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COMMUTATORS OF CURRENT DENSITY OPERATORS:
ASYMPTOTIC EXPANSION WITH FIXED NEGATIVE PHOTON MASS SQUARED

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A B S T R A C T

We present a group theoretical approach to the $O(3,1)$ analysis of commutators of conserved and non-conserved currents.

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1. COVARIANCE

We consider the absorptive part of the amplitude for the forward scattering of a spacelike "charged" photon with momentum k

$$k^2 = -\mu^2 < 0, \mu > 0$$

on an elementary particle with spin s and mass M

$$\begin{aligned} M_{\mu\nu}^{ab}(k, p)_{q_2 q_1} &= \\ &= N^{-2} \int d^4x e^{ikx} \langle p, q_2 | [j_\mu^a(x), j_\nu^b(0)] | p, q_1 \rangle \end{aligned} \quad (1)$$

N is a normalization factor [in the notations of ¹⁾]

$$N^2 = \frac{2M}{(2\pi)^3}, \quad \langle p_2, q_2 | p_1, q_1 \rangle = 2p_1^0 \delta^{(3)}(p_2 - p_1) \delta_{q_2 q_1}$$

The superscripts a, b describe the "charge" in a very general sense. For example, they may label the parity and a component of an $SU(3)$ multiplet.

We show that the amplitude (1) generates a covariant operator of $SL(2, C)$ in the sense of the definition introduced in ²⁾. This operator is decomposed into irreducible covariant operators by means of an expansion theorem derived in ²⁾. Such decomposition can be used to deduce an asymptotic expansion of the amplitude (1) in powers of

$$\nu = \frac{k p}{M} \quad (2)$$

for $\nu \rightarrow \pm\infty$ and for fixed photon mass μ . In Section 6 we show that this expansion can be interpreted as an expansion of the type invented by Toller: the group $SL(2, C)$ is the little group for an object with momentum zero which is exchanged between the photon and the proton (in the case of $s = \frac{1}{2}$). The case of a proton and currents

of equal parities with or without current conservation is explicitly calculated.

Our treatment of the amplitude (1) makes use of the following properties:

- a. its covariance under the proper orthochronous Lorentz group $L_+^\uparrow$ [or $SL(2, \mathbb{C})$], under parity reflection and time reversal;
- b. its crossing symmetry;
- c. current conservation.

In fact we exploit time reversal invariance and current conservation only for the special case of a spin $\frac{1}{2}$ particle and equal current parities. It is obviously possible to generalize our analysis for amplitudes like the covariant T product of current density operators, which have similar covariance properties. We close our introductory remarks by emphasizing that helicity amplitudes lose their group theoretical meaning as soon as one external momentum becomes spacelike.

With the rotation matrix $D_{q_2 q_1}^S(u)$ and the Clebsch-Gordan coefficient of $SU(2)$ ³⁾ we define

$$\begin{aligned} \Gamma_{\mu\nu}^{ab}(k, p)_q^S &= \\ &= \sum_{q_1 q_2 q_2'} D_{q_2 q_2'}^S(-i\sigma_2)(sq_2; sq_1 | Sq) M_{\mu\nu}^{ab}(k, p)_{q_2' q_1} \end{aligned} \quad (3)$$

Under a homogeneous Lorentz transformation $\Lambda \in L_+^\uparrow$ the amplitude (3) exhibits the covariance [Ref. 1), Eqs. (2) and (A.4) to (A.8)]

$$\begin{aligned} \Gamma_{\mu\nu}^{ab}(k, p)_q^S &= \sum_{\mu' \nu'} (\Lambda^{-1})_{\mu}^{\mu'} (\Lambda^{-1})_{\nu}^{\nu'} \\ &\times \sum_{q'} D_{q' q}^S(R(\Lambda, p)) \Gamma_{\mu' \nu'}^{ab}(\Lambda k, \Lambda p)_{q'}^S \end{aligned} \quad (4)$$

