CM-P00058221

Ref.TH.1104-CERN

DUAL LOOP AMPLITUDES WITH GENERAL REGGE TRAJECTORIES

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A B S T R A C T

Previously we have obtained constraints on Regge intercepts from factorization of the N point dual amplitude. Here we consider more general factorization properties, and obtain an expression for the loop amplitude with general (factorizable) Regge intercepts. It turns out that the loop amplitude is finite if the Regge intercepts satisfy a certain condition. If this condition is not satisfied we give a continuation procedure which in principle allows one to calculate the loop amplitude (even if all intercepts are universal). However, in this case one encounters a cut which does not correspond to a physical singularity.

1. INTRODUCTION

It has been shown by Fubini and Veneziano ¹⁾, Bardakçi and Mandelstam ²⁾ that the N point Veneziano amplitude factorizes if all Regge trajectories have the same slope and intercepts. This result was generalized by Mandelstam ³⁾ and by us ⁴⁾ to the case when the intercepts depend on the channels involved in a given many-body scattering process. In particular, we have shown in I that the requirement of factorization imposes constraints on the Regge intercepts, which lead to certain sum rules.

In the present paper we wish to extend the factorization given in I to more complicated problems, where the N point amplitude is factored into several groups of external particles, each group being connected to the other groups by super-states (i.e., excited states). We apply the harmonic oscillator formalism ⁵⁾ to write the factorized amplitude in a simple form. In Section 2 we investigate the general factorization properties of the N point amplitude when the Regge intercept in an arbitrary channel ik is a quadratic function of the constituent external particles

$$\alpha_{ik}(0) = b + c'(b'_i + b'_{i+1} + \dots + b'_k) + c''(b''_i + b''_{i+1} + \dots + b''_k)^2. \quad (1.1)$$

In I we have argued on the basis of the Hagedorn temperature ⁶⁾ that the level density corresponding to the trajectories of the type (1.1) is most likely to be in agreement with the experimental hadron distribution. However, the factorization properties obtained in Section 2 are valid when $\alpha_{ik}(0)$ is a polynomial in the constituent particles. The quantum numbers b'_ℓ and b''_ℓ in Eq. (1.1) are not specified by the requirement of factorization, and many models are possible. At the moment we prefer to leave open the question concerning how one associates specific quantum numbers with the b 's, and instead we consider Eq. (1.1) as a general framework which is compatible with the N point dual amplitude.

Having obtained the expression for a factorized amplitude we can proceed to construct planar loop amplitudes. For the case of universal intercepts ($c' = c'' = 0$) the loop amplitude has been obtained by many authors ^{7),8),9)}. In Section 3 we generalize the elegant harmonic oscillator treatment by Amati, Bouchiat and Gervais ⁹⁾ to the case of general intercepts (1.1). It turns out that the loop amplitude is modified in a rather essential way. The authors in Refs. ^{7),8),9)} showed that the integrand in the amplitude contains a multiplicative factor

$$F(w) = \prod_{r=1}^{\infty} (1 - w^r)^{-4}, \quad (1.2)$$

which is independent of the kinematical variables. Since the range of integration includes the point $w = 1$ the loop integral is badly divergent ^{7),8),9)}. With general trajectories (1.1) the function $F(w)$ is modified by a factor which for $w \sim 1$ can be written

$$\prod_{r=1}^{\infty} \left\{ (1 - w^r)^{-5} \exp \left[\frac{2 c'' \left(\sum_{i=1}^N b_i'' \right)^2}{r (1 - w^r)} \right] \right\}, \quad (1.3)$$

where c'' and b_i'' are defined in Eq. (1.1). For $w \sim 1$ the expression (1.3) behaves like

$$\exp \left\{ \frac{\pi^2}{6} \frac{1}{1-w} \left[5 + 2 c'' \left(\sum_{i=1}^N b_i'' \right)^2 \right] \right\}, \quad (1.4)$$

which converges provided

$$c'' \left(\sum_{i=1}^N b_i'' \right)^2 < -\frac{5}{2}. \quad (1.5)$$

Thus, if the N particle Regge intercept is sufficiently small or negative (for $N \geq 4$) the loop amplitude is well-defined. In Section 4 we discuss the constraint (1.5) in more detail. It turns out that in order that (1.5) is satisfied the quantum numbers b_i'' should not distinguish between particles and anti-particles.

