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EXTRAPOLATION OF SCATTERING AMPLITUDES
IV. EXTRAPOLATION OFF A PART OF THE BOUNDARY
TO THE REST OF THE BOUNDARY

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A B S T R A C T

The problem of extrapolating an amplitude (or a form factor) off a part of the boundary of the analyticity region to the rest of the boundary is considered. To obtain a stable procedure we extrapolate to average values over an arc on the boundary and to values in a point. In defining the average value we introduce weight functions analytic in the analyticity region of the amplitude. The expression for the average value is then simply obtained from the Cauchy theorem applied to the product of the weight function and the amplitude. The optimal weight functions (corresponding to a given arc) are found for cases of simply and doubly connected analyticity regions. For the latter case we present (in Appendix C) also the optimal solution to the problem of extrapolation to points inside the analyticity region.

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1. INTRODUCTION

The present paper concludes the series ¹⁾⁻³⁾ (referred to as I-III hereafter) dealing with analytic extrapolations of form factors and scattering amplitudes. In I and II we discussed the extrapolation of an amplitude from a segment (a "physical region") lying inside the analyticity region. The extrapolation off a part of the boundary to points inside the analyticity region was dealt with in III. To complete the picture we shall consider here the extrapolation off a part of the boundary to the rest of the boundary. It is easy to find situations corresponding to such a mathematical problem. Suppose, for instance, that a forward scattering amplitude of some reaction is known in the low energy region. To infer thereof an information about the behaviour of the amplitude in another (say, intermediate - or high) energy region means to extrapolate the amplitude from one part of the boundary to another one. The same problem is encountered in an attempt to determine the discontinuity of a partial wave amplitude over the left-hand cut if phase shift and elasticity are known in the physical region (on the right-hand cut).

In any realistic situation, one starts with the data subject to experimental errors and the stability of the extrapolation procedure becomes a crucial point ^{*)}.

The problem of stability of an extrapolation up to the boundary is much more difficult than that of extrapolating to points inside the analyticity region. In a context of an application to particle physics this was clearly realized by Bertero and Viano ⁷⁾. Using the Nevanlinna principle [see also Ref. 5] and making no additional assumptions about the "decent" behaviour of the extrapolated function on the boundary, they showed that the extrapolation to the boundary is basically unstable.

^{*)} In fact, the present increase of interest to problems of analytic extrapolation was stimulated by Cutkosky and Deo ⁴⁾ and Ciulli ⁵⁾ (extrapolations based on polynomial expansions). Their method leads to the best asymptotic rate of convergence and, as shown by Ciulli and Cutkosky ⁶⁾, also to the best stability.

A simple illustrative example of this fact is given in the introduction of II. The proof given in Ref. 7) applies to an extrapolation off a segment inside the analyticity region. It is easy to obtain the same result for an extrapolation off a part of the boundary. One only needs to use the form of the Nevanlinna principle appropriate for this case (see, e.g., Section 3 in III) and repeat almost literally the procedure given in Refs. 5) and 7). The simple example presented in the introduction of II is also readily modified to the new situation.

There are essentially two ways to overcome this difficulty. The former, analogous to the method used in Ref. 5) would consist in imposing some additional requirements upon the behaviour of the extrapolated function on the boundary. The latter discussed in II *) calculates as a result of extrapolation, not a value of the extrapolated function in a point on the "unknown" part of the boundary, but an average value over a prechosen arc.

The problem in which we are concerned is generally described in the following way.

Let an amplitude or a form factor $f(w)$ be analytic in a region D . The boundary of D is denoted as Γ and consists of Γ_e [the "experimental" or "known" part on which there are some data about $f(w)$ available] and Γ_u (the "unknown" part). Denoting the data by $f_e(w)$ and corresponding errors by $e(w)$ we can write **)

$$f(w) = f_e(w) + e(w) \quad (1)$$

for $w \in \Gamma_e$.

Let γ be an arc on Γ_u and suppose that we are interested in the average value of $f(w)$ over γ . The class of weight functions which can be used for this purpose is denoted as $W[D, \gamma]$ and defined as follows. A function $g(w)$ belongs to $W[D, \gamma]$ if

*) Related ideas have also been expressed by Bowcock et al. 8). Our approach is, in a sense, the optimization of the procedure of Bowcock and John 8). We are indebted to Professor A. Martin for this reference.

**) A brief discussion about this way of representing the data is given in Section 4.

- i) $g(w)$ is analytic in D ;
- ii) the expression $g(w)dw$ is real and positive on γ ;
- iii) the Cauchy theorem along the contour Γ is valid ^{*)} for the product $f(w)g(w)$.

Using the notation

$$N \equiv N(g, \gamma) = \int_{\gamma} g(w) dw \quad (2)$$

and

$$[f] = \frac{1}{N} \int_{\gamma} f(w) g(w) dw \quad (3)$$

we obtain as a consequence of the Cauchy theorem the following relation

$$[f] + \frac{1}{N} \int_{\Gamma_e} f e(w) g(w) dw = A[\gamma, g(w)] \quad (4)$$

where

$$A[\gamma, g(w)] = -\frac{1}{N} \int_{\Gamma_e} e(w) g(w) dw - \frac{1}{N} \int_{(\Gamma_u - \gamma)} f(w) g(w) dw \quad (4')$$

The term $[f]$ is interpreted ^{**)} as an average value of $f(w)$ over the arc γ , the second term on the left-hand side of Eq. (4) is the estimate of $[f]$ obtained from the experimental data and $A[\gamma, g]$ is the extrapolation error done by using the weight function $g(w)$.