Next we map the momentum p on its boost $a(p)$, which parametrizes a right coset of $SU(2)$ in $SL(2, \mathbb{C})$ and similarly k on $a(k)$, which describes a right coset of $SU(1,1)$ in $SL(2, \mathbb{C})$. $SU(1,1)$ is the little group for spacelike momenta if we choose the reference momentum as

$$k^R = (0, 0, 0, \mu) \quad (5)$$

The functions Γ can be extended from the boosts to the whole $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by the definition

$$\Gamma_{\mu\nu}^{ab}(va_2, ua_1)_q^S = \sum_{q'} D_{q'q}^S(\bar{u}^1) \Gamma_{\mu\nu}^{ab}(a_2, a_1)_{q'}^S \quad (6)$$

where $a_{1,2} \in SL(2, \mathbb{C})$, $u \in SU(2)$ and $v \in SU(1,1)$. Denoting the homomorphism from $SL(2, \mathbb{C})$ onto $L_+^\uparrow$ as $\Lambda = \Lambda_a$, we may use the relations

$$\begin{aligned} a(\Lambda_a p) &= u(a, p) a(p) a^{-1} \\ R(\Lambda_a, p) &= \Lambda_{u(a, p)} \end{aligned} \quad (7)$$

to rewrite (4) as

$$\begin{aligned} \Gamma_{\mu\nu}^{ab}(a_2, a_1)_q^S &= \\ &= \sum_{\mu'\nu'} (\Lambda_a^{-1})_{\mu}^{\mu'} (\Lambda_a^{-1})_{\nu}^{\nu'} \Gamma_{\mu'\nu'}^{ab}(a_2 \bar{a}^{-1}, a, \bar{a}^{-1})_q^S \end{aligned} \quad (8)$$

Finally, we decompose the tensor representation into irreducible representations χ' and use the canonical basis ²⁾, and Appendix A to span the χ' spaces. We get

$$\Gamma_Q^J(a_2, a_1 | \chi')_q^S = \sum_{\gamma' Q'} D_{\gamma' Q' Q}^{\chi'}(a) \Gamma_{Q'}^{J'}(a_2, a_1 | \chi')_q^S \quad (9)$$

and

$$\Gamma_Q^J(u a_2, u a_1 | \chi')_q^S = \sum_{q'} D_{q' q}^S(u) \Gamma_{Q'}^J(a_2, a_1 | \chi')_{q'}^S \quad (10)$$

2. THE EXPANSION THEOREM

The functions $\Gamma_Q^J(a_2, a_1 | \chi')_q^S$ with the properties (9), (10) generate a covariant operator in the sense of the definition given in ²⁾. We want to expand it into irreducible covariant operators. For this purpose we make the working hypothesis that the covariant operator is smooth, that is to say: $\Gamma_Q^J(a_2, a_1 | \chi')_q^S$ is infinitely differentiable in a_2 and a_1 , and $\Gamma_Q^J(d, e | \chi')_q^S$ falls off faster than any power of $\text{ch } \eta$, $d = \exp \frac{1}{2} \eta \sigma_3$, for $|\eta| \rightarrow \infty$. This hypothesis has to be given up for physical reasons if we treat the amplitude (1). Contrary to the functions studied in ²⁾ which were covariant on right cosets of $SU(2)$ in both arguments a_2 and a_1 [Ref. 2), Eq. (19)], we have now to deal with functions which are covariant on right cosets of $SU(1,1)$ in a_2 [strictly speaking they are invariant, since the representation of $SU(1,1)$ is one-dimensional] and covariant on right cosets of $SU(2)$ in a_1 . The occurrence of the trivial representation of $SU(1,1)$ in this context does not simplify but complicates the problem. As we shall see in a moment this is due to the fact that the trivial representation has Plancherel measure zero, namely it is not contained in the decomposition of L^2 functions on $SU(1,1)$ which carry the regular representation of $SU(1,1)$ into irreducible unitary representations of $SU(1,1)$.