The divergent factor (1.2) is closely connected to the large density of excited states at a given angular momentum. One might therefore wonder why it helps to take $c'' \neq 0$, since in that case the density of states at a given angular momentum is larger than if $c'' = 0$. The point is, however, that the divergence in Eq. (1.2) is also due to the vertex function which couples two super-states with one external particle. In the case of general trajectories (1.1) this vertex function is modified, and the inequality (1.5) expresses the condition for the vertex function to be sufficiently well-behaved at large masses of the states involved.

Condition (1.5) might not be satisfied in nature, of course. In Section 4 it is shown that if (1.5) is violated one encounters a cut which does not correspond to a physical singularity. The meaning of this cut is therefore unclear, and in a sense unitarity demands that condition (1.5) is satisfied.

In Section 5 we formulate our results in the most general way compatible with the Hagedorn fit ⁶⁾.

2. GENERAL FACTORIZATION PROPERTIES

In I we have shown that the integrand for the $(m+n)$ point amplitude (see Fig. 1) can be factorized in the form

$$I_{m+n} = I_{m+1} I_{n+1} w_1^{-\alpha_{in}(s_1)-1} (1-w_1)^{b-1} \\ \times \exp \left[\sum_{r=0}^{\infty} \frac{w_1^r}{r} \bar{P}^{(r)} \bar{Q}'^{(r)} \right], \quad (2.1)$$

where the five-dimensional vectors $\bar{P}^{(r)}$ and $\bar{Q}'^{(r)}$ are given by

$$\begin{aligned}\bar{P}^{(r)} &= \left(\sqrt{2a} \sum_{j=1}^{n-1} p_j \prod_{k=1}^{j-1} v_k^r, \sqrt{2c''} \sum_{j=1}^{n-1} b_j'' \prod_{k=1}^{j-1} v_k^r \right), \\ \bar{Q}'^{(r)} &= \left(\sqrt{2a} \sum_{j=1}^{m-1} q_j \prod_{k=1}^{j-1} u_k^r, \sqrt{2c''} \sum_{j=1}^{m-1} b_j'' \prod_{k=1}^{j-1} u_k^r \right).\end{aligned}\quad (2.2)$$

In the following we use the convention that a five-vector is distinguished from a four-vector by a bar. In I we also generalized the harmonic oscillator formalism by Fubini, Gordon and Veneziano ⁵⁾ by introducing a set of harmonic oscillators $\bar{a}_\mu^{(r)}$ with five dimensions. Equation (2.1) could then easily be written as a vacuum expectation value of the product of one propagator and two vertices. Here we shall generalize these results to the more complicated factorization problem, where the $(m+1)$ point amplitude is factorized into a $(p+2)$ point amplitude and a $(f+1)$ point amplitude. For convenience we use the notation shown in Fig. 2. Writing

$$\bar{Q}''^{(r)} = \bar{Q}^{(r)} + z^r w_2^r \bar{K}^{(r)}, \quad z = \prod_{i=1}^{p-1} u_i, \quad (2.3)$$

where

$$\begin{aligned}\bar{Q}^{(r)} &= \left(\sqrt{2a} \sum_{j=1}^p q_j \prod_{k=1}^{j-1} u_k^r, \sqrt{2c''} \sum_{j=1}^p b_j'' \prod_{k=1}^{j-1} u_k^r \right), \\ \bar{K}^{(r)} &= \left(\sqrt{2a} \sum_{j=1}^{f-1} k_j \prod_{k=1}^{j-1} r_k^r, \sqrt{2c''} \sum_{j=1}^{f-1} b_j'' \prod_{k=1}^{j-1} r_k^r \right),\end{aligned}\quad (2.4)$$

