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ANALYTIC EXTRAPOLATIONS AND THE DETERMINATION
OF PION-PION PHASE SHIFTS^{*×)}

Ján Pišut^{†)}

CERN, Geneva, Switzerland

ABSTRACT

Analytic extrapolations directly or implicitly involved in obtaining the information on the pion-pion scattering are briefly discussed.

A qualitative description is given of ambiguities and instabilities arising in some of the methods.

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- *) An extended Appendix to the companion review by D. Morgan and the author on the pion-pion interactions at low energies.
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 - †) On leave of absence from the Department of Theoretical Physics, Comenius University, Bratislava, Czechoslovakia.

1. INTRODUCTION

Owing to the lack of direct experimental evidence, a reliable picture of pion-pion interaction can only emerge as a consistent result of various indirect methods. These methods are, in general, making substantial use of analyticity properties of amplitudes -- either of those for pion-pion scattering or of others in which the pion-pion interaction plays an important role. On a technical level this amounts to saying that information on pion-pion scattering has to be obtained by some kind of analytic extrapolation.

The instabilities inherent in some of the types of extrapolation are probably the reason why even phenomenological models of pion-pion scattering led to a broad spectrum of frequently contradictory results.

The purpose of the present talk is to give a brief review of various types of analytic extrapolations, and to indicate their connection with some of the methods used to obtain the information about $\pi\pi$ scattering. We shall concentrate mainly on the qualitative description of instabilities and ambiguities.

In the next section we shall give a classification of different types of analytic extrapolation and mention briefly some of the methods in which they are involved. In Section 3 we shall discuss, on the same qualitative level, the ambiguities in the loosely related problem of the determination of phase shifts from enhancement factors in final-state interactions.

2. A CLASSIFICATION OF ANALYTIC EXTRAPOLATIONS

The boundary of the analyticity region differs in many respects from the interior of the analyticity region. It is therefore natural to classify the analytic extrapolations according to the following scheme^{*)}.

2.1 I $\rightarrow$ I extrapolation

In this case the extrapolation is performed from a segment inside the analyticity region to a point inside the analyticity region (in a shorthand notation I stands for Inside). The problem is displayed in a

*) Unless specified otherwise, we have in mind the case of a simply connected region.

schematic way in Fig. 1 for both the typical case of a cut plane and for the unit circle. The mapping of a cut plane onto a finite region frequently simplifies the discussion and provides an insight into the problem^{*)}.

This type of extrapolation is stable in the following sense.

Let $f(z)$ denote the function to be extrapolated; let errors of the data be of the order ϵ ; and let $|f(z)|$ be bounded by a constant M in the whole analyticity region. Then the error of extrapolation to the point a is given as

$$\Delta f(a) \sim \epsilon^{\omega(a)} M^{1-\omega(a)}, \quad (1)$$

where $\omega(z)$ is a so-called harmonic measure, i.e. a real function which satisfies the equation

$$\frac{\partial^2}{\partial x^2} \omega(z) + \frac{\partial^2}{\partial y^2} \omega(z) = 0$$

in the analyticity region (with an additional cut along the segment on which the data are given) and the boundary conditions

$$\begin{aligned} \omega(z) &= 1 && \text{for } z \text{ in "experimental region"} \\ \omega(z) &= 0 && \text{for } z \text{ on the boundary.} \end{aligned} \quad (2)$$

Instead of being more precise in statements contained in Eqs. (1) and (2) or reproducing the proofs^{**)} , we shall stress here the most important points and give a simple illustrative example.

Equations (1) and (2) show that any estimate of the error requires either information or a hypothesis about the bound of the extrapolated function ($|f(z)| < M$) in the whole analyticity region. To show this in a simple way, suppose that the analyticity region is the unit circle,

*) The mapping due to Cutkosky and Deo [1] and Ciulli [2], optimal from the point of view of polynomial expansions, will be mentioned shortly.

