into accordance with the $\rho - \omega$ exchange degeneracy. Of course, the integral representation for the N point function is strictly speaking not valid for positive intercepts, and some ghost-killing mechanism is necessary. Anticipating that this can be constructed without changing the factorization properties, we may tentatively assume that Eq. (3.18) is valid for positive intercepts. Requiring exact degeneracy between ρ and ω and taking the external particles to be pions, we obtain with the input ^{*)}
 $\alpha_\rho(0) = \frac{1}{2}, \alpha_\pi(0) = 0$ (we ignore finer details arising from $am_\pi^2 \approx 0.02$)

$$\alpha_n(0) = -1 - \frac{1}{4}(n^2 - 5n). \quad (3.19)$$

In Table I we have shown some of the intercepts constructed from Eq. (3.19) as a function of the number of external pions to which the trajectory couples. Many of these trajectories are degenerate, and for $\alpha(0) \geq -1$ there are only three different trajectories, α_ρ, α_π , and a secondary two-body trajectory with $\alpha_{2\pi}(0) = -1$ (corresponding to a spin 0 resonance with mass ≈ 1 GeV). It is seen that this type of spectrum is rather reasonable from a phenomenological point of view.

Finally we mention that one can assign the ϵ 's in many different ways to the external particles in an N point function.

^{*)} We note that the solution (3.19) satisfies the necessary consistency requirement $c_2 < 0$. This requirement is a consequence of the Froissart bound $\alpha(0) < 1$ for any trajectory, as one can see from Eq. (3.18) by taking n large.

However, the different ways of assigning ϵ 's lead to different trajectories in the various channels contained in a given N point function. This is illustrated in the simplest way for the four-point function in Fig. 3. In Fig. 3a we exchange the same trajectory in the s and t channels, whereas in Fig. 3b the trajectories in the two channels are different. The corresponding amplitudes are given by

$$A_a(s,t) = \beta \Gamma(-\alpha_0(s)) \Gamma(-\alpha_0(t)) / \Gamma(-\alpha_0(s) - \alpha_0(t)), \quad (3.20)$$

$$A_b(s,t) = \beta \Gamma(-\alpha_0(s)) \Gamma(-\alpha_2(t)) / \Gamma(-\alpha_0(s) - \alpha_2(t)), \quad (3.21)$$

For the N point function similar remarks are true. We mention that since any N point function constructed with the trajectories (3.18) automatically factorizes with different trajectories in different channels, there exists the possibility that N point functions with positive intercepts in some channels can be constructed with a fairly simple ghost-killing mechanism. This question is now under investigation.

4. SELECTION RULES FOR THE TOTAL TRAJECTORY QUANTUM NUMBER B_N : PARTIALLY CONSERVED QUANTUM NUMBERS

In the two previous Sections we have established the rather rigorous result that any factorization of the N point dual amplitude which is in exact agreement with a level density

$$d(J) \sim e^{6.2\sqrt{J}} \quad (J \rightarrow \infty), \quad (4.1)$$

necessarily leads to conserved trajectory quantum numbers, $B_N'' = 0$. One might still ask the question whether it is possible to have a theory where $B_N'' \neq 0$. It is clear that such a theory cannot be in exact agreement with Eq. (4.1). On the other hand, one can replace the trajectory (3.13) with a polynomial of degree M in N_{ik} , and as shown in I this leads to a level density different from (4.1). However, one can generalize the arguments in Sections 2 and 3 and show that the trajectory quantum numbers are conserved. It therefore follows that the only possibility of obtaining non-conserved trajectory quantum numbers is to construct a model where the level density (4.1) is approximately correct for some of the trajectories, whereas other trajectories have more complicated spectra. The trajectory quantum numbers are partially conserved in such models.

We have not been able to investigate in a systematic way all such models. Instead we discuss one model with partially conserved quantum numbers. The model is essentially the same as Mandelstam's quark model ¹⁾, but we shall point out a few features which are relevant for the present discussion. Also, the external particles do not necessarily have to be quark, but could equally well be mesons. One of the differences between the model in this Section and the model in the previous Section is that in the present model only the four-point function (3.21) could occur in the meson model (or scalar baryons etc.). For convenience we refer to the external particles as quarks.