we obtain easily

$$I_{m+n} = I_{f+1} I_{p+2} I_{n+1} w_1^{-\alpha_{1n}(s_1)-1} w_2^{-\alpha_{1f}(s_2)-1} \\ \times [(1-w_1)(1-w_2)]^{b-1} \exp \left\{ \sum_{r=0}^{\infty} \frac{1}{r} [w_1^r \bar{P}^{(r)} \bar{Q}^{(r)} \right. \\ \left. + w_1^r w_2^r z^r \bar{P}^{(r)} \bar{K}^{(r)} + w_2^r \bar{Q}^{(r)} \bar{K}^{(r)}] \right\}. \quad (2.5)$$

We now introduce the five-dimensional harmonic oscillators $\bar{a}^{(r)}$ and $\bar{a}^{+(r)}$, which satisfy

$$[\bar{a}_\mu^{(r)}, \bar{a}_\nu^{+(r')}] = \delta_{rr'} g_{\mu\nu}, \quad (2.6)$$

where $g_{\mu\nu}$ is diagonal and $g_{55} = +1$ if $c'' > 0$ and $g_{55} = -1$ if $c'' < 0$ [in this case $\sqrt{2c''}$ should be replaced by $\sqrt{-2c''}$ in Eqs. (2.2) and (2.4)]. Integrating Eq. (2.5) over the independent variables the $(m+n)$ point amplitude A_{m+n} can then be written

$$A_{m+n} = \langle 0 | \mathcal{G}(k, \bar{a}) P(H, \alpha_{1f}(s_{1f})) \\ \times \Gamma(q, \bar{a}^+, \bar{a}) P(H, \alpha_{1n}(s_{1n})) \mathcal{G}(p, \bar{a}^+) | 0 \rangle \quad (2.7)$$

where

$$P(H, \alpha(s)) = \int_0^1 dx (1-x)^{b-1} x^{-\alpha(s)-1+H}, \\ H = \sum_r r \bar{a}^{+(r)} \bar{a}^{(r)}, \quad (2.8)$$

$$G(k, \bar{a}) = \int \prod_{i=1}^{f-2} dr_i I_{f+1}(r, k) \times \exp \left[\sum_{r=1}^{\infty} \bar{K}^{(r)} \bar{a}^{(r)} / \sqrt{r} \right], \quad (2.9)$$

$$\Gamma(q, \bar{a}^+, \bar{a}) = \int \prod_{k=1}^{p-1} du_k I_{p+2}(u_k, q) \times \exp \left[\sum_{r=1}^{\infty} \bar{Q}^{(r)} \bar{a}^{+(r)} / \sqrt{r} \right] z^H \exp \left[\sum_{r=1}^{\infty} \bar{Q}^{(r)} \bar{a}^{(r)} / \sqrt{r} \right]. \quad (2.10)$$

The vertex Γ is the amplitude with p external scalar particles and two super-states. We can now continue to factorize the various vertices. It is easily seen that the multiperipheral configuration shown in Fig. 3 corresponds to the following expression for the $(p+2)$ amplitude [the notation is defined in Fig. 3]

$$A = \langle 0 | V(\bar{k}_{p+1}) P(H, \alpha_{1,p}(s_{1,p})) V(\bar{k}_p) \times P(H, \alpha_{1,p-1}(s_{1,p-1})) V(\bar{k}_{p-1}) P(H, \alpha_{1,p-2}(s_{1,p-2})) \dots V(\bar{k}_3) P(H, \alpha_{1,2}(s_{1,2})) V(\bar{k}_2) | 0 \rangle, \quad (2.11)$$

where

$$V(\bar{k}_i) = \exp \left[\sum_{r=1}^{\infty} \bar{k}_i \bar{a}^{+(r)} / \sqrt{r} \right] \exp \left[\sum_{r=1}^{\infty} \bar{k}_i \bar{a}^{(r)} / \sqrt{r} \right], \quad (2.12)$$