**) The Eqs. (1) and (2) are the Nevanlinna principle known from the theory of functions. Its first application to problems in particle physics is apparently due to Bertero and Viano [3]. The derivation of the Nevanlinna principle is given, for example in Refs. [2] and [3].

the experimental region is, say, the segment $\langle -c, c \rangle$, $c < 1$ and one extrapolates to the point a , a real, $c < a < 1$.

Consider now the function

$$\chi(z) = \varepsilon' \exp \left\{ \left[z - \frac{a+c}{2} \right] \frac{2}{a-c} \ln \frac{K}{\varepsilon'} \right\}, \quad (3)$$

where K and ε' are positive and otherwise arbitrary. By construction, $\chi(z)$ is analytic in the unit circle,

$$|\chi(z)| < \varepsilon' \quad \text{for } z \in (-c, c) \quad (4)$$

$$\chi(a) = K.$$

Choosing $\varepsilon' \ll \varepsilon$, we can add $\chi(z)$ to any solution of the extrapolation problem without spoiling the fit [since errors on $(-c, c)$ are supposed to be of the order of ε]. Because of Eq. (4) we change at the same time the result of extrapolation by K (and so far K is arbitrary). Requiring $|f(z)|$ to be bounded by M in the whole unit circle, one restricts K , and as a consequence restricts also the error of extrapolation. The resulting error of course agrees with Eqs. (1) and (2). A somewhat more complicated example which just "saturates" the bounds given by the Nevanlinna principle is given by Bertero and Viano [3].

A glance at Eqs. (1) and (2) (or the preceding example) shows that the extrapolation to a point on the boundary is basically unstable since the error of extrapolation becomes proportional to the bound M on $|f(z)|$ and not to the experimental error ε . This is the reason why the extrapolation to the boundary is a more delicate problem which has to be treated separately.

The most important application of the $I \rightarrow I$ extrapolation to $\pi\pi$ physics is the Chew-Low-Goebel [4,5] extrapolation, which we shall now discuss together with methods suitable for this type of extrapolation.

2.1.1 The Chew-Low-Goebel extrapolation

The problem can roughly be described as follows: a function $f(t)$ is analytic in the complex t -plane with a cut $t_0 < t < \infty$ (pion mass units, the cut is due to the 3π intermediate state in the $\bar{N}\bar{N} \rightarrow 3\pi$ channel, and for simplicity we forget about possible left hand cut due to crossed channel).

At $t = -1$, $f(t)$ is simply related to the pion-pion cross-section or to one of the coefficients in the Legendre polynomial decomposition of the angular distribution for the dipion decay. The experimental data are available along the segment $0 < t < t_0$. The situation is displayed in Fig. 2a.

In applications of the CLG method performed so far, $f(t)$ has been constructed as a polynomial in t and the coefficients determined by the fit to the data in the experimental region^{*)}.

It is, however, well known that such a procedure is not the best thing to follow. About ten years ago Frazer [6] and Ciulli and Fischer [7] proposed to use conformal mapping techniques for phenomenological extrapolation. Frazer dealt with the problem of determining the coupling constant of a particle which is exchanged in a given reaction by extrapolation in the variable $z = \cos \theta$, where θ is the scattering angle. He proposed to fit the data not by a polynomial in the original variable z but by a polynomial in $v = v(z)$, where $v(z)$ is a function which maps conformally the original analyticity region onto the unit circle (see Fig. 2b). He demonstrated by a numerical example that a fit by a polynomial of the fourth order in the original variable may be no better than a fit by a polynomial of the second order in a new variable v . The procedure is, of course, equivalent to performing first the conformal mapping $v = v(z)$ and then expanding the experimental data into a polynomial in the variable v .

Recently, Cutkosky and Deo [1] and Ciulli [2] improved the original proposal of Frazer [6] and of Ciulli and Fischer [7], and found the conformal mapping leading to the optimal asymptotic convergence of polynomial expansions. Even if the formal mathematical proofs assure that one has only the asymptotically optimal solutions, the method, as already exemplified by Frazer, may lead to marked improvements in fits by low-order polynomials. The use of these methods in future practical extrapolations is to be highly recommended.