Let us consider the quark diagram in Fig. 4, and let us imagine that the line l_m is not present, i.e., only the lines l_N and $j, j+1$ cross the dividing line D . In channel lj the total quark number (i.e., the number of quarks plus anti-quarks which crosses the dividing line D) is 2. Obviously, this can be described essentially by the model in the previous Section if we assign ϵ 's as shown in Fig. 4. The only trouble comes in if we have a connecting line like $j, j+1$. Let us define, following Mandelstam ¹⁾, the trajectory as

$$\alpha_{ik}(0) = b + c_1 \nu_{ik} + c_2 \nu_{ik}^2, \quad (4.2)$$

where ν_{ik} is the total quark number in channel ik (i.e., ν_{ik} is the number of quarks plus anti-quarks which crosses a line L drawn just above the particles i and k , as shown in Fig. 4). If i and k are not connected we have

$$\nu_{ik} = \epsilon_i + \epsilon_k + \nu_{i+1, k-1},$$

$$\nu_{ik} = \epsilon_i + \nu_{i+1, k}, \quad (4.3)$$

$$\nu_{ik} = \epsilon_k + \nu_{i, k-1},$$

where

$$\epsilon_\alpha = \begin{cases} +1 & \text{if the line } \alpha \text{ goes towards the top of the diagram} \\ -1 & \text{if the line } \alpha \text{ goes towards the bottom of the diagram} \end{cases} \quad (4.4)$$

where the "top" and "bottom" of the diagram are defined with respect to the division D of the N point function in Fig. 4. Inserting Eqs. (4.3) in Eq. (2.1) we immediately obtain

$$\delta(i, k) = -2c_2 \epsilon_i \epsilon_k, \quad (4.5)$$

in complete accordance with the model in the previous Section. The only trouble comes in if we consider the line l_m , which is connected. Here $\nu_{l_m} = 0$, $\nu_{l+1, m-1} = 0$, $\nu_{l+1, m} = 1$, $\nu_{l, m-1} = 1$, and hence

$$\delta(l, m) = 2c_2 + 2c_1, \quad (4.6)$$

showing that the linear term in Eq. (4.2) contributes. Also according

to (4.6) $\epsilon_l \epsilon_m = -1$. However, if we consider the channel $l k$ (k is above m as in Fig. 4), we see by a simple argument that

$$\delta(l, k) = 2c_2 \epsilon_k, \quad (4.7)$$

showing that $\epsilon_l = -1$. On the other hand, taking k below m , we would obtain $\epsilon_l = +1$. Thus, no matter how we assign the ϵ 's we can never get the right answer for all the particles. We therefore have to adopt a convention and make corrections for those particles that are not treated correctly by this convention. In the following we accept the convention in Fig. 4, where connected particles always have ϵ 's equal to -1 , i.e., $\epsilon_l = \epsilon_m = -1$. To see the correction terms arising from this convention we have to go back to the fundamental equation for the dual integrand [see Eq. (2.15) in I]

$$I_N = I_{j+1} I_{N-j} w^{-\alpha_i(s)-1} (1-w)^{b-1} \times \prod_{i=2}^j \prod_{k=j+1}^{N-1} (1-w y_i y_k)^{-2\alpha_i p_k + \delta(i, k)}, \quad (4.8)$$

where the y 's are defined in terms of the standard multiperipheral variables u_{js} and u_{Ns} by

$$y_i = \prod_{s=i}^{j-1} u_{1s}, \quad y_k = \prod_{s=j+2}^k u_{Ns}. \quad (4.9)$$

Defining

$$f_{ik} = (1 - w y_i y_k)^{-2\alpha p_i p_k + \delta(i,k)} \quad (4.10)$$

we have

$$\begin{aligned} \prod_{i=2}^j \prod_{k=j+1}^{N-1} f_{ik} &= \left(\prod_{i=2}^{l-1} \prod_{k=j+1}^{m-1} f_{ik} \right) \left(\prod_{i=2}^{l-1} \prod_{k=m+1}^{N-1} f_{ik} \right) \\ &\times \left(\prod_{i=l+1}^j \prod_{k=j+1}^{m-1} f_{ik} \right) \left(\prod_{i=l+1}^j \prod_{k=m+1}^{N-1} f_{ik} \right) \left(\prod_{i=2}^{l-1} f_{im} \right) \\ &\times \left(\prod_{i=l+1}^j f_{im} \right) \left(\prod_{k=j+1}^{m-1} f_{lk} \right) \left(\prod_{k=m+1}^{N-1} f_{lk} \right) f_{lm}. \end{aligned} \quad (4.11)$$

It is easily seen that the only factors on the right-hand side of Eq. (4.11) for which our convention for ϵ_l and ϵ_m does not work are the factors

$$\prod_{i=l+1}^j f_{im}, \quad \prod_{k=j+1}^{m-1} f_{lk}, \quad f_{lm}. \quad (4.12)$$