The function $g(w)$ should naturally be chosen so as to minimize $|A[\gamma, g]|$. In an actual situation, we are lacking the accurate knowledge of $f(w)$ on $(\Gamma_u - \gamma)$ and of $e(w)$ on γ . We can then only minimize an upper bound of $|A[\gamma, g]|$ which can be calculated on the basis of this incomplete knowledge of $f(w)$ and $e(w)$; such an upper bound can be obtained in different ways. Here we shall adhere to the kind of estimate used in I-III. Denoting

*) A full discussion of questions related to item iii) is given in Privalov's book ⁹⁾.

**) The possibility to interpret $[f]$ as an average value over γ is ensured by requirement ii) on functions $g(w)$ belonging to the class $W[\gamma, D]$.

$$\int_{(\Gamma_u - \delta)} |f(w)| |dw| = M, \quad \int_{\delta} |e(w)| |dw| = \varepsilon M$$

$$\max_{(\Gamma_u - \delta)} |g(w)| = Q, \quad \max_{\delta} |g(w)| = Q \quad (5)$$

we can write

$$|A[\delta, g]| \leq \frac{M}{N} (\varepsilon Q + Q) = E \quad (6)$$

This estimate is rather conservative and corresponds to the minimal information ^{*)} about $f(w)$ on $(\Gamma_u - \delta)$ and about $e(w)$ on δ . The function $g(w) \in W[\bar{D}, \delta]$ which minimizes the bound E of the extrapolation error [given by Eq. (6)] will hereafter be called the optimal weight function.

The purpose of the present paper is to find these optimal weight functions for the case of a simply connected region D and for a special case of a doubly connected D . The solution to the former problem is given in Section 2 and the solution to the latter is presented in Section 3. The results obtained are briefly commented on in Section 4. A few technical details are deferred to Appendices.

2. EXTRAPOLATION TO AVERAGE VALUES OVER ARCS OF THE BOUNDARY OF A SIMPLY CONNECTED REGION

In the present section we shall specify the general formulation of the problem to the case of a simply connected region and find the optimal solution. Applications of this case to particle physics cover, for instance, the case of a forward scattering amplitude analytic in the w plane with cuts $-\infty < w < w_-$ and $w_+ < w < \infty$ and the case of

*) A different estimate of E was used, e.g., by Ciulli and Fischer ¹⁰⁾ in a context related to III. To compare the results obtained with different estimates of E , one can compare the results of Ref. 10) and III.

real (or imaginary) part of a scattering amplitude at a fixed energy taken as an analytic function of variable $w = \cos\theta$ where θ is the scattering angle in the centre-of-mass system.

Let $f(w)$ be analytic in a simply connected region D . The boundary Γ of D consists of two parts: the "experimental" or "known" one Γ_e , and the unknown one Γ_u . We are interested in the average value of $f(w)$ along an arc γ , $\gamma \in \Gamma_u$. The situation is displayed in Fig. 1a. Before actually performing the extrapolation, it is advantageous to use a conformal mapping $z = z(w)$ which maps a region D onto the upper half of the unit circle ^{*}) in a way shown in Fig. 1b. Images of the region D and of arcs Γ_e , γ and $(\Gamma_u - \gamma)$ are denoted as S, C_1, C_2 and C_3 respectively.

Proceeding as indicated in Section 1, we now take a function $g(z) \in W[C_2, S]$ (e.g., analytic in S and real positive on C_2) and introduce the notation

$$f(z) = f_e(z) + e(z) \quad \text{for } z \in C_1 \quad (7)$$

$$\int_{C_3} |f(z)| |dz| = M(\pi - \alpha) \quad , \quad \int_{C_1} |e(z)| |dz| = \epsilon \alpha M \quad (8)$$

$$Q = \max_{z \in C_1} |g(z)| \quad , \quad q = \max_{z \in C_3} |g(z)| \quad (9)$$

and

$$[f] = \frac{1}{N} \int_{-1}^1 g(a) f(a) da \quad N = \int_{-1}^1 g(a) da \quad (10)$$

*) The mapping which is likely to be useful for practical situations is given in some detail in Appendix A.

The Cauchy theorem applied to the product $f(z)g(z)$ along the contour $C_1 \cup C_2 \cup C_3$ leads after simple manipulations to the following inequality

$$|[f] + \frac{1}{N} \int_{C_1} f e(z) g(z) dz| \leq E \equiv \frac{M}{N} [(\pi - \alpha) q + \alpha \varepsilon Q] \quad (11)$$

The optimal weight function is by definition that one which minimizes E . Note first that $g(z)$ is analytic not only in the region S but at least in the unit circle $K: |z| < 1$.

This follows from the fact that $g(z)$ is real for z real, $-1 < z < 1$ and from the Schwarz reflection principle. In the lower half of the unit circle $g(z)$ is then given by the prescription $g(z) = g^*(z^*)$.

In a search of the optimal $g(z)$ we can restrict ourselves to functions with no zeros in $\bar{S}$ (and consequently in K). To prove that, suppose first the $g(z)$ has a zero of l^{th} order in a point z_0 , $\text{Im } z_0 \neq 0$. We can then define a function

$$G(z) = \left[\left(\frac{1 - z z_0}{z - z_0} \right) \left(\frac{1 - z z_0^*}{z - z_0^*} \right) \right]^l g(z) \quad (12)$$

For $|z| = 1$ it holds $|G(z)| = |g(z)|$ and consequently Q and q corresponding to $G(z)$ and to $g(z)$ are the same. It is also simple to see that N given by Eq. (10) is larger for $G(z)$ than for $g(z)$. The upper bound E corresponding to $G(z)$, which has no zeros, is thus smaller than E corresponding to the original function $g(z)$. The same conclusion holds true if $g(z)$ has a zero of $2l^{\text{th}}$ order on the real segment $-1 \leq z \leq 1$. One only needs to put $z_0 = z_0^*$ into Eq. (12). Note that zeros of an odd order on the real axis are forbidden since $g(z)$ has to be positive for $-1 \leq z \leq 1$.