Therefore, we first study functions which instead of (10) have the property

$$\begin{aligned} \Gamma_Q^J(v a_2, u a_1 | \chi')_{q_2 q_1}^{J' S} &= \\ &= \sum_{q_3 q_4} D_{q_2 q_4}^{J'}(v) D_{q_3 q_1}^S(u^{-1}) \Gamma_Q^J(a_2, a_1 | \chi')_{q_4 q_3}^{J' S} \end{aligned} \quad (11)$$

where J' belongs to the principal series of $SU(1,1)$, $\text{Re } J' = -\frac{1}{2}$, and q_2 is an integer. For the functions $D_{q_2 q_1}^{J'}(v)$ we use the definition of ⁴⁾, Chapters 6 and 7. We project the functions (11) on cosets of $SU(2)$ as in ²⁾, Eq. (18)

$$\begin{aligned} \Gamma_Q^J(a_2, a_1 | \chi')_{q_2 q_1}^{J' S} &= \\ &= \sum_{j=0}^{\infty} (2j+1) \sum_{q=-j}^{+j} \int d\mu(u) D_{qq}^j(u^{-1}) \Gamma_Q^J(u a_2, a_1 | \chi')_{q_2 q_1}^{J' S} \end{aligned} \quad (12)$$

where the sums over j and q are restricted to integers. Every such projection can be expanded by means of the formulae (36), (40) of ²⁾ yielding

$$\begin{aligned} \Gamma_Q^J(a_2, a_1 | \chi')_{q_2 q_1}^{J' S} &= \sum_{j=0}^{\infty} (2j+1) \sum_{q=-j}^{+j} i \int d\lambda \sum_{M, \alpha} \\ &\times \langle \chi; j q | T_{a_2}^X T_Q^J(\alpha) T_{a_1^{-1}}^{X-\alpha} | \chi - \alpha; S q_1 \rangle \\ &\times C_{\alpha}(\chi) \int d\mu(a_4 a_3^{-1}) \sum_{j_1, Q_1} (-1)^{j_1 - M'} G_{j_1} \\ &\times \int d\mu(u) D_{q_3 q}^j(u^{-1}) \Gamma_{Q_1}^{j_1}(u a_4, a_3 | \chi')_{q_2 q_1}^{J' S} \\ &\times \langle \chi; j q_3 | T_{a_4}^X T_{Q_1}^{j_1}(\alpha^*) T_{a_3^{-1}}^{X-\alpha^*} | \chi - \alpha^*; S q_1 \rangle \end{aligned} \quad (13)$$

Here

$$\chi = (n, \lambda), \chi' = (n', \lambda'), n' = j_2 - j_1, \lambda' = j_1 + j_2 + 1$$

The sum over M extends over a finite set of integers, the restriction being due to S even after the summations over j and M are interchanged. The integrand under the λ integral consists of the rational function $C_\alpha(\chi)$ and the remainder which is entire analytic with a rapid fall off parallel to the imaginary λ axis. In addition to the integral over the imaginary axis in (13), the poles of $C_\alpha(\chi)$ are connected with a discrete series of irreducible covariant operators whose contributions must be added to (13). After the introduction of representation functions with an exponential decrease in one half plane we can easily include these terms. We may therefore neglect them for the moment.

The integration over u is performed by replacing a_4 by $u^{-1}a_4$ in the integral over $a_4 a_3^{-1}$ which yields for the integral over u

$$(2j+1)^{-1} \Gamma_{Q_1}^{j_1}(a_4, a_3 | \chi')_{q_1 q_1}^{j' S} \overline{\langle \chi; j_1 q_1 | T_{a_4}^\chi \dots | \dots \rangle}$$

The next simplification consists in the replacement of the completeness sum over the canonical basis by the completeness sum over the pseudo-basis

$$\begin{aligned} \sum_{j_1 q_1} |\chi; j_1 q_1\rangle \langle \chi; j_1 q_1| &= \\ &= \int d\phi_\chi(j') \sum_{j_1' q_1'} |\chi; j_1' q_1'\rangle \langle \chi; j_1' q_1'| \end{aligned} \quad (14)$$

In the space of the representation χ of the principal series of $SL(2, \mathbb{C})$ the canonical basis and the pseudobasis are defined up to a normalization by their properties [Ref. 4), Chapter 7]

$$\begin{aligned} T_u^X |X; j' q_1\rangle &= \sum_{q_2} D_{q_2 q_1}^{j'}(u) |X; j' q_2\rangle \\ T_v^X |X; j' \zeta q_1\rangle &= \sum_{q_2} D_{q_2 q_1}^{j'}(v) |X; j' \zeta q_2\rangle \end{aligned} \quad (15)$$