In the first factor ϵ_m ought to be +1, whereas our convention gives -1, and in the second factor ϵ_l ought to be +1. In the last factor $\epsilon_l \epsilon_m$ ought to be -1. Correcting these errors we obtain

$$\begin{aligned}
 I_N &= I_{j+1} I_{N-j} w^{-\alpha_{ij}(s)-1} (1-w)^{b-1} \\
 &\times \prod_{i=2}^j \prod_{k=j+1}^{N-1} (1-w y_i y_k)^{-2\alpha_{pi} p_k - 2c_2 \epsilon_i \epsilon_k} \\
 &\times \prod_{i=l+1}^j (1-w y_i y_m)^{-4c_2 \epsilon_i} \prod_{k=j+1}^{m-1} (1-w y_l y_k)^{-4c_2 \epsilon_k} \\
 &\times (1-w y_l y_m)^{4c_2 + 2c_1} (1-w)^{4c_2 + 2c_1}.
 \end{aligned}
 \tag{4.13}$$

The last factor comes from the line $j, j+1$. With more than one line which connects the two parts of the diagram we get more additional factors.

In general the asymptotic density is only affected by the additional factors if one has a large number of such quarks crossing the diagram. If the external particles are mesons it means that the low-lying secondary many-body meson trajectories are more densely populated than the primary trajectories. However, for the primary trajectories the spectrum is the same as in Section 3.

It is clear that the model discussed in this Section does not have conserved trajectory quantum numbers. The price one has to pay for this is that the level density of the low-lying secondary trajectories is a function of the spectrum of the trajectories. Also, in a harmonic oscillator formalism the additional terms in (4.13) require additional scalar modes. It is unfortunately not easy to estimate the exact contribution to the level density from these extra modes. With only a few lines crossing the diagram the asymptotic level density is unaffected.

One can easily see from the Froissart bound that $c_2 < 0$. Thus the main term in Eq. (4.13) can be written

$$\prod_{i=2}^j \prod_{k=j+2}^{N-1} (1 - w y_i y_k)^{-2a p_i p_k - 2c_2 \epsilon_i \epsilon_k}$$

$$= \prod_{i=2}^j \prod_{k=j+2}^{N-1} (1 - w y_i y_k)^{-\bar{p}_i \bar{p}_k} \quad (4.14)$$

where $\bar{p}_i$ and $\bar{p}_k$ are five-dimensional vectors

$$\bar{p}_i = (\sqrt{2a} p_i, \sqrt{-2c_2} \epsilon_i), \quad \bar{p}_k = (\sqrt{2a} p_k, \sqrt{-2c_2} \epsilon_k), \quad (4.15)$$

where $g_{55} = -1$. It is very important that the fifth component is space-like because of $c_2 < 0$, since this implies no ghosts arising from the fifth component. The corrections to (4.14) in (4.13) are, however, in general of mixed signs and can lead to ghosts.

Other models with partially conserved trajectory quantum number may exist, but we have not been able to construct such models. However, we want to point out that if one divides the diagram for the N point function by more than one cut, there exists the possibility of another model. The model in Section 3 can easily be generalized to multifactorization, and it is likely that the quark model can also be generalized. However, there exists a third model, which is a mixture of the model in Section 3 and the quark model. We refer to this model as the "mixed model".

In the mixed model the external particles are not quarks, and we refer to these particles as mesons. First let us note that with the quark-like diagram in Fig. 4, where the external particles are now mesons and the only meaning of the diagram is to keep track of the construction of the trajectories, we can avoid corrections for the lines ℓ_m and $j, j+1$ by introducing the condition ^{*)}

^{*)} We cannot, of course, avoid the other corrections in Eq. (4.13).