If $g(z)$ has more zeros, one defines the function $G(z)$ in a way similar to (12) with the only change that the multiplicative factor in front of $g(z)$ is replaced by a corresponding product.

The optimal weight function may then be written in a form

$$g(z) = \exp[P(z)] \quad (13)$$

where $P(z)$ is real analytic in the unit circle K . Denoting

$$B = \max_{C_1} \operatorname{Re} P(z), \quad b = \max_{C_3} \operatorname{Re} P(z) \quad (14)$$

we have

$$Q \equiv \max_{C_1} |g(z)| = e^B, \quad q \equiv \max_{C_3} |g(z)| = e^b. \quad (14')$$

A function $g(z)$ is said to belong to the class $C(b, B)$ if

- i) $g(z)$ is real analytic in the unit circle K ;
- ii) $g(z)$ has no zeros in K ;
- iii) Eqs. (13), (14) and (14') are valid.

According to Eq. (11), the weight function $g(z)$ optimal within the class $C(b, B)$ is the one leading to the maximal value of

$$N \equiv N(b, B) = \int_{-1}^1 e^{P(a)} da \equiv \int_{-1}^1 e^{\operatorname{Re} P(a)} da. \quad (15)$$

Since $\operatorname{Re} P(z)$ is harmonic in K one can use the maximum principle ^{*}) for harmonic functions to show that the $g(z)$ optimal within the class $C(b, B)$ satisfies the following relations

$$\operatorname{Re} P(z) = B \quad \text{for } z \in C_1, \quad \operatorname{Re} P(z) = b \quad \text{for } z \in C_3. \quad (16)$$

^{*}) Only the following consequence of the maximum principle is actually used: if $P_1(z)$ and $P_2(z)$ are harmonic in a region D and if it holds (on the boundary Γ of D) $P_1(z) \leq P_2(z)$, then this inequality is valid for any $z \in D$.

Due to the properties of $P(z)$, Eqs. (16) define $P(z)$ in an unambiguous way. Proceeding in a way similar to that used in III, one obtains the explicit expression

$$P(z) = b + \frac{\alpha}{\pi}(B-b) - \frac{B-b}{i\pi} \lg \frac{1+z^2 e^{i\alpha}}{1+z^2 e^{-i\alpha}} \quad (17)$$

Inserting Eq. (17) into Eq. (13), we arrive at $g(z)$ optimal within the class $C(b, B)$. The upper bound E on the extrapolation error corresponding to this $g(z)$ is calculated from Eq. (11) :

$$E \equiv E(c) = M \frac{\alpha \varepsilon \exp(\frac{\alpha c}{\pi}) + (\pi - \alpha) \exp(\frac{\alpha - \pi}{\pi} c)}{\int_{-1}^1 da \exp \left\{ - \frac{2c}{\pi} \arg(1 + a^2 e^{i\alpha}) \right\}} \quad (18)$$

where $c = B - b$.

To find the optimal weight function $g(z)$ then amounts to finding the minimum of $E(c)$ as a function ^{*)} of $c = B - b$. This problem cannot be solved in an analytic way but it is not a difficult task for a computer.

In order to discuss the stability of the proposed procedure, we have to use an estimate of the optimal value (denoted as c_0) of c . Since c_0 is expected to depend strongly on ε and the denominator in Eq. (18) is independent of ε , one hopes to obtain a reasonable estimate of c_0 by minimizing the numerator in Eq. (18). This results in

$$c_0 \approx \lg \frac{1}{\varepsilon} \quad (19)$$

*) Adding the same constant to both b and B means to multiply $g(z)$ as given by Eqs. (13) and (16) by a constant. This naturally does not change results of extrapolation. The bound E depends then only on $c = B - b$.

Inserting Eq. (19) into Eq. (18) and denoting $E(c_0)$ as E_0 we obtain

$$E_0 \leq M \varepsilon^{\alpha/\pi} \left[\int_0^1 da \varepsilon^{\frac{2}{\pi} \arg(1+a^2 e^{i\alpha})} \right]^{-1} \quad (20)$$

Making use of inequalities

$$\arg(1+a^2 e^{i\alpha}) = \arctg \frac{a^2 \sin \alpha}{1+a^2 \cos \alpha} \leq a$$

and

$$\begin{aligned} \int_0^1 da \exp \left\{ -\frac{2}{\pi} \left(\lg \frac{1}{\varepsilon} \right) \arg(1+a^2 e^{i\alpha}) \right\} &\geq \\ &\geq \int_0^1 \exp \left\{ -\frac{2}{\pi} a \lg \frac{1}{\varepsilon} \right\} da \geq \frac{1}{1 + \frac{2}{\pi} \lg \frac{1}{\varepsilon}} \end{aligned}$$

we get from Eq. (20) the following upper bound on E_0 (for $\varepsilon < 1$):

$$E_0 \leq M \varepsilon^{(\alpha/\pi)} \left(1 + \frac{2}{\pi} \lg \frac{1}{\varepsilon} \right) \quad (21)$$

If $\alpha > 0$, e.g., if the arc along which the data are given does not reduce to a point, the extrapolation is stable. This follows from Eq. (21) since $E_0 \rightarrow 0$ for $\varepsilon \rightarrow 0$. It is also easily seen that if the arc over which the average value is taken is enlarged, the angle α in Eq. (21) (see also Fig. 1b) increases and the stability of the extrapolation is improved.