$\zeta = \pm 1$ labels the shells of a two-shell hyperboloid. The measure $d\varnothing_X^{(J')}$ is non-zero on the principal series and at most a finite number of representations of the discrete series of $SU(1,1)$. Then we decompose the Haar measure on $SL(2, \mathbb{C})$ into the product of the Haar measure on $SU(1,1)$ and a quotient measure on the set of right cosets of $SU(1,1)$ in $SL(2, \mathbb{C})$

$$d\mu(a_4 a_3^{-1}) = d\mu(v) \frac{d\mu(a_4 a_3^{-1})}{d\mu(v)} \quad (16)$$

A convenient explicit form of this product decomposition is given in (21). The integration over $SU(1,1)$ having been performed we are left with

$$\begin{aligned} \Gamma_Q^j(a_2, a_1 | X')_{q_2 q_1}^{j' s} &= i \int \alpha \lambda \sum_{M, \zeta} \sum_{\alpha} F_{\alpha}(X, \zeta) \\ &\times \langle X; j' \zeta q_2 | T_{a_2}^X T_Q^j(\alpha) T_{a_1^{-1}}^{X-\alpha} | X-\alpha; s q_1 \rangle \end{aligned} \quad (17)$$

and

$$\begin{aligned} F_{\alpha}(X, \zeta) &= C_{\alpha}(X) \int \frac{d\mu(a_4 a_3^{-1})}{d\mu(v)} \sum_{j, Q} (-1)^{j-M'} G_j \\ &\times \sum_{q_2} \Gamma_Q^j(a_4, a_3 | X')_{q_2 q_1}^{j' s} \frac{\langle X; j' \zeta q_2 | T_{a_4}^X T_Q^j(\alpha^*) T_{a_3^{-1}}^{X-\alpha^*} | X-\alpha^*; s q_1 \rangle}{|X-\alpha^*; s q_1 \rangle} \end{aligned} \quad (18)$$

The two particles with momenta p and k can be looked upon in a system where the timelike particle is at rest and the space-like particle moves in the direction of the positive third axis

$$p = (M, 0, 0, 0)$$
$$k = (k_0, 0, 0, k_3), \quad k_3 > 0$$

We set

$$k_0 = -\mu \sinh \eta$$
$$k_3 = +\mu \cosh \eta$$
$$\eta \leq 0$$
(19)

It is easily verified that this amounts to the decomposition

$$a_3 = u^{-1} a$$
$$a_4 = v d(\eta) a$$
$$d(\eta) = e^{\frac{1}{2}\eta \sigma_3}$$
(20)

In these parameters we can give the quotient measure as

$$d\mu(a_4 a_3^{-1}) = d\mu(u) \left[2\pi \frac{d\mu(u)}{d\psi} \right] \frac{1}{4\pi} \cosh^2 \eta d\eta$$
(21)

where the second factor is a measure (normalized to one) on the right cosets of $U(1)$ in $SU(2)$. Inserting (20) and (21) into (18) yields

$$F_{\alpha}(\chi, \zeta) = C_{\alpha}(\chi) [4\pi(2S+1)]^{-1} \times \int_{-\infty}^{+\infty} d\eta \, \chi^2 \eta \sum_{jQ} (-1)^{j-M'} G_j \sum_{q_1, q_2} \Gamma_Q^j(d(\eta), e|\chi')_{q_2 q_1}^{j's} \times \langle \chi; j's q_2 | T_{d(\eta)}^{\chi} T_Q^j(\alpha^*) | \chi - \alpha^*; S q_1 \rangle \quad (22)$$

In (17) and (22) we continue analytically in J' till $J'=0$. Given the function $\Gamma_Q^j(d(\eta), e|\chi')_q^S$ we define for every complex J' the trivial continuation

$$\Gamma_Q^j(d(\eta), e|\chi')_{q_2 q_1}^{j's} = \delta_{q_2 0} \Gamma_Q^j(d(\eta), e|\chi')_{q_1}^S \quad (23)$$

and its extension on the manifold $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by means of (11). The analytic continuation in J' reduces then completely to the continuation of the matrix elements appearing in the integrals (17) and (22). Since the limits of the continuation are sufficiently well behaved, we can simply continue under the integral sign. This guarantees that (17) (together with the terms of the discrete series to be added) remains the inverse transformation of the Fourier transformation (22).