and

$$\bar{k}_i = (\sqrt{2a} k_i, \sqrt{2c''} b_i''). \quad (2.13)$$

Equation (2.11) is similar to the expression obtained by Fubini, Gordon and Veneziano ⁵⁾ with the important difference that the vectors in the vertex operator (2.12) are five-vectors. Again, we have tacitly understood that Eq. (2.13) is valid when $c'' > 0$, corresponding to the metric $g_{55} = +1$, and if $c'' < 0$ we replace $\sqrt{2c''}$ by $\sqrt{-2c''}$ and choose the metric $g_{55} = -1$.

Following Amati, Bouchiat and Gervais ⁹⁾ we introduce the coherent states ¹⁰⁾ [the annihilation and creation operators correspond to one particular mode with r and μ fixed]

$$|z\rangle = e^{za^+} |0\rangle. \quad (2.14)$$

For the reader's convenience we list some of the well-known properties of these states ^{9),10)}

$$\begin{aligned} \langle z' | z \rangle &= e^{zz'^*}, \\ a|z\rangle &= z|z\rangle, \quad e^{fa}|z\rangle = e^{fz}|z\rangle, \\ e^{z'a^+}|z\rangle &= |z+z'\rangle, \quad x^{a^+a}|z\rangle = |xz\rangle, \\ 1 &= \frac{1}{\pi} \int d\text{Re}z \, d\text{Im}z \, e^{-|z|^2} |z\rangle \langle z|. \end{aligned} \quad (2.15)$$

Using Eqs. (2.15) we can easily check explicitly that Eq. (2.11) reduces to the ordinary five-point function for $p+2 = 5$. We have

$$\begin{aligned}
 B_5(k_1 \dots k_5) &= \langle 0 | V(\bar{k}_4) P(H, \alpha_{13}(s_{13})) \\
 &\times V(\bar{k}_3) P(H, \alpha_{12}(s_{12})) V(\bar{k}_2) | 0 \rangle \\
 &= \int_0^1 dx_1 dx_2 x_1^{-\alpha_{12}(s_{12})-1} x_2^{-\alpha_{13}(s_{13})-1} [(1-x_1)(1-x_2)]^{b-1} \\
 &\times \prod_{r=1}^{\infty} M_r(x_1, x_2, \bar{k}_2, \bar{k}_3, \bar{k}_4), \tag{2.16}
 \end{aligned}$$

where

$$\begin{aligned}
 M_r &= \langle 0 | e^{\bar{k}_4 \bar{a}^{(r)}/\sqrt{r}} x_2^{r \bar{a}^{(r)+} \bar{a}^{(r)}} e^{\bar{k}_3 \bar{a}^{(r)+}/\sqrt{r}} \times \\
 &\times e^{\bar{k}_3 \bar{a}^{(r)}/\sqrt{r}} x_1^{r \bar{a}^{(r)+} \bar{a}^{(r)}} e^{\bar{k}_2 \bar{a}^{(r)+}/\sqrt{r}} | 0 \rangle \\
 &= \exp\left(\frac{1}{r} [\bar{k}_3 \bar{k}_4 x_2^r + \bar{k}_2 \bar{k}_4 x_1^r x_2^r + \bar{k}_2 \bar{k}_3 x_1^r]\right). \tag{2.17}
 \end{aligned}$$

Using

$$\begin{aligned}
 \bar{k}_2 \bar{k}_3 &= 2\alpha k_2 k_3 + 2c'' b_2'' b_3'' \\
 &= \alpha(k_2 + k_3)^2 + 2b + c' (b_2' + b_3') + c'' (b_2'' + b_3'')^2 \\
 &= \alpha_{23}(s_{23}) + b, \tag{2.18}
 \end{aligned}$$

where we used that the external particles are spinless,

$$a k_2^2 + b + c' b_2' + c'' b_2''^2 = 0, \quad (2.19)$$

as well as the relation

$$\alpha_{23}(s_{23}) + \alpha_{34}(s_{34}) - \alpha_{24}(s_{24}) = -\bar{k}_2 \bar{k}_4, \quad (2.20)$$