$$c_1 = -2c_2. \quad (4.16)$$

To illustrate the mixed model consider the 18-point function in Fig. 5a. We first make the division D , and associate ϵ 's to the particles in accordance with Eq. (4.4), and assign -1 to the lines that cross the line D . We notice that these assignments of ϵ 's fix the trajectories in the channels ik , where $1 \leq i \leq 9$, $10 \leq k \leq 18$. However, these assignments of ϵ 's do not necessarily have to fix trajectories in channels ii' and kk' , where $1 \leq i' \leq 9$, $10 \leq k' \leq 18$. Channels of the type ii' and kk' only enter in the integrand I_{j+1} and I_{N-j} in Eqs. (4.8) and (4.13), respectively, and factorization along the line D does of course not imply anything for these channels. Now perform a second factorization along the line D' in Fig. 5a. Then there exists two possibilities for the four-point function consisting of particles 14, 15, 16, and 17. Either one can consider the original assignments of ϵ 's, and use the model with conserved quantum numbers in Section 3, or one can use the lines. These two alternatives are illustrated in Figs. 5b and 5c, respectively. In the case of Fig. 5b one gets a symmetric four-point function (3.20), whereas in the case of Fig. 5c one gets the non-symmetric function (3.21). In the case of a general N point function one can similarly use the model with conserved trajectory quantum numbers whenever one factorizes a subgraph where the total quantum number is conserved. The mixed model is important for loop diagrams. The spectrum of meson Regge trajectories is again the same as in Section 3.

5. DUAL LOOPS WITH UNEQUAL INTERCEPTS

Previously ⁵⁾ we have investigated dual loop amplitudes with unequal intercepts. In this Section we shall investigate the general structure of the formulae derived in Ref. 5) from the point of

view of the models developed in Sections 3 and 4. The loop amplitude for planar diagrams with external momenta $p_1, p_2, \dots, p_n$ is given by

$$L = \int d^4 k \, M(k, p_1, \dots, p_n), \quad (5.1)$$

where

$$M = \sum_T \beta_T \int_0^1 \prod_{i=1}^n dx_i x_i^{-\alpha_{T,i-1}(k_{i-1}^2)-1} (1-x_i)^{b-1} \\ \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{2 \sum_{i,j=1}^n C_{ij}^r (a_i p_j + c'' b_i'' b_j'')}{r(1-w^r)} \right] \right\} \\ \times A(x_i). \quad (5.2)$$

Here $A(x_i)$ is a factor which takes care of the additional modes needed in the quark model ¹⁾. The summation over T goes over all states consisting of external particles into which the superstate can decay [see Ref. 5) for further explanation of the terminology]. The sum over T must be carried out in such a way that double counting is avoided. β_T is an over-all normalization constant for the coupling of the trajectory $\alpha_T(k^2)$ to the state T .

In the models discussed in Sections 3 and 4 the various different trajectories are given by

$$\alpha_n(0) = b + c_1 n + c_2 n^2. \quad (5.3)$$

The interpretation of n differs in the three models. Equation (5.2) can then be written

$$\begin{aligned}
 M = & \int_0^{\infty} \prod_{i=1}^n dx_i \, x_i^{-\alpha(k_{i-1}^2)-1} (1-x_i)^{b-1} \Delta(x_i) \\
 & \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{2 \sum_{i,j=1}^n c_{ij}^r (a p_i p_j + c_2 \epsilon_i \epsilon_j)}{r(1-w^r)} \right] \right\} \\
 & \times A(x_i),
 \end{aligned}
 \tag{5.4}$$

where

$$\alpha(k^2) = a k^2 + b,
 \tag{5.5}$$

$$\Delta(x_i) = \sum_{n_0} \beta_{n_0} \sum_{i=2}^n x_i^{-n_{i-1}^2 c_2}.
 \tag{5.6}$$

The sum over n_0 in Eq. (5.6) is to be performed over all possible values $0, 1, 2, \dots, \infty$ of n_0 in such a way that double counting is avoided, and the sum over n_i goes over all values compatible with the value assigned to n_0 . The quantity $\Delta(x_i)$ represents the correction to the propagator for universal intercepts

$$x_i^{-\alpha(k_{i-1}^2)-1} (1-x_i)^{b-1}.$$

The correction factor is finite provided the sum (5.6) is convergent, which can be achieved in several ways (note that $0 \leq x_i \leq 1$). If, e.g., all β_{n_0} have the same sign we must require

$$\lim_{n_0 \rightarrow \infty} \beta_{n_0} = 0.
 \tag{5.7}$$

This requirement is rather weak.

A much more serious point is the behaviour of the infinite product in (5.4). In Ref. 5) it was pointed out that in order to have a well-defined infinite product one must require

$$c_2 \left(\sum_{i=1}^n \epsilon_i \right)^2 < -\frac{5}{2}. \quad (5.8)$$

Since $c_2 < 0$ in all models from the Froissart bound the condition (5.8) can be satisfied (with a sufficiently large value of $|c_2|$) if the ϵ 's do not sum up to zero.