3. REPRESENTATION FUNCTIONS IN A MIXED BASIS

The sum over j

$$\langle \chi; j's q_2 | T_{d(\eta)}^{\chi} T_Q^j(\alpha) | \chi - \alpha; S q_1 \rangle = \sum_j \langle \chi; j's q_2 | T_{d(\eta)}^{\chi} | \chi; j q_2 \rangle \langle \chi; j q_2 | T_Q^j(\alpha) | \chi - \alpha; S q_1 \rangle \quad (24)$$

is restricted to that finite set for which the Clebsch-Gordan coefficient $(jq_2; JQ | Sq_1)$ of $SU(2)$ differs from zero. The matrix element (24) can therefore be computed explicitly if we know the representation functions in a mixed basis

$$d_{j' \zeta j q}^{\chi}(\eta) = \langle \chi; j' \zeta q | T_{d(\eta)}^{\chi} | \chi; j q \rangle \quad (25)$$

We must find an integral representation for these functions which can be continued in J' .

For this purpose we use a realization of the χ space as a space of functions on the cosets $SU(2)/U(1)$. In this case the basis elements of the canonical basis are spherical harmonics and the elements of the pseudobasis are known hypergeometric functions. We obtain the integral representation [all notations as in Ref. 4)]

$$d_{j' \zeta j q}^{\chi}(\eta) = \frac{1}{2} e^{\frac{i}{2}\pi(M-q)} (2j+1)^{\frac{1}{2}} \int_0^1 dz z^{-\lambda-1} \times (ch\eta - \zeta z sh\eta)^{\lambda-1} d_{\zeta M, q}^{j'}(\bar{z}^{-1}) d_{M, q}^j\left(\frac{\zeta z - th\eta}{1 - \zeta z th\eta}\right) \quad (26)$$

If both λ and J' belong to the respective principal series ($\text{Re } \lambda = 0$, $\text{Re } J' = -\frac{1}{2}$), the expression (26) implies the relation

$$d_{j' \zeta j q}^{(M, \lambda)}(\eta) = (-1)^{M-q} d_{j' \zeta j q}^{(M, -\lambda)}(\eta) \quad (27)$$

which can be exploited for the analytic continuation in both J' and λ . A further symmetry relation

$$d_{j' \zeta j q}^{(M, \lambda)}(\eta) = e^{i\pi(j+q+M)} d_{j' - \zeta j q}^{(-M, \lambda)}(-\eta) \quad (28)$$

can easily be deduced from (26).

We are only interested in the limit $J' \rightarrow 0$ of the functions

$$d_{j' \zeta j 0}^{(0, \lambda)}(\eta) \equiv d_{j' \zeta j 0}^{\lambda}(\eta)$$

The integral representation (26) for $d_{j' \zeta j q}^{(M, \lambda)}(\eta)$ converges absolutely at $z=0$ as long as

$$\min(-\operatorname{Re} j', \operatorname{Re} j' + 1) > \operatorname{Re} \lambda$$

In order to achieve absolute convergence along the whole path of continuation we assume $\operatorname{Re} \lambda < 0$. With the Legendre polynomial $P_j(z)$ we find

$$\begin{aligned} d_{0 \zeta j 0}^{\lambda}(\eta) &= \\ &= \frac{1}{2} (2j+1)^{\frac{1}{2}} \int_0^1 dz z^{-\lambda-1} (ch\eta - \zeta z sh\eta)^{\lambda-1} P_j\left(\frac{\zeta z - th\eta}{1 - \zeta z th\eta}\right) \end{aligned} \quad (29)$$

This integral can be expressed by a Jacobi polynomial, namely in the form

$$\begin{aligned} d_{0 \zeta j 0}^{\lambda}(\eta) &= (2j+1)^{\frac{1}{2}} (-\lambda)^{-1} \zeta^j \frac{e^{-5\eta(\lambda+j)}}{(2ch\eta)^{j+1}} \\ &\quad \times {}_2F_1(-j, -j-\lambda, 1-\lambda; e^{-25\eta}) \end{aligned} \quad (30)$$

or the equivalent form

$$\begin{aligned} d_{0 \zeta j 0}^{\lambda}(\eta) &= (2j+1)^{\frac{1}{2}} \frac{(-1)^j j! \Gamma(-\lambda)}{\Gamma(j-\lambda+1)} \frac{e^{-5\lambda\eta}}{2ch\eta} \\ &\quad \times P_j^{(5\lambda, -5\lambda)}(th\eta) \end{aligned} \quad (31)$$