we obtain from Eqs. (2.16) and (2.17)

$$B_5(k_1, \dots, k_5) = \int_0^1 \int_0^1 dx_1 dx_2 x_1^{-\alpha_{12}(s_{12})-1} x_2^{-\alpha_{13}(s_{13})-1} \\ (1-x_1)^{-\alpha_{23}(s_{23})-1} (1-x_2)^{-\alpha_{34}(s_{34})-1} \\ (1-x_1 x_2)^{\alpha_{23}(s_{23}) + \alpha_{34}(s_{34}) - \alpha_{24}(s_{24})} \quad (2.21)$$

which is the Bardakçi-Ruegg amplitude ¹¹⁾ with general trajectories of the form

$$\alpha_{ik}(s) = as + b + c'(b_i' + \dots + b_k') + c''(b_i'' + \dots + b_k'')^2 \quad (2.22)$$

In a similar way one can check that the factorized expression (2.11) yields the standard expression for the N point function. It is most convenient to give a proof by induction, using the formula

$$A_N = \langle 0 | V(\bar{k}_{N-1}) P(H, \alpha_{N,N-1}) V(\bar{k}_{N-2}) \times \\ \times P(H, \alpha_{N,N-2}) \dots V(\bar{k}_3) P(H, \alpha_{N,3}) V(\bar{k}_2) | 0 \rangle$$

$$\begin{aligned}
 &= \int_0^1 dx \, x^{-\alpha_{N,N-1}-1} (1-x)^{-\alpha_{N-1,N-2}-1} \\
 &\times \prod_{r=1}^{\infty} \langle 0 | V(\bar{k}_{N-2} + \frac{x^r \bar{k}_{N-1}}{\sqrt{r}}) P(H, \alpha_{N,N-2}) V(\bar{k}_{N-3}) \\
 &\quad \dots V(\bar{k}_3) P(H, \alpha_{N,3}) V(\bar{k}_2) | 0 \rangle
 \end{aligned}
 \tag{2.23}$$

where Eqs. (2.15) have been used. Under the integral sign a product of $(N-1)$ point functions enters with k_{N-2} replaced by $k_{N-2} + x^r k_{N-1}/\sqrt{r}$ and b_{N-2}'' replaced by $b_{N-2}'' + x^r b_{N-1}''/\sqrt{r}$.

3. CONSTRUCTION OF PLANAR LOOP AMPLITUDES

In this section we construct the planar loop amplitude with external momenta $p_1, p_2, \dots, p_n$ and loop momenta

$$k_i = k + p_1 + \dots + p_i, \quad k_n \equiv k. \tag{3.1}$$

We follow the same procedure as in Refs. 7), 8), 9). We write the loop amplitude as

$$L = \int d^4 k \, M(k, k_1, \dots, k_{n-1}) \tag{3.2}$$

where M can be obtained from the factorized amplitude (2.11) by considering the poles for $\alpha_T(k^2) = J$. Here T denotes a state of external particles into which the super-state with momentum k can decay. The situation is shown in Fig. 4. We have from Eq. (2.11)

$$M = \text{Tr} \left[P(H, \alpha_T(k^2)) V(\bar{p}_1) \dots \right. \\ \left. \dots P(H, \alpha_{T,n-2}(k_{n-2}^2)) V(\bar{p}_{n-1}) P(H, \alpha_{T,n-1}(k_{n-1}^2)) \right. \\ \left. \times V(\bar{p}_n) \right], \quad (3.3)$$

$$\bar{p}_i = (\sqrt{2a} p_i, \sqrt{2c''} b_i''). \quad (3.4)$$

We rewrite Eq. (3.3) as

$$M = \int \prod_{i=1}^n dx_i x_i^{-\alpha_{T,i-1}(k_{i-1}^2)-1} (1-x_i)^{b-1} \\ \times \prod_{r=1}^{\infty} \text{Tr} \left[x_1^{r\bar{a}^{(r)}\bar{a}^{(r)}} V^{(r)}(\bar{p}_1) \dots x_n^{r\bar{a}^{(r)}\bar{a}^{(r)}} V^{(r)}(\bar{p}_n) \right]. \quad (3.5)$$