*) The error is then simply derived from statistical errors of these coefficients. We hope that the example given above indicates clearly that this estimate of error is not conservative enough.

In a rough but hopefully transparent way, the idea of the optimal mapping can be described as follows^{*)}: suppose that instead of knowing the data on a segment (as it is always in realistic cases) we know that a function $f(z)$ is analytic in a region shown in Fig. 3a and we have an exact information about the first N -derivatives of $f(z)$ in the origin. Denoting

$$C_n = f^{(n)}(0) [n!]^{-1}$$

we can write an approximate expression for $f(a)$ as

$$f(a) \approx f_N(a) = \sum_0^N C_n a^n. \quad (5)$$

If $f(z)$ is bounded by M in the analyticity region $\mathcal{D}$ and on the boundary L , we have from the Cauchy theorem

$$C_m = \frac{1}{2\pi i} \int \frac{f(z')}{z'^{m+1}} dz', \quad (5')$$

and hence

$$|C_n| < M b^{-n}, \quad (6)$$

where b is the distance from the origin to the nearest singularity. For the remainder of the approximation in Eq. (5) we can easily obtain the estimate

$$|f(a) - f_N(a)| \leq M \frac{b}{b-a} \left(\frac{a}{b}\right)^{n+1}. \quad (7)$$

Suppose now that a mapping $v = v(z)$ is performed [such that $v(0) = 0$]. In the new variable one can again construct the Taylor series and find the remainder similar to that given by Eq. (7). The new variable will be "asymptotically better" if the remainder becomes asymptotically smaller than in the original case. This is equivalent to the requirement that $|a'/b'|$ be smaller than a/b , where a' is the image of a and b' is the distance from the origin (in the v -plane) to the nearest singularity.

*) For a full account of relevant mathematics, the reader is referred to original papers by Cutkosky and Deo [1] and Ciulli [2], and to the excellent book by Walsh [8].

It is easy to believe^{*)} that the "asymptotically optimal" mapping for this case is the one shown in Fig. 3b, e.g. the mapping of the cut z -plane onto the unit circle. This mapping has two important properties:

- i) The Taylor series converges in the new variable for any point inside the analyticity region (compare points d and d' in Figs. 3a and 3b).
- ii) The upper bound of the remainder given in Eq. (7) is asymptotically the smallest in the v -plane.

The oversimplified point in the preceding example lies in the fact that the information (or the data) were assumed to be given in only one point. In a more realistic case when data are given on a segment, the optimal mapping was given by Cutkosky and Deo [1] and by Ciulli [2]. In Ciulli's formulation one makes an additional cut along the "experimental region" (see Fig. 1) and maps the whole analyticity region (with two cuts) shown in Fig. 1 onto the annular region shown in Fig. 4a. If, as in most cases, the function is real in the data region in the original variable, it will be real on the unit circle in Fig. 4a. The appropriate expression for expanding the amplitude is then

$$w = \frac{1}{2}(v + v^{-1}) \quad (8)$$

which is by construction real for $|v| = 1$. Interpreting Eq. (8) as a conformal mapping, one obtains the form of the optimal mapping given by Cutkosky and Deo [2] (shown in Fig. 4b). Cutkosky and Deo's and Ciulli's mappings are completely equivalent, and both have the virtues (i) and (ii) mentioned above. For practical applications the Cutkosky and Deo mapping is perhaps more convenient since the amplitude is then decomposed (fitted) on the segment $-1 \leq w \leq 1$ by more familiar orthogonal polynomials.

As an alternative to polynomial expansions, a method based on using the weight functions was proposed in Ref. [9]. The basic idea goes as follows: denote as C_e the "experimental region" shown in Fig. 4a, and as C the boundary of the analyticity region. Let $f(v)$ be the function which should be extrapolated from C_e to the point v_1 . Let $g(v)$ be any function analytic for $1 \leq |v| \leq R$ and such that $g(v_1) = 1$.

*) From the symmetry arguments.