From (31) we deduce the Weyl symmetry relation

$$d_{05j0}^{\lambda}(\eta) = \frac{\Gamma(-\lambda)\Gamma(j+\lambda+1)}{\Gamma(\lambda)\Gamma(j-\lambda+1)} d_{0-5j0}^{-\lambda}(\eta) \quad (32)$$

The functions (30), (31) possess first order poles at $\lambda = 0, 1, 2, \dots, j$. These poles and the poles of $C_{\alpha}(\chi)$ (22) interfere to give the contributions of the discrete series.

The Weyl symmetry (32) and the Weyl symmetry of the matrix elements of the irreducible tensor operators in the canonical basis [Ref. 2), Eq. (44)] imply

$$\begin{aligned} \langle \chi; 050 | T_{d(\eta)}^{\chi} T_Q^j(\alpha) | \chi - \alpha; SQ \rangle &= \\ &= \frac{\Gamma(-\lambda)\Gamma(S+\lambda-\Delta_{\alpha}\lambda+1)}{\Gamma(\lambda)\Gamma(S-\lambda+\Delta_{\alpha}\lambda+1)} \\ &\times \langle -\chi; 0-50 | T_{d(\eta)}^{-\chi} T_Q^j(-\alpha) | -\chi + \alpha; SQ \rangle \end{aligned} \quad (33)$$

Note that we made use of the fact that j_2 is an integer. Since $\chi = (0, \lambda)$ and therefore $\chi = \hat{\chi}$, we have a symmetry under parity conjugation; another relation follows from (28)

$$\begin{aligned} \langle \chi; 050 | T_{d(\eta)}^{\chi} T_Q^j(\alpha) | \chi - \alpha; SQ \rangle &= \\ &= (-1)^{M'+\Delta_{\alpha}M} \langle \chi; 050 | T_{d(\eta)}^{\chi} T_{-Q}^j(\hat{\alpha}) | \chi - \hat{\alpha}; S, -Q \rangle \\ &= (-1)^{j-S} \langle \chi; 0-50 | T_{d(\eta)}^{\chi} T_{-Q}^j(\alpha) | \chi - \alpha; S, -Q \rangle \end{aligned} \quad (34)$$

4. WEYL SYMMETRY, PARITY INVARIANCE, AND CROSSING SYMMETRY

We introduce an abbreviation

$$\begin{aligned} \Gamma_Q^j(d(\eta), e | \chi')_q^S &= \delta_{Qq} \Gamma_Q^j(d(\eta), e | \chi')_Q^S \\ &= \delta_{Qq} \Gamma_Q^j(\chi') \end{aligned} \quad (35)$$

Inserting the representation functions of the mixed basis into (22) we get $(\chi^* = (0, -\lambda))$

$$\begin{aligned} F_\alpha(\lambda, \xi) &= C_\alpha(0, \lambda) [4\pi(2S+1)]^{-1} \int_{-\infty}^{+\infty} d\eta \, \chi^2 \eta \\ &\quad \times \sum_j (-1)^{j-M'} G_j \sum_Q \Gamma_Q^j(\chi') \\ &\quad \times \langle \chi^*; 050 | T_{d(\eta)}^{\chi^*} T_Q^j(\alpha^*) | \chi - \alpha^*; SQ \rangle \end{aligned} \quad (36)$$

and its inversion (still without the discrete series)

$$\begin{aligned} \Gamma_Q^j(\chi') &= i \int d\lambda \sum_{\xi, \alpha} F_\alpha(\lambda, \xi) \\ &\quad \times \langle 0\lambda; 050 | T_{d(\eta)}^{(0, \lambda)} T_Q^j(\alpha) | \chi - \alpha; SQ \rangle \end{aligned} \quad (37)$$