The trace in Eq. (3.5) has been evaluated for a particular mode (r, μ) by Amati, Bouchiat and Gervais ⁹⁾, and using their result we obtain easily

$$M = \int \prod_{i=1}^n dx_i x_i^{-\alpha_{T,i-1}(k_{i-1}^2)-1} (1-x_i)^{b-1} \\ \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{\sum_{i,j=1}^n \bar{p}_i \bar{p}_j (C_{ij})^r}{r(1-w^r)} \right] \right\}, \quad (3.6)$$

where

$$C_{ij} = \begin{cases} x_1 \dots x_i & \text{if } j = n \\ (x_1 \dots x_i)(x_{j+1} \dots x_n) & \text{if } i \leq j \leq n-1 \\ x_{j+1} \dots x_i & \text{if } i > j \end{cases} \quad (3.7)$$

$$w = \prod_{i=1}^n x_i \quad (3.8)$$

Instead of the five-vectors $\bar{p}_i$ we can now go back to ordinary four-vectors. Using Eq. (3.4) we obtain

$$M = \int \prod_{i=1}^n dx_i x_i^{-\alpha_{T,i-1}(k_{i-1}^2)-1} (1-x_i)^{b-1} \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{2 \sum_{i,j=1}^n C_{ij}^r (a_i p_i + c'' b_i'' b_j'')}{r(1-w^r)} \right] \right\}. \quad (3.9)$$

4. CONSTRAINTS ON REGGE INTERCEPTS FROM CONVERGENCE OF PLANAR LOOPS

The expression for the planar loop obtained in the previous section is only well defined under certain conditions. To see this we notice that in the limit $w \rightarrow 1$ all $x_i \rightarrow 1$. Hence $C_{ij} \rightarrow 1$, and from Eq. (3.9) we obtain by energy-momentum conservation

$$\begin{aligned}
 & \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{2 \sum_{i,j=1}^n c_{ij}^r (a_{pi} p_j + c'' b_i'' b_j'')}{r (1-w^r)} \right] \right\} \\
 & \sim_{w \sim 1} \exp \left[\frac{5\pi^2}{6} \frac{1}{1-w} + 2c'' \left(\sum_{i=1}^n b_i'' \right)^2 \sum_{r=1}^{\infty} \frac{1}{r(1-w^r)} \right] \\
 & \sim_{w \sim 1} \exp \left[\frac{\pi^2}{6} \frac{1}{1-w} \left(5 + 2c'' \left(\sum_{i=1}^n b_i'' \right)^2 \right) \right]. \quad (4.1)
 \end{aligned}$$

The loop amplitude is therefore only well defined if

$$c'' \left(\sum_{i=1}^n b_i'' \right)^2 < -\frac{5}{2}. \quad (4.2)$$

If all intercepts are universal, i.e., $c' = c'' = 0$, the expression (4.1) is badly divergent, and the loop integral is not well-defined as shown by the authors of Refs. 7), 8), 9). In the following we assume that not all intercepts are equal. The question then arises whether the condition (4.2) is reasonable. The rest of this section is therefore devoted to a discussion of (4.2).

As a first condition for (4.2) to be satisfied we must of course demand that the quantum numbers b_i'' do not sum up to zero. This implies that b_i'' cannot distinguish between particle and anti-particle.