3. THE CASE OF A DOUBLY CONNECTED REGION

The approach described above can also be applied to doubly connected regions. The idea of the procedure is the same, so we shall start with introducing the notation. Let $f(w)$ be analytic in a doubly

connected region D bounded by two curves Γ_e and Γ_u (see Fig. 2a). The curve Γ_e is supposed to be the "experimental" part of the boundary and Γ_u the "unknown" part. We are interested in the average value of $f(w)$ along an arc γ , $\gamma \in \Gamma_u$. Prior to an actual extrapolation it is advantageous to map conformally D onto the annular region S in the z plane. Images of Γ_e , γ and $(\Gamma_u - \gamma)$ are denoted as C_1, C_2 and C_3 respectively. The radius of C_1 is chosen to be 1, and the radius of the internal circle is denoted as r , $r < 1$. The problem consists in finding the average value of $f(z)$ over C_2 if data and errors are given on C_1 (see Fig. 2b). Note [see 5) and II] that this formulation includes also extrapolations of a segment inside a simply connected analyticity region to the boundary (see Fig. 2c).

Mathematical formulation parallels that of Section 2. It is assumed that

$$f(z) = f_e(z) + e(z), \quad z \in C_1 \quad (22)$$

where f_e represents data and $e(z)$ errors.

Constants M and ε are defined as

$$\int_{C_3} |f(z)| |dz| = M(\pi - \alpha), \quad \int_{C_1} |e(z)| |dz| = \varepsilon \alpha M. \quad (23)$$

A weight function $g(z)$ is required to be analytic in S and real and positive on C_2 . The average value $[f]$ of $f(z)$ along C_2 is defined as

$$[f] = \frac{1}{N} \int_{C_2} f(a) g(a) \frac{da}{ia} \quad (24)$$

where

$$N = \int_{C_2} g(a) \frac{da}{ia} \quad (24')$$

The factor $(ia)^{-1}$ is taken out of the weight function to make the term $da/(ia)$ real and positive. The Schwarz reflection principle enables to continue $g(z)$ through C_2 into the region G shown in Fig. 3. The radius of C_4 is equal to r^2 .

The Cauchy theorem

$$\int_C f(z)g(z) \frac{dz}{iz} = 0, \quad C = C_1 \cup C_2 \cup C_3 \quad (25)$$

leads to the inequality

$$\left| [f] - \frac{1}{N} \int_{C_1} f_e(z)g(z) \frac{dz}{iz} \right| \leq \frac{M}{N} \left[(\pi - \alpha) \frac{q}{r} + \alpha \varepsilon Q \right] = E \quad (26)$$

where

$$q = \max_{C_3} |g(z)| \quad \text{and} \quad Q = \max_{C_1} |g(z)| \quad (27)$$

The optimal weight function is again defined as the one which minimizes the upper bound E on the extrapolation error.

It is easy to show that E corresponding to $g(z)$ is not smaller than that corresponding to $g'(z) = (\frac{1}{2})[g(z) + g^*(z^*)]$ which is real analytic by construction. In a search for the optimal $g(z)$ we can therefore restrict ourselves to real analytic functions. Proceeding as in Section 2, we define a class $C(q, Q)$ consisting of functions which are real analytic in S and satisfy Eqs. (27) with q and Q kept fixed. The weight function $g(z)$ optimal within the class $C(q, Q)$ is according to Eq. (26) the one leading to the maximal value of N . A basis for a construction of this function is provided by the work of Golusin¹¹⁾. In order to apply his results, we map conformally the region S shown in Fig. 2b onto the region S' in the v plane shown in Fig. 4. [The mapping $v = v(z)$ is given in Appendix B.] Curves C_i are mapped onto K_i ; $i = 1, 2, 3$. The radius of K_1 is chosen to be 1 and that of K_3 is denoted as ϱ , $\varrho < 1$. The cut K_2 is the segment $\varrho \leq v \leq \varrho + a$.

The optimal function $g(z)$ is real analytic in S and real on C_2 . Due to the symmetry of regions S and S' it holds, $v(z) = v^*(z^*)$. The optimal function $g(z)$ does not have the cut K_2 and is analytic in the annular region S'' bounded by two concentric circles K_1 and K_3 . The function $g(z)$ optimal within the class $C(q, Q)$ should thus satisfy the following conditions :

- i) $g(v)$ is real analytic in S'' and positive for $\rho < v < \rho + a$;
 - ii) $\max_{K_3} |g(v)| = q$, $\max_{K_1} |g(v)| = Q$;
 - iii) the integral $\int_{\rho}^{\rho+a} g(v) \left| \frac{dz}{dv} \right| |dv|$ attains the maximal value.
- This means that N corresponding to the optimal $g(z) \in C(q, Q)$ is maximal.