On the basis of the Cauchy theorem, one has

$$f(v_1) = \frac{1}{2\pi i} \int_{C_e} \frac{f(v)g(v)}{v - v_1} dv = \frac{1}{2\pi i} \int_C \frac{f(v)g(v)}{v - v_1} dv . \quad (9)$$

Choosing now $g(v)$ so that it is small on C [to suppress our ignorance of the function $f(v)$ on the boundary] we can obtain the estimate of $f(v_1)$. A simple example of a weight function $g(v)$ with this property is

$$g(v) = (v_1/v)^n \quad (9')$$

with n being a positive integer. The optimal weight function of this kind is given in Ref. [10].

2.1.2 The estimate of errors in $I \rightarrow I$ extrapolation

The most serious pitfall of analytic extrapolations lies in the estimate of errors. Let us start again with a simple example. Suppose that $f(z)$ is analytic in the unit circle and that the derivatives at the origin are known with experimental errors up to the order N , so that one can write

$$C_n \equiv f^{(n)}(0) [n!]^{-1} = C'_n + \varepsilon_n \quad n = 0, 1, \dots, N .$$

Here C_n are the true values, C'_n the experimental data, and ε_n the errors. For a value of $f(z)$ in a point a one then obtains

$$f(a) - f'_N(a) = \sum_0^N \varepsilon_n a^n + \sum_{N+1}^{\infty} C_n a^n , \quad (10)$$

where

$$f'_N(a) = \sum_0^N C'_n a^n , \quad (10')$$

is the estimate of $f(a)$ obtained from the "data", e.g. from the information about C'_n for $n = 0, 1, \dots, N$. The first term on the right-hand side of Eq. (10) can be treated as a statistical error, while the second term is a typical extrapolation error. As shown in Eqs. (5), (6), and (7) one can estimate the extrapolation error if $f(z)$ is known to be bounded on the boundary by a constant M . If such information is not available one has to guess the behaviour of the coefficients C_n for $n > N$ from the behaviour of C_n with $n \leq N$.

A proposal by Cutkosky and Deo [1] consists essentially in estimating the constant M in Eq.(6) from the coefficients C_n with $n \leq N$ and using this M also for $n > N$. Even if this is a guess it is apparently the best that can be done when *a priori* information about $f(z)$ on the boundary is not available.

The situation gets a bit worse in a realistic situation when the data are given on a segment and not idealized as a set of derivatives in a point. The discussion of the estimates of errors is, however, similar to the simplified example just described, and an interested reader is referred to Bertero and Viano [3], for example, or to the last section of Ref. [9] for details^{*)}.

An important question connected with estimating the error of extrapolation can be formulated as follows:

When are the data good enough for performing the Chew-Low-Goebel extrapolation?^{**))}

In performing a CLG extrapolation one actually separates the contribution of the one-pion exchange from that of other mechanisms. It is therefore advisable to choose only reactions which are known (from the study of t -dependence of cross-sections and from the behaviour of density matrix elements) to be dominated by OPE^{***)}. A quantitative insight into the question whether the data are able to reveal the presence of the pion pole and give some information about its residue can be obtained along the following lines.

Coming back to a simple example. Suppose that $f(z)$ is analytic in the unit circle; we know the coefficients C_n in its Taylor expansion in

*) The troubles with estimating errors of analytic extrapolations come, in general, from difficulties connected with merging statistical concepts with analyticity. A pioneering step in this direction was recently taken by Cutkosky [1].

**) The author is indebted to F. James for valuable discussions concerning this point. The following argument follows the reasoning by Cutkosky and Deo [1] on similar points.

***) The related questions are discussed in detail in excellent reviews by G. Kane [12].

the origin and we suspect that the function has a pole in a point b , $b < 1$. According to Eq. (6) it holds that

$$|C_n| < M \quad (11a)$$

if there is no singularity inside the unit circle, and

$$|C_n| < Mb^{-n} \quad (11b)$$

if there is a singularity in a point b . If the information about C_n is good enough to favour at a pragmatic level the alternative (11b), it implies that the data "feel" the presence of the pole and the extrapolation makes sense. In a more realistic situation when the data are given along a segment, the argument is unchanged; one only needs to study the dependence on n of coefficients in polynomial representation of the data, instead of C_n 's described here.