If (4.2) is satisfied, it follows from

$$\alpha_{1n}(0) = b + c' (b_1' + \dots + b_n') + c'' (b_1'' + \dots + b_n'')^2 \quad (4.3)$$

that the intercept of the n body trajectory in general is small or negative. Note, however, that the trajectories on which the external particles lie are not very much restricted by (4.2), since in

$$\alpha_i(0) = b + c' b_i' + c'' b_i''^2, \quad 1 \leq i \leq n, \quad (4.4)$$

we can have b sufficiently large. In particular, if all the external particles are identical, the one-body intercepts $\alpha_i(0)$ are universal, but the many-body intercepts are different from the one-body intercepts. As mentioned in the Introduction this mechanism makes the vertex function a decreasing function of angular momentum in the neighbourhood of $w \sim 1$, as is seen from Eq. (4.1). Therefore the large number of states at a given angular momentum is compensated by a decreasing vertex operator.

One can of course imagine that (4.2) is not satisfied in nature. In that case, one can use a formal variation of c'' to define the amplitude. To see this let us change variables in the loop integrand (3.9) from $x_1, \dots, x_n$ to $x_1, \dots, x_{n-1}, w$. We obtain

$$M = \prod_{i=1}^{n-1} \int_0^1 dx_i \int_0^{x_1 \dots x_{n-1}} dw \, x_i^{-\alpha_{T,i-1}(k_{i-1}^2) - 2} \times (1-x_i)^{b-1} \left(\frac{w}{x_1 \dots x_{n-1}} \right)^{-\alpha_{T,n-1}(k_{n-1}^2) - 1} \left(1 - \frac{w}{x_1 \dots x_{n-1}} \right)^{b-1} \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{2 \sum_{i,j=1}^n C_{ij}^r (a p_i p_j + c'' b_i'' b_j'')}{r(1-w^r)} \right] \right\}. \quad (4.5)$$

Close to $w \sim 1$ we obtain the behaviour ($\epsilon_i < 1, \epsilon < 1$)

$$\prod_{i=1}^{n-1} \int_{1-\epsilon_i}^1 dx_i (1-x_i)^{b-1} \int_{1-\epsilon}^1 dw \left(1 - \frac{w}{x_1 \dots x_{n-1}} \right)^{b-1} \times \exp \left(\frac{\alpha}{1-w} \right), \quad \alpha = \frac{\pi^2}{6} \left[5 + 2c'' \left(\sum_{i=1}^n b_i'' \right)^2 \right]. \quad (4.6)$$

It is easily seen that in the w integral we can approximate $x_1 \sim 1, \dots, x_{n-1} \sim 1$, and as a function of the parameter α we can consider just the single integral

$$I(\alpha) = \int_{1-\epsilon}^1 dw (1-w)^{b-1} e^{\alpha/(1-w)} \\ = -(-\alpha)^b \int_{-\alpha/\epsilon}^{\infty} dy e^{-y} y^{-b-1} . \quad (4.7)$$

For negative values of the parameter α this integral is of course well defined. The question is whether one can continue $I(\alpha)$ to $\alpha > 0$. It is easily seen that if $b < 0$ the y integral in (4.7) is well defined. Hence, let us take $b < 0$, and consider $I(\alpha)$ for $\alpha > 0$. We find

$$I(\alpha) = -e^{i\pi b} \alpha^b \int_0^{\infty} dy e^{-y} y^{-b-1} \\ + \alpha^b \int_{-\alpha/\epsilon}^0 dy e^{-y} (-y)^{b-1} . \quad (4.8)$$

Hence, for $\alpha > 0$ $I(\alpha)$ has a cut. Now the region $w \sim 1$ does not correspond to any dynamical singularities, and the amplitude should therefore not develop singularities for $\alpha \geq 0$. On the other hand, it is clear from the divergences found in Refs. 7), 8), 9) that something must happen when $\alpha > 0$. Since the basic motivation for introducing loop amplitudes is to satisfy unitarity (in a perturbative sense), it is clearly highly undesirable to have singularities which do not correspond to physical singularities. It therefore seems that unitarity demands that condition (4.2) is satisfied. If it turns out that for some reason (4.2) is violated, one is forced to find some interpretation of the "non-dynamical cut" exhibited in (4.8). We mention that if b is a negative integer the cut contracts into a pole. With $b = -1$ we obtain

$$I(\alpha) = (1/\alpha) e^{\alpha/\epsilon} . \quad (4.9)$$

The pole at $\alpha = 0$ is of course also non-dynamical.