The problem formulated in i)-iii) is easily reduced to the one studied by Golusin ¹¹⁾ and the solution is given as

$$g_0(v) = Q v^n G(v, p) \quad (28)$$

where n is an integer and p is a real number, $\rho < p \leq 1$. Both p and n are determined in an unambiguous way from the condition

$$q = Q \rho^n p \quad (28')$$

and the function $G(v, p)$ is expressed ¹¹⁾ in terms of the Riemann theta function as

$$G(v, p) = v \frac{\Theta\left(\frac{\rho v}{p}, p\right)}{\Theta\left(\frac{pv}{\rho}, p\right)} \quad (28'')$$

Coming back to the variable z we can write the optimal weight function $g_0(z)$ [still within the class $C(q, Q)$] in the following form

$$g_0(z) = Q v^n(z) G[v(z), p] \quad (29)$$

where p and n are determined from Eq. (28'). Noting that a multiplication of $g(z)$ by a constant does not change the results of extrapolation we can restrict ourselves to the class $C(1, Q)$. The optimal

function within $C(1, Q)$ is denoted as $g_{0,Q}(z)$ and the corresponding bound E on the extrapolation error as $E(Q)$. In order to find the optimal weight function, we have to find the value of Q which leads to minimal $E(Q)$. The minimization can again be performed only in a numerical way.

In order to discuss, without extensive numerics, the stability of the method we have to content ourselves with an approximation to the optimal function $g_0(z)$. The complicated Golusin function $G(v, p)$ satisfies according to Ref. 11) the following inequality :

$$\rho < p \leq |G(v, p)| \leq 1 \quad \text{for } z \in S. \quad (30)$$

We can thus hope to obtain a reasonable approximation to the optimal weight function by replacing $G(v, p)$ by a constant. Multiplicative constants are irrelevant and we can therefore write

$$g_0(z) \approx g_n(z) = v^n(z) \quad (31)$$

The value of n (an integer) is determined by minimizing E . Inserting Eq. (31) into the right-hand side of Eq. (26), we obtain

$$E \equiv E(n) = M \frac{(\pi - \alpha)(\rho^n/r) + \varepsilon \alpha}{2 \int_0^\alpha v^n(re^{i\varphi}) d\varphi} \quad (32)$$

In deriving Eq. (32), use was made of the relation $q = Q\rho^n$, valid for $g_n(z)$ and of the equality

$$\int_{C_2} g_n(a) \frac{da}{ia} = 2 \int_0^\alpha v^n(re^{i\varphi}) d\varphi.$$

What remains is to minimize $E(n)$ with respect to n . Even this simplified problem can only be solved in a numerical way. A reasonable approximation of optimal n is obtained from the requirement that both terms in the numerator in (32) be equal. This leads to the relation

$$n_0 \approx \frac{\lg\left(\frac{\pi-\alpha}{\varepsilon r^\alpha}\right)}{\lg(1/\rho)} \quad (33)$$

To estimate the integral in the denominator in (32) we have to approximate the function $v(z)$. Within a rough approximation we can simply put

$$v(re^{i\varphi}) \approx \rho + a - \frac{a}{\alpha} \varphi, \quad 0 \leq \varphi \leq \alpha \quad (34)$$

This linear ansatz satisfies requirements $v(r) = \rho + a$, $v(re^{i\varphi}) = \rho$ following from the definition of the mapping $v = v(z)$. The approximation is expected to be realistic at least for small α and a . Inserting Eq. (34) into the integral in Eq. (32) we obtain

$$\int_0^\alpha v^m(re^{i\varphi}) d\varphi = \frac{\alpha}{a(m+1)} \rho^{m+1} \left[\left(1 + \frac{a}{\rho}\right)^{m+1} - 1 \right]$$

Inserting that into Eq. (32) and expressing n_0 by Eq. (33) we arrive after simple manipulations at the estimate

$$E(n_0) \approx M \varepsilon \frac{a}{\rho \lg(1/\rho)} \quad (35)$$

For a small arc C_2 over which the average value is taken, the α is also small and it holds $a \approx k \alpha$ with k being a constant. The estimate of the upper bound on the extrapolation error can thus be written as

$$E \lesssim M \varepsilon L^\alpha \quad (35')$$

L constant. The inequality in Eq. (35') is due to the fact that we have always replaced optimal quantities by estimates. Equation (35') shows distinctly that an extrapolation to an average value over an arc is stable and that the stability is the better the longer the arc is.

It is instructive to compare the stability attained by using the optimal weight function with those obtained by using weight functions restricted from the outset to certain simple classes. If $g(z)$ is restricted to be a polynomial, one obtains (see Appendix B of II)

$$E \sim \lg\left(\frac{1}{\varepsilon}\right) \quad (36)$$

while for $g(z)$ of the form

$$g(z) = \exp\{C_1[z + z^{-1}] + C_2\}$$

with C_1 and C_2 constant, the upper bound E is estimated to be [see Eq. (16) of II]

$$E \sim \varepsilon^{D\alpha^2}$$

where D is a constant and α has the same meaning as in Eq. (35'). In both cases the stability is worse than that given by Eq. (35). This is not quite surprising since both $g(z)$ in a form of a polynomial or in the form (36') are not flexible enough to approach closely the shape of the optimal weight function.