2.2 I → B extrapolation

The extrapolation from a segment inside the analyticity region to the boundary is shown in a schematic way in Fig. 5. The nature of the problem is illustrated by the following example. Suppose that $f(z)$ is analytic in the unit circle, data are given on the segment $(-\frac{1}{2}, \frac{1}{2})$, and we wish to extrapolate to the point $z = 1$. Let errors of the data be of the order ϵ .

Adding a function

$$\chi(z) = K \frac{\epsilon'}{1 + \epsilon' - z} \quad (12)$$

to any solution of the extrapolation problem does not spoil the fit to the data on the segment $(-\frac{1}{2}, \frac{1}{2})$, provided that $\epsilon' < \epsilon$, but changes the result of extrapolation to the point $z = 1$ by an arbitrary^{*)} number K .

The derivatives of $\chi(z)$ for $z = 1$ are given by the relation

$$\chi^{(n)}(1) = \frac{K}{(\epsilon')^n} \quad (13)$$

*) If $f(z)$ is known to be bounded by M for $|z| \leq 1$, $|K|$ should naturally be less than M .

This relation illustrates one possible way of avoiding the instability of the extrapolation to the boundary. If the derivatives of the extrapolated function are required to exist on the boundary up to a given order n_0 and be bounded by a constant, this in turn restricts the value of K in Eq. (13) and assures stability. A method for extrapolation to the boundary along these lines was discussed by Ciulli [2]. In cases when bounds on derivatives are not available, one has to use another approach^{*)} which consists in extrapolating, not to values of a function at points on the boundary, but to average values over arcs on the boundary. The reason why the extrapolation to average values over arcs is stable is intuitively clear from the example of $\chi(z)$ as given by Eq. (12). Even if $\chi(1) = K$, any average value of $\chi(z)$ along an arc longer than ε' would be of the order of $(\varepsilon'K)$.

The results of Refs. [14] and [10] show that the stability of extrapolation increases with the increase of the length of the arc over which the average value is calculated.

Instances where $I \rightarrow B$ extrapolation is employed, in the context of $\pi\pi$ scattering are

- i) models for $\pi\pi$ scattering above threshold incorporating the constraints of current algebra (hard-pion theory);
- ii) models for $\pi\pi$ scattering based on Martin inequalities and other rigorous constraints valid below the threshold.

2.3 $B \rightarrow I$ extrapolation

This is shown in Fig. 6. The methods applicable to this situation are described in Refs. [15] to [18]. The error of the extrapolation is formally given by Eq. (1) where the harmonic measure is this time given by the conditions

$$\begin{aligned}\omega(z) &= 1 && \text{on the "experimental" part of the boundary} \\ \omega(z) &= 0 && \text{on the "unknown" part of the boundary.}\end{aligned}$$

The extrapolation becomes perfectly stable (in the sense that the error of extrapolation is proportional to the error of the data) when the "experimental" region is identical to the whole boundary. This is the case

*) This is actually the approach used by Bowcock et al. [13] and in a different formulation in Refs. [14], [10].

of various sum rules expressing coupling constants (or values of amplitudes at a particular point) as integrals of scattering amplitudes over the whole physical region.

The $B \rightarrow I$ problem is just the inverse of $I \rightarrow B$ extrapolation. Thus $B \rightarrow I$ extrapolation can, for example, be used for testing consistency of $\pi\pi$ phase shifts with Martin inequalities or with current algebra predictions.

2.4 $B \rightarrow B$ extrapolation (see Fig. 7)

Mathematically the problem of the stability is similar to that of the $I \rightarrow B$ case. The solution for the case where restrictions on derivatives are imposed on the boundary is most probably feasible, but to the best of our knowledge has not, so far, been discussed in the literature. The optimal solution for extrapolation to average values along arcs on the boundary is given by Prešnajder et al. [10].