It should be pointed out that we have not investigated the restrictions from the various symmetries of the N point function ^{1),2)}. The work in Ref. 8) indicates that the known symmetries only modify the loop integrand in an inessential manner.

It is also worth pointing out that the non-planar diagrams have not been investigated. At present, it is not known whether these diagrams diverge, but it is not unlikely that they may give further information on the Regge intercepts. There also exists the possibility that the non-dynamical cut disappears if one adds planar and non-planar diagrams, in which case it would be possible to get rid of condition (4.2) without conflict with unitarity. However, we have no arguments in favour of such a possibility.

5. DISCUSSION OF MORE GENERAL CONSTRAINTS ON REGGE INTERCEPTS

In I we have pointed out that the N point dual amplitude factorizes even with more general intercepts than given by Eq. (1.1). In order to have a model with a level density which is in agreement with the Hagedorn fit ⁶⁾, one finds that the most general solution for $\alpha_{ik}(0)$ which leads to a factorizable amplitude is given by the constraint [see I, Section 3]

$$\begin{aligned} & -\alpha_{ik}(0) - \alpha_{i+1,k-1}(0) + \alpha_{i,k-1}(0) + \alpha_{i+1,k}(0) \\ & = -\delta_i \delta_k \end{aligned} \tag{5.1}$$

where δ_i refer to channel i . With the constraint (5.1) the loop integrand becomes

$$M = \int \prod_{i=1}^n dx_i x_i^{-\alpha_{T,i-1}(k_{i-1}^2) - 1} \times \prod_{r=1}^{\infty} \left\{ (1-w^r)^{-5} \exp \left[\frac{\sum_{i,j=1}^n C_{ij}^r (2\alpha p_i p_j + \delta_i \delta_j)}{r(1-w^r)} \right] \right\} \quad (5.2)$$

where the cases $i = j$ and $i = j+1$ have to be treated in a special way which is explained in I. As before, Eq. (5.2) is convergent if

$$\left(\sum_{i=1}^n \delta_i \right)^2 < -5. \quad (5.3)$$

This inequality constitutes the most general condition of convergence of the planar loop which is consistent with the Hagedorn density of states.

6. DISCUSSION

The main result of this paper is that the loop diagram can be made well-defined also in the region where all the independent variables approach one, provided one imposes the restriction (5.3) on the Regge intercepts [where δ_i is defined in Eq. (5.1)]. If (5.3) is not satisfied one can still define the amplitude, but a cut emerges and the value which one assigns to the amplitude is therefore rather arbitrary. The way to fix which prescription one should follow is to demand asymptotic Regge behaviour. The asymptotic behaviour of the box diagram is presently under investigation.

Obviously, the most favourable situation is when condition (5.3) is satisfied. In this case there is no conflict with unitarity, and we therefore strongly advocate that this condition should be adopted. Since, in the formalism of I,

$$\alpha_{ik}(0) = b + c'(b'_i + \dots + b'_k) + c''(b''_i + \dots + b''_k)^2, \quad (6.1)$$

condition (5.3) [or (4.2)] imposes the rather weak restriction

$$\alpha_{ik}(0) < b + c'(b'_i + \dots + b'_k) - \frac{5}{2}. \quad (6.2)$$

By suitably adjusting b , c' and b'_i it should be possible to have (6.2) satisfied.

ACKNOWLEDGEMENT

The author takes great pleasure in thanking Professor D. Amati for stimulating criticism.

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FIGURE CAPTIONS

- Figure 1 : Factorization of the $(n+m)$ point amplitude.
- Figure 2 : Factorization of the $(p+f+1)$ point amplitude.
- Figure 3 : Total factorization in the multiperipheral configuration.
- Figure 4 : Factorization corresponding to the loop integrand.

FIG. 1

FIG. 2

FIG. 3

FIG. 4