4. COMMENTS AND CONCLUSIONS

A feature common to the present paper and to I-III is the representation of experimental data and errors in the form of Eqs. (1), (2) and (22). This presumes the data and upper bounds on errors to be given as continuous functions. Such an assumption is in a sense opposite to the usual representation of data by points and error bars at definite values of the variable in question (say, energy, or the scattering angle). This "point-like" representation of data is in many cases inexact, too, since the variable is only known with some inaccuracy. The "continuous" description we are using is well suited for the data given in a form of a histogram. In cases when the variable is measured accurately and data

are given in points separated by large distances, the present method can only be used if additional assumptions about the smooth behaviour of the extrapolated function are accepted. If the function is assumed to be smooth, one can construct continuous functions $f_e(z)$ and $e(z)$ representing data and errors. It should, however, be stressed that this "smoothness" assumption is always (even if tacitly) done. If for instance the data are given in n points and in the extrapolation or fitting procedure one uses polynomial of n^{th} order, one tacitly assumes that all the derivatives starting with the $(n+1)^{\text{th}}$ one are equal to zero. It is clear that with data given only in n points, smoothness assumptions (tacit or explicit) are simply a necessity, since an analytic function is never determined by values in a finite number of points.

A general feature of the present approach which should be improved lies in the way the error of extrapolation was estimated. Both here and in I-III we have calculated upper bounds on errors. In deriving bounds on extrapolation error, we have in fact assumed that errors of the data or some quantity simply related thereto (e.g., the average of absolute values of errors along the "experimental" part of the boundary) cannot exceed a fixed quantity. In a similar way, we supposed that the average of the absolute value of the extrapolated function over the "unknown" part of the boundary cannot exceed a given number. These assumptions do not reflect properly the statistical nature of experimental errors. The drawback could be remedied by using the approach proposed recently by Cutkosky ¹²⁾ based on probability concepts rather than on upper bounds.

Optimal solutions to problems formulated in Sections 2 and 3 involve either in the conformal mapping used or in the final result complicated functions which necessitates a computer handling of the problem. At the price of lowering the stability, one could use simpler methods described, e.g., in II and sketched briefly in the last section of III. In order to obtain the same accuracy of results by using a less stable extrapolation method one needs to increase the accuracy of the data what practically implies the increase of the number of events. An estimate of the labour and cost of the experimental work involved might show in many cases the preference of using a more stable, even if also more cumbersome, extrapolation procedure. This is apparently one of the reasons why so much attention was recently paid to problems of stability, Refs. 3)-6), 10).

Possible applications of the present approach to some problems in particle physics were already mentioned in I-III, where an interested reader is referred to.

Finally, turning to another subject, let us complete the list of references given in III. After having read the proofs of III, we have learnt that the idea of the method proposed there was known in mathematical literature, where it was used in solving the Laplace equation in two dimensions with Cauchy boundary conditions. The method is described by Lavrentjev ¹³⁾ and the idea can be traced back to Carleman ¹⁴⁾.

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A P P E N D I X A

We shall briefly sketch here a way how to perform the mapping leading from Fig. 1a to Fig. 1b. According to the Riemann theorem it is always possible to map the region D of Fig. 1a onto the region D' shown in Fig. 5a, where the angle δ is given by the geometry in Fig. 1a. The mapping depends on the shape of D and is well known for practically important cases (e.g., when D is the whole plane with cuts $\langle -\infty, w_- \rangle$ and $\langle w_+, +\infty \rangle$). As a next step, we map the region D' of Fig. 5a onto the region D'' of Fig. 5b. Note that images of Γ_e and γ are symmetrical in Fig. 5b with respect to the imaginary axis. The mapping is performed as follows. In general a mapping of the unit circle onto itself $v = v(w)$ is given unambiguously by three conditions $v_i = v(w_i)$, $i = 1, 2, 3$ and $|v_i| = |w_i| = 1$. The explicit formula reads

$$\frac{v - v_1}{v - v_2} \cdot \frac{v_3 - v_2}{v_3 - v_1} = \frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} \quad (\text{A.1})$$

Inserting here $w_1 = 1$, $w_2 = -1$, $w_3 = i$ and $v_1 = 1$, $v_2 = -1$, $v_3 = \exp i(\frac{\pi + \beta}{2})$ we obtain

$$\frac{v - 1}{v + 1} \cdot \frac{1 + \exp[i(\pi + \beta)/2]}{-1 + \exp[i(\pi + \beta)/2]} = \frac{1}{i} \cdot \frac{w - 1}{w + 1} \quad (\text{A.2})$$

The free parameter β is then determined from the requirement that $w = \exp(i\delta)$ is mapped into $v = \exp[i(\pi - \beta)/2]$. This leads to the condition

$$\operatorname{tg}^2\left(\frac{\pi - \beta}{4}\right) = \operatorname{tg} \frac{\delta}{2} \quad (\text{A.3})$$

which assigns a unique β , $0 < \beta < \pi$ to any δ , $0 < \delta < \pi/2$.

To obtain the situation shown in Fig. 1b from that in Fig. 5b, we map first the unit circle of Fig. 5b onto the upper half of the t plane :

$$t = \frac{1}{i} \frac{v-1}{v+1} \quad (\text{A.4})$$

Taking the square root $u = \sqrt{t}$ and performing the mapping

$$z = - \frac{u-i}{u+i} \quad (\text{A.5})$$

we finally arrive at the situation shown in Fig. 1b. The angle α in Fig. 1b is related to β as follows

$$\operatorname{tg} \frac{\alpha}{2} = \frac{1 - \operatorname{tg} \frac{\pi - \beta}{4}}{2 \sqrt{\operatorname{tg} \frac{\pi - \beta}{4}}} \quad (\text{A.6})$$

A P P E N D I X B

In constructing the mapping $v=v(z)$ used in Section 3, we shall proceed in three steps. The region S shown in Fig. 2b is first mapped onto the u plane with two cuts along the real axis. The cut u plane is then mapped onto the c plane with again two cuts along the real axis. The cut c plane is finally mapped onto the region S' shown in Fig. 4. Conformal mappings used in this procedure are defined as follows.:

The function

$$z = Z_k(u) = i \exp \left\{ \frac{i\pi}{2K} \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \right\} \quad (B.1)$$

maps the upper half of the cut u plane shown in Fig. 6a onto the upper half of the annular region given in Fig. 2b. The symbol K entering Eq. (B.1) is defined as

$$K \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (B.1')$$

and the square root in Eq. (B.1) is chosen to be

- i) real positive for $-1 < t < 1$;
- ii) purely imaginary positive for $-1/k < t < -1$ and $1 < t < 1/k$;
- iii) real negative for $-\infty < t < -1/k$ and $1/k < t < \infty$.