In practice, one uses suitable weight functions [14-19], and one applies the Cauchy theorem to the product of the weight function and the function to be extrapolated. The results [10] can be summarized as follows. The region shown in Fig. 7 is mapped onto the one shown in Fig. 8. Then if the error in the experimental region is of the order ε , the error of the mean value over the segment $< -1, 1 >$ in Fig. 8, obtained with the optimal weight function, is proportional to $\varepsilon^{(\alpha/\pi)}$. Note that for $\alpha = \pi$ the situation (as expected) is perfectly stable.

Actual extrapolations to the boundary (both cases $I \rightarrow B$ and $B \rightarrow B$) have, so far, been performed almost exclusively in the following way: the amplitudes were first written in forms containing a few free parameters which were determined later on from fits in the "experimental region". Due to the instabilities inherent in the problem, results are expected to be very sensitive to the form of the parametrization chosen.

An honest solution to these complications was given by Lovelace, Heinz and Donnachie [20] in their determination of δ_0^0 from πN backward dispersion relations^{*)}. Figure 9 shows a limited sample of the set of phase

*) For other references pertaining to this approach, see the companion paper by D. Morgan and the author.

shifts obtained by them in different parametrizations. It is clear that average values of corresponding amplitudes over the range from threshold to about 1 GeV are all more or less the same, while averages taken over smaller intervals, say from threshold up to 350 MeV, are still dependent on the parametrization chosen. As an alternative to this type of solution, one could calculate directly the average values of the $I = J = 0$ $\pi\pi \rightarrow N\bar{N}$ partial wave amplitude.

The problems where one is faced with a $B \rightarrow B$ type of extrapolation are numerous. For instance:

- a) the determination of the left-hand cut singularities from the right-hand cut ones (Bowcock and John [21]);
- b) predictions of $\pi\pi$ phase shifts from πN backward scattering;
- c) predictions of $\pi\pi$ phase shifts from πN partial wave dispersion relations;
- d) the determination of $\pi\pi$ low-energy behaviour from forward dispersion relations. In this case the extrapolation ($B \rightarrow B$ type) is not the full story, since at low energies s-waves dominate and the requirement of the unitarity will probably reduce the ambiguities to something like CDD-pole ambiguities. At present there is, however, no mathematical theory for this problem^{*)}.

In all these applications quite contradictory results can be found in the literature. The major explanation lies probably in the fact that the extrapolation to the boundary is, in a local sense, basically instable, and as a consequence the results of calculations are strongly dependent on the parametrizations used.

3. AMBIGUITIES IN THE DETERMINATION OF PHASE SHIFTS FROM FINAL STATE ENHANCEMENT FACTORS

Suppose, for simplicity, that there are only two strongly interacting particles in the final state. A common procedure is to make an ansatz for the phase shift δ (the two particles are supposed to be in a state with given angular momentum) containing a few free parameters, then to construct an enhancement factor^{**)} .

*) The author is indebted to D. Atkinson for a few agreeable discussions about these questions.

**) Various enhancement factors and their relations are discussed in, for example, Jackson and Kane [22], and Jacob, Mahoux and Ommès [23].

$$E(s) = \exp \{R(s) + i\delta(s)\} \quad (14)$$

with

$$R(s) = \frac{s - s_0}{\pi} P \int_{s_0}^{\infty} \frac{\delta(s')}{(s' - s_0)(s' - s)} ds' \quad (15)$$

and insert it into the amplitude^{*)}. If there is no other final-state interaction, the invariant mass distribution of the two strongly interacting particles is given as

$$\frac{d\sigma}{dm} \sim |E(s)|^2 = \exp \{2R(s)\} . \quad (16)$$

Using Eq. (16) one fits the data in a finite region, say $s_0 < s < s_1$, determines the free parameters, and obtains the phase shift $\delta(s)$. It is natural to ask whether the procedure leads to an unambiguous result, or whether the result may be biased by the parametrization chosen.