For the model in Section 3 it is easily seen that the sum over ϵ 's in (5.8) is zero, since the trajectory quantum numbers are conserved. In this model it is therefore impossible to obtain finite loops by use of the device (5.8). The model can then be made finite by renormalization of the masses of the lowest state on all the trajectories, as shown by Neveu and Scherk ⁶⁾. Since there is an infinite number of trajectories this means that one has to introduce an infinite number of renormalization constants.

The quark model due to Mandelstam ¹⁾ is more complicated because of the additional modes in the harmonic oscillator formalism. We have also found various other difficulties arising when one tries to carry out a multifactorization in the quark model. In particular, it is difficult to assign ϵ 's in such a way that the assignment is consistent for all possible factorizations of the N point function. It might be that there exists various new Ward identities which allow one to overcome these difficulties.

Instead we have investigated the mixed model, discussed at the end of Section 4, for the simple planar loop shown in Fig. 6a. The loop can be constructed in two ways, as shown in Figs. 6b and 6c. We have assigned ϵ 's in the symmetric cut D . In the case of Fig. 6b, $\epsilon_1 = \epsilon_2 = -1$, whereas in Fig. 6c $\epsilon_1 = \epsilon_2 = +1$.

However, from Eq. (5.4) it is seen that the loop is invariant under an over-all change of sign of the ϵ 's. Furthermore $(\epsilon_1 + \epsilon_2)^2 = 4$, so the inequality (5.8) is satisfied if $c_2 < -5/8$. The only thing one has to check is that the two diagrams, Figs. 6b and 6c, lead to the same value for the loop, i.e., that they are mutually consistent. This can be shown by a very simple calculation. One can easily see that Fig. 6b gives $\bar{a}$ denotes the five-vector $(a, \sqrt{-2c_2} \epsilon_a)$ with metric $g_{55} = -1$

$$I = I_R I_L u_2^{-\alpha_1(s_2)-1} (1-u_2)^{b-1} \times \exp \sum_{r=1}^{\infty} \frac{1}{r} u_2^r [\bar{q}_1 \bar{q}_2 + \bar{q}_1 \bar{p}_1 u_3^r + \bar{k}_1 \bar{q}_2 u_1^r + \bar{k}_1 \bar{p}_1 u_1^r u_3^r], \quad (5.9)$$

where I_R is the four-point integrand with external particles p_1, p_2, q_2 and the superstate corresponding to the trajectory α_1 . Similarly I_L is the four-point integrand with external particles k_2, k_1, q_1 and the superstate corresponding to α_1 . These four-point integrands have the same trajectory α_0 in both channels, i.e.,

$$I_R = u_3^{-\alpha_0(s_3)-1} (1-u_3)^{-\alpha_0((q_2+p_1)^2)-1} = u_3^{-\alpha_0(s_3)-1} (1-u_3)^{b-1} \exp \sum_{r=1}^{\infty} \frac{1}{r} u_3^r \bar{q}_2 \bar{p}_1, \quad (5.10)$$

$$I_L = u_1^{-\alpha_0(s_1)-1} (1-u_1)^{b-1} \exp \sum_{r=1}^{\infty} \frac{1}{r} u_1^r \bar{q}_1 \bar{k}_1. \quad (5.11)$$

Inserting Eqs. (5.10) and (5.11) in Eq. (5.9) we obtain easily the usual expression

$$\begin{aligned}
 I = & u_1^{-\alpha_0(s_1)-1} u_2^{-\alpha_1(s_2)-1} u_3^{-\alpha_6(s_3)-1} \\
 & \times [(1-u_1)(1-u_2)(1-u_3)]^{b-1} \\
 & \times \prod_{r=1}^{\infty} \langle 0 | e^{\bar{p}_1 \bar{a}^{(r)}/\sqrt{r}} u_3^r \bar{a}^{+(r)} a^{(r)} e^{\bar{q}_2 \bar{a}^{+(r)}/\sqrt{r}} e^{\bar{q}_2 \bar{a}^{(r)}/\sqrt{r}} \\
 & \times u_2^r \bar{a}^{+(r)} \bar{a}^{(r)} e^{\bar{q}_1 \bar{a}^{+(r)}/\sqrt{r}} e^{\bar{q}_1 \bar{a}^{(r)}/\sqrt{r}} u_1^r \bar{a}^{+(r)} \bar{a}^{(r)} e^{\bar{k}_1 \bar{a}^{+(r)}/\sqrt{r}} | 0 \rangle,
 \end{aligned}
 \tag{5.12}$$

which leads to Eq. (5.4) for the special diagram in Fig. 6a. For the diagram in Fig. 6c we obtain