The radius of the external circle in Fig. 2b is fixed to be 1 and that of the internal one is denoted as r . According to (B.1) it holds

$$r = \exp \left\{ -\frac{\pi}{2} \cdot \frac{K'}{K} \right\} \quad (B.2)$$

where

$$K' = \int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} \quad (\text{B.2'})$$

The mapping inverse to $z = Z_k(u)$ is denoted as

$$u = U_k(z) \quad (\text{B.3})$$

Recalling the definition of the angle α in Figs. 2b and 4 we further denote

$$u_\alpha \equiv U_k(re^{i\alpha/2})$$

As a further step we need a mapping of the cut u plane shown in Fig. 6a onto the cut c plane shown in Fig. 6b. The part of the cut lying between 1 and $1/q$ will finally be mapped onto the cut K_2 in Fig. 4. The mapping shown in Figs. 6a and 6b is given by the relation [cf., Eq. (A.1)] .:

$$\frac{c+1}{c-1} \frac{1+q}{1-q} = \frac{u+1}{u-1} \frac{1+k}{1-k} \quad (\text{B.4})$$

where q is determined from the condition $c(u_\alpha) = 1/q$. This means

$$\left(\frac{1+q}{1-q} \right)^2 = \frac{(u_\alpha+1)(1+k)}{(u_\alpha-1)(1-k)} \quad (\text{B.4'})$$

Using the mapping given by Eqs. (1) we now map the upper half plane of the cut c plane onto the upper half of the region S' shown in Fig. 4. Explicitly, we have

$$v = V_q(c) = i \exp \left\{ \frac{i\pi}{2Q} \int_0^c \frac{dt}{\sqrt{(1-t^2)(1-q^2t^2)}} \right\} \quad (\text{B.5})$$

where square roots are chosen as described after Eq. (B.1') and

$$Q = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-q^2 t^2)}} \quad (\text{B.5'})$$

Radius ρ of the internal circle in Fig. 4 is given as

$$\rho = \exp \left\{ -\frac{\pi}{2} \frac{Q'}{Q} \right\} \quad (\text{B.6})$$

where

$$Q' = \int_1^{1/2} \frac{dt}{\sqrt{(t^2-1)(1-q^2 t^2)}} \quad (\text{B.6'})$$

the function

$$v = v(z) = V_q [\mathcal{C}(U_k(z))] \quad (\text{B.7})$$

maps the upper part of the annular region S onto the upper part of the region S' . The lower part of S is mapped onto the lower part of S' by the following relation

$$v = v^*(z^*) = V_q^* [\mathcal{C}(U_k(z^*))] \quad (\text{B.7'})$$

The mapping $v = v(z)$ given by Eqs. (B.7) and (B.7') is the desired result.

Note marginally that the region G shown in Fig. 3 is mapped by Eqs. (B.7) and (B.7') onto a Riemann surface with two sheets connected across the cut $K_2 = \langle \rho, \rho+a \rangle$, where $a = v[r \exp(i\alpha/2)]$.

A P P E N D I X C

The problem we are here concerned with is formulated as follows ^{*)}

- i) the function $f(z)$ is analytic in the annular region D . The boundary of D consists of two circles : C (with the radius $r_C = 1$) and C' (with the radius $r_{C'} \equiv r$, $r < 1$);
- ii) experimental data $f_e(z)$ together with errors $e(z)$ are given along the circle C thus

$$f(z) = f_e(z) + e(z) \quad \text{for } z \in C, \quad (C.1)$$

the "average error" is known to be bounded

$$\int_C |e(z)| |dz| = 2\pi \varepsilon M \quad (C.2)$$

where the constant M is given by the relation

$$\int_{C'} |f(z)| |dz| = 2\pi M r \quad (C.2')$$

and is supposed to be known or, at least, an upper bound on M is available;

- iii) we are interested in the value of $f(z)$ in a point a inside the analyticity region, a real, $r < a < 1$. Practically we wish to express $f(a)$ in terms of $f_e(z)$, $z \in C$ and estimate the extrapolation error.

Let $g(z)$ be analytic in D and $g(a) = 1$. Functions of this type will be said to belong to the class W . Since $f(z)g(z)$ is analytic in D it holds

$$f(a) = \frac{1}{2\pi i} \int_{C \cup C'} \frac{f(z)g(z)}{z-a} dz \quad (C.3)$$

*) Appendix C presents an improvement on the method given in I. The formulation of the problem and discussion of its relevance for practical purposes are given in I.