An obvious ambiguity is exhibited by the transformation

$$\delta(s) \rightarrow \delta'(s) = \delta(s) + C\sqrt{s - s_0} \quad (17)$$

since the principal value integral with the square root term explicitly gives zero. The example may be considered as unrealistic since $\delta'(s) \rightarrow \infty$ for $s \rightarrow \infty$. It is, however, easy to construct a general class of functions $\delta(s)$ which will lead to the same $R(s) = |E(s)|^2$ for $s_0 < s < s_1$. Let us first perform the conformal mapping $z = z(s)$ shown in Fig. 10 and take any function of the form

$$\Delta(s) = \text{Re} \left\{ \sum_{n=0}^N a_{2n} z^{2n} \right\} . \quad (18)$$

*) The expression in Eq. (14) corresponds to $D^{-1}(s)$ in the N/D method. Other enhancement factors contain also effects of the left-hand cut or other, slowly varying functions of s .

The corresponding part of the enhancement factor is then given as^{*)}

$$R(s) = -\text{Im} \left\{ \sum_0^N a_{2n} z^{2n} \right\} . \quad (19)$$

Taking the coefficients a_{2n} as real, it obviously follows that

$$\Delta(s) = \sum_0^N a_{2n} [z(s)]^{2n} \quad \text{for } s_0 < s < s_1 \quad (20)$$

and

$$R(s) = 0 \quad \text{for } s_0 < s < s_1 . \quad (21)$$

The addition of $\Delta(s)$ to the phase shift $\delta(s)$ does not change $R(s) \equiv |E(s)|^2$ on the interval $s_0 < s < s_1$, while it can change the phase shift in a substantial way^{**)}. Note further that $\Delta(s)$ is finite by construction for $s \rightarrow \infty$, although it shows oscillations, which can become physically unacceptable for large N . In a fitting procedure the phase shift $\delta(s)$ may not approach the true phase shift but one of the class

$$[\delta_{\text{true}}(s) + \Delta(s) + c\sqrt{s - s_0}] ,$$

depending on the way in which $\delta(s)$ was parametrized.

From this point of view the results on $\pi\pi$ phase shifts coming from the analysis of $\bar{n}p \rightarrow 3\pi$ are preferable to those from $\pi N \rightarrow \pi\pi N$ below the threshold for $\Delta(1238)$ production, since in the former case the phase shifts δ_ℓ^I are responsible for the interference of various amplitudes, and the preceding argument does not apply. On the other hand, a three-body final state offers better possibilities of neglecting important three-body diagrams.

4. COMMENTS

The complications that arise when obtaining reliable information on low-energy parameters by explicit or tacit analytic extrapolations are

*) Note that $\Delta(s)$ and $R(s)$ are the imaginary and the real parts of an analytic function for $s_0 < s < s_1$.

***) The whole ambiguity is due to the trivial fact that an analytic function is not given in an unambiguous way by values of its real part along a part of the boundary.

not limited to the pion-pion case. They are, however, most pronounced here since a clear-cut resolution of ambiguities by experimental data is difficult and perhaps impossible owing to the lack of direct evidence. A critical attitude towards some of the results obtained is probably necessary if a consistent picture is finally to emerge. In the present brief review of pitfalls of extrapolations, we have restricted ourselves only to the description of a few basic points; for details about the extrapolation methods the reader should consult the references quoted above, and for illustrative examples from the field of pion-pion interactions the companion lecture by D. Morgan and the author on pion-pion scattering at low energies.