$$I = I_L I_R u_2^{-\alpha_0(s_1)-1} (1-u_2)^{b-1} \exp \sum_{r=1}^{\infty} \frac{1}{r} u_2^r \bar{L}^{(r)} \bar{R}^{(r)},
 \tag{5.13}$$

$$\left. \begin{aligned}
 \bar{L}^{(r)} &= \bar{q}_1 + u_1^r \bar{k}_1 + (u_1 v)^r \bar{k}_2, \\
 \bar{R}^{(r)} &= \bar{q}_2 + u_3^r \bar{p}_1 + (u_3 w)^r \bar{p}_2.
 \end{aligned} \right\}
 \tag{5.14}$$

Here I_R is the five-point function with the external particles p_3, p_2, p_1, q_2 and the superstate α_0 . Similarly for I_L . Now, for I_R and I_L we can use the conserved model, and we obtain

$$I_R = I(p_1, p_2, w) u_3^{-\alpha_1(s_3)-1} (1-u_3)^{b-1} \exp \sum_{r=1}^{\infty} \frac{u_3^r}{r} \bar{q}_2 \bar{P}^{(r)},
 \tag{5.15}$$

$$I_L = I(k_1, k_2, v) u_1^{-\alpha_1(s_1)-1} (1-u_1)^{b-1} \exp \sum_{r=1}^{\infty} \frac{u_1^r}{r} \bar{q}_1 \bar{K}^{(r)},
 \tag{5.16}$$

$$\bar{P}^{(r)} = \bar{p}_1 + w^r \bar{p}_2, \quad \bar{K}^{(r)} = \bar{k}_1 + v^r \bar{k}_2. \quad (5.17)$$

Collecting results we obtain

$$\begin{aligned} I = & I(p_1, p_2, w) I(k_1, k_2, v) u_1^{-\alpha_1(s_1)-1} u_2^{-\alpha_0(s_1)-1} u_3^{-\alpha_1(s_3)-1} \\ & \times [(1-u_1)(1-u_2)(1-u_3)]^{b-1} \exp \sum_{r=1}^{\infty} \frac{1}{r} [u_2^r \bar{q}_1 \bar{q}_2 + (u_1 u_2)^r \bar{q}_2 \bar{K}^{(r)} \\ & + (u_2 u_3)^r \bar{q}_1 \bar{P}^{(r)} + (u_1 u_2 u_3)^r \bar{K}^{(r)} \bar{P}^{(r)} + u_3^r \bar{q}_2 \bar{P}^{(r)} \\ & + u_1^r \bar{q}_1 \bar{K}^{(r)}]. \end{aligned} \quad (5.18)$$

This we can rewrite in the harmonic oscillator formalism, and the result leads to the same loop as Eq. (5.12). Hence, we have shown that the loop in Fig. 6a can be constructed in a manner which is consistent with the two diagrams in Figs. 6b and 6c. Furthermore, the loop converges if $c_2 < -5/8$.

Of course, this simple example does not prove that all loops can be made finite. To investigate the general case one has to find all rules in the mixed model which are applicable to multifactorization. This problem is now under investigation.

6. DISCUSSION

The models discussed in Sections 3 and 4 provides us with a set of trajectories which are consistent with factorization. The spectrum in all the models is quite rich, and by imposing some boundary conditions (e.g., $\rho - \omega$ degeneracy) one can obtain a set of meson trajectories which resemble those found in nature.

These models might therefore be of interest in connection with the problem of constructing dual N point function with positive intercepts.

The "mixed model" discussed in Section 4 might lead to finite planar dual loops. At least, we have given one concrete example of a finite loop.

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T A B L E I

n	$\alpha_{n\pi}(0)$
1	0
2	$\frac{1}{2}, -1$
3	$\frac{1}{2}, 0$
4	$\frac{1}{2}, 0, -1$
5	$\frac{1}{2}, 0, -1$
6	$\frac{1}{2}, 0, -1, -\frac{5}{2}$

R E F E R E N C E S

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F I G U R E C A P T I O N S

Figure 1: Factorization of the N point function.

Figure 2: Consistency conditions on α_2 .

Figure 3: Trajectories in the four-point function.

Figure 4: Factorization in the quark model.

Figure 5: Double factorization in the mixed model.

Figure 6: Construction of a planar dual loop.

FIG. 1

FIG. 2

FIG. 3

a)

b)

c)

FIG. 5

a)

=

b)

=

c)

FIG.6