Let us write $g(z)$ as a product

$$g(z) = g_1(z) g_2(z) \quad , \quad g_1, g_2 \in W$$

and choose g_1 so that it holds

$$\begin{aligned} |g_1(z)| &= k \left| 1 - \frac{z}{a} \right| & \text{for } z \in C' \\ |g_1(z)| &= k \left| 1 - \frac{a}{z} \right| & \text{for } z \in C \end{aligned} \quad (C.4)$$

where k is a constant. To find $g_1(z)$ we construct a function $H(z)$ which : i) is analytic in D [and need not obey the condition $H(a)=1$] and ii) satisfies Eq. (C.4) with $k=1$. In terms of $H(z)$ we can write

$$g_1(z) = \frac{H(z)}{H(a)} \quad , \quad k = \frac{1}{H(a)} \quad (C.5)$$

Note that $g_1(z)$ is given only by the position of the point a and has nothing to do with the particular shape of the function to be extrapolated. The function $H(z)$ is explicitly given by the formula

$$H(z) = \prod_{k=1}^{\infty} \left\{ \frac{[1 - (az/r^2)r^{2k}][1 - (1/az)r^{2k}]}{[1 - (a/z)r^{2k}][1 - (z/a)r^{2k}]} \right\}^{1/2} \quad (C.5')$$

Equation (C.5') is derived in the following way : we construct first a function

$$\Omega(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

analytic in D and satisfying the requirements

$$\begin{aligned} \operatorname{Re} \Omega(z) &= \lg \left| 1 - \frac{z}{a} \right| & \text{for } z \in C' \\ \operatorname{Re} \Omega(z) &= \lg \left| 1 - \frac{a}{z} \right| & \text{for } z \in C \end{aligned}$$

Making the Taylor expansions in the variable (z/a) for $z \in C'$ and in (a/z) for $z \in C$ and comparing them with the Laurent series for $\Omega(z)$ we determine coefficients a_n and obtain

$$\Omega(z) = - \sum_1^{\infty} \left\{ \frac{1}{2m} \left[(az)^m \left(1 - \frac{r^{2m}}{a^{2m}} \right) + \left(\frac{r^2}{az} \right)^m (1 - a^{2m}) \right] \frac{1}{1 - r^{2m}} \right\}$$

Expanding $(1 - r^{2n})^{-1}$ into the Taylor series and changing the order of summations, we obtain after elementary manipulations the expression

$$\Omega(z) = \frac{1}{2} \lg \prod_{k=1}^{\infty} \left\{ \frac{[1 - (az/r^2)r^{2k}][1 - (1/az)r^{2k}]}{[1 - (a/z)r^{2k}][1 - (z/a)r^{2k}]} \right\}$$

Putting $H(z) = \exp \Omega(z)$ we arrive at Eq. (C.5').

The function $g_1(z)$ was introduced to facilitate the search for the optimal weight function $g(z) = g_1(z)g_2(z)$. Now we proceed further and determine the function $g_2(z)$. Using the notation

$$q = \max_{c'} |g_2(z)|, \quad Q = \max_c |g_2(z)| \quad (C.6)$$

we obtain from Eqs. (4)-(6) the following relation

$$\left| f(a) - \frac{1}{2\pi i} \int_c f(z) g_1(z) g_2(z) dz \right| \leq kME \quad (C.7)$$

where

$$E = (r/a)q + \varepsilon Q \quad (C.8)$$

The function $g_2(z)$ is now chosen so as to minimize the upper bound E .

In solving this problem, we shall use the Golusin function [see Eq. (28)]

$$G(z, p) = z \frac{H\left(\frac{rz}{p}, r\right)}{H\left(\frac{pz}{r}, r\right)} \quad (C.9)$$

Within the class of functions $G(z)$ analytic in D and satisfying inequalities

$$\begin{aligned} |G(z)| &\leq 1 \quad \text{for } z \in C, \\ |G(z)| &\leq p \quad \text{for } z \in C' \end{aligned} \quad (C.10)$$

$G(z, p)$ has the maximal absolute value at the point $z = a$. This enables us to find a function $G_n(z, p)$ which

- i) is analytic in D ;
- ii) obeys inequalities

$$\begin{aligned} |G_n(z, p)| &\leq 1 \quad \text{for } z \in C \\ |G_n(z, p)| &\leq p r^n \quad \text{for } z \in C' \end{aligned} \quad (C.11)$$

where n is an integer and $r < p \leq 1$;

- iii) attains the maximal value at the point $z = a$ (under conditions stated above).

This function is given by the relation

$$G_n(z, p) = z^n G(z, p). \quad (C.12)$$

The function $g_2(z)$ is required to be

- i) analytic in D
 - ii) equal to one for $z = a$.
- (C.13)

Proceeding as before, we define a class C_Q , so that $g_2(z)$ belongs to C_Q if $g_2 \in W$, Eq. (C.13) is valid and for $z \in C$, $\max |g_2(z)| = Q$. By definition, $W = UC_Q$ holds.

Requiring $g_2(z) \in C_Q$ to minimize E we find

$$g_2(z) = (z/a)^n G(z, p) [G(a, p)]^{-1} \quad \text{for } g_2 \in C_Q$$

values of n and p are determined from relations

$$Q = (r/a)^n \frac{f_0}{G(a,p)}, \quad Q = \left(\frac{1}{a}\right)^n \frac{1}{G(a,p)}, \quad \frac{Q}{Q} = r^n p \quad (C.14)$$

The search for optimal values of n and p parallels that discussed in the text. The final result for the extrapolation error is again found to agree with the Nevanlinna principle. Even if the actual extrapolation based on this method requires computer work, it leads to more accurate results than the one proposed in I, where the class of weight functions considered was rather restrictive.

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FIG.1a

FIG.1b

FIG.2a

FIG.2b

FIG.2c

FIG.3

FIG.4

FIG.5a

FIG.5b

n-plane

FIG.6a

c-plane

FIG.6b