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REFERENCES

- 1) R.E. Cutkosky and B.B. Deo, Phys. Rev. 174, 1859 (1968), and Phys. Rev. Letters 22, 1272 (1968).
- 2) S. Ciulli, Nuovo Cimento 61 A, 787 (1969); Nuovo Cimento 62 A, 301 (1969).
- 3) M. Bertero and G.A. Viano, Nuovo Cimento 38, 1915 (1965).
- 4) G.F. Chew and F.E. Low, Phys. Rev. 113, 1640 (1959).
- 5) C. Goebel, Phys. Rev. Letters 1, 337 (1958).
- 6) W.R. Frazer, Phys. Rev. 123, 2180 (1961).
- 7) S. Ciulli and J. Fischer, Nuclear Phys. 24, 465 (1961).
- 8) J.L. Walsh, Interpolation and approximation by rational functions in the complex domain, Amer. Math. Soc. Coll. Publ. Vol. XX, published by the Amer. Math. Soc., Providence Rhode Island, 1956.
- 9) J. Pišut and P. Prešnajder, Nuclear Phys. B12, 110 (1969).
- 10) P. Prešnajder and J. Pišut, CERN preprint TH.1158 (1970).
- 11) R.E. Cutkosky, Ann. Phys. (N.Y.) 54, 350 (1969).
- 12) G.L. Kane, Proc. Argonne Conf. on $\pi\pi$ and πK Scattering, 1969. Also, talk given at the Philadelphia Conference on $\pi\pi$ and πK Scattering, 1970 (Michigan preprint).
- 13) J.E. Bowcock, W.N. Cottingham and J.G. Williams, Nuclear Phys. B3, 95 (1967).
- 14) J. Pišut, P. Prešnajder and J. Fischer, Nuclear Phys. B12, 586 (1969).
- 15) T. Carleman, Sur les fonctions quasianalytiques, Paris 1926.
- 16) M.M. Laurentief, On some incorrect problems of mathematical physics (Russian), published by Acad. of Sci., Novosibirsk 1962 (see also Springer Tracts, Vol. 11).
- 17) P. Prešnajder and J. Pišut, Nuclear Phys. B14, 489 (1969).
- 18) S. Ciulli and J. Fischer, NORDITA preprint 1970.
- 19) J. Fischer, J. Pišut, P. Prešnajder and J. Šebesta, Czech. J. Phys. 19, 1486 (1969).
- 20) C. Lovelace, R.M. Heinz and A. Donnachie, Phys. Letters 22, 332 (1966).

- 21) J.E. Bowcock and G. John, Nuclear Phys. B11, 659 (1969).
- 22) J.D. Jackson and G.L. Kane, Nuovo Cimento 23, 444 (1962).
- 23) M. Jacob, G. Mahoux and R. Omnès, Nuovo Cimento 23, 838 (1962).

Figure captions

- Fig. 1 : Extrapolation from data (denoted by flowers) inside the analyticity region to a point a inside the analyticity region. The extrapolation is referred to as $I \rightarrow I$ type. The upper half shows a typical case of a cut plane, the lower half is the conformal mapping of the cut plane onto the unit circle.
- Fig. 2 : a) Analytic properties of functions extrapolated by the Chew-Low-Goebel method. The region where data are available is denoted by flowers.
b) The conformal mapping of the cut t -plane of Fig. 2a onto the unit circle.
- Fig. 3 : A simple example illustrating the idea of the optimal mapping.
- Fig. 4 : a) Ciulli's form of the optimal mapping for cases similar to the Chew-Low-Goebel extrapolation.
b) Cutkosky and Deo's form of the optimal mapping.
- Fig. 5 : Extrapolation from data inside the analyticity region to the boundary. The case is referred to as $I \rightarrow B$ extrapolation.
- Fig. 6 : Analytic extrapolation from a part of the boundary to a point inside the analyticity region (referred to as $B \rightarrow I$ extrapolation).
- Fig. 7 : Extrapolation from a part of the boundary to the rest of the boundary ($B \rightarrow B$ extrapolation).
- Fig. 8 : The conformal mapping used in extrapolation to average values along arcs on the boundary in the $B \rightarrow B$ case.

Fig. 9 : $\pi\pi \delta_0^0$ phase shift obtained by Lovelace, Heinz and Donnachie from πN backward dispersion relations.

Fig.10 : Ambiguities in determining phase shifts from enhancement factors. The upper part shows the original s -plane, the lower part its mapping onto the semicircle in the z -plane. The enhancement factor is supposed to be known experimentally for $s_0 < s < s_{1+}$.

FIG. 1

FIG. 2a

FIG. 2b

FIG. 3a

FIG. 3b

v - plane
FIG. 4a

w - plane
FIG. 4b

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG. 9

FIG.10