From (36) and the Weyl symmetry of the matrix element (33) we deduce the Weyl symmetry of the Fourier transform

$$F_\alpha(\lambda, \xi) = \frac{\Gamma(\lambda) \Gamma(S - \lambda + \Delta_\alpha \lambda + 1)}{\Gamma(-\lambda) \Gamma(S + \lambda - \Delta_\alpha \lambda + 1)} F_{-\alpha}(-\lambda, -\xi) \quad (38)$$

As a consequence of the Weyl symmetries (33) and (38) we may select a certain ζ instead of summing over both, so that the functions under the integral sign of (37) show an exponential fall-off in a half plane which depends only on the sign of η . For example, we choose $\zeta = +1$ such that

$$\langle 0\lambda; 0+0 | T_{d(\eta)}^{(0,\lambda)} T_Q^J(\alpha) | \chi-\alpha; SQ \rangle$$

tends to zero exponentially for $\text{Re } \lambda \rightarrow -\infty$ if $\eta < 0$ and for $\text{Re } \lambda \rightarrow +\infty$ if $\eta > 0$. These functions have poles on the real axis at most at $\lambda = 0, 1, 2, \dots, S+J$, whereas the Fourier transform may have poles at the corresponding negative points. Instead of integrating over the imaginary axis in (37), we introduce a contour C_+ (C_-) which consists of two infinite intervals on the imaginary axis with upward direction, and a finite piece going around the poles just mentioned and the poles of $C_\alpha(\lambda)$ in the right (left) sense. We replace the preliminary form (37) by

$$\begin{aligned} \Gamma_Q^J(\chi') &= 2i \int_{C_\pm} d\lambda \sum_\alpha F_\alpha(\lambda, +) \\ &\times \langle 0\lambda; 0+0 | T_{d(\eta)}^{(0,\lambda)} T_Q^J(\alpha) | \chi-\alpha; SQ \rangle \end{aligned} \quad (39)$$

This formula is justified by the fact that it gives the correct asymptotic behaviour for $\Gamma_Q^J(\chi')$. If we deform the contours $C_\pm$ into the imaginary axis, the poles yield the contributions of the discrete series. Of course the integral over $C_+ - C_-$ must vanish.

We extend the group $SL(2, \mathbb{C})$ by parity s just as in ²⁾ Section 5. The extension is denoted G . The momentum k^R (5) is not stationary under s , therefore the little group of k^R in G does not contain s but s' ,

$$s' = s e^{\frac{i}{2}\pi\sigma_2}$$

(40)

First we consider the representations of the type $\chi' = [j_1, j_2]$ with $j_1 = j_2$ like the symmetric traceless tensor of rank two $\chi' = [1, 1]$ and the trivial representation $\chi' = [0, 0]$. After an extension they yield a pair of associate representations of G . As in ²⁾ we label them by (λ', σ) where $\sigma = +1$ denotes the true scalar and tensor and $\sigma = -1$ the pseudoscalar and pseudo-tensor representations. We choose the trivial representation $s' \rightarrow +1$ for the little group of k^R in G . The functions Γ_Q^J can be extended on G by the postulates

$$\begin{aligned} \Gamma_Q^J(a_2, a, 1\lambda', \sigma)_q^S &= \Gamma_Q^J(s'a_2, a, 1\lambda', \sigma)_q^S \\ &= \pi_1 \Gamma_Q^J(a_2, sa, 1\lambda', \sigma)_q^S \\ &= \sigma(-1)^J \Gamma_Q^J(a_2\bar{s}', a, \bar{s}' 1\lambda', \sigma)_q^S \end{aligned}$$

(41)

These conditions (41) are compatible if and only if the relation of parity invariance

$$\begin{aligned} \Gamma_Q^J(d(\eta), e 1\lambda', \sigma)_Q^S &= \Gamma_Q^J(d(\eta), e 10, \lambda')_Q^S \\ &= \sigma\sigma_1 \Gamma_{-Q}^J(d(\eta), e 10, \lambda')_{-Q}^S \end{aligned}$$

(42)

with

$$\pi_1 = \sigma_1(-1)^S$$

holds.

With the G covariant operators of ²⁾, (93), (94) we make the following ansatz for the functions (41)

$$\begin{aligned} \Gamma_Q^j(q_2, q_1, \lambda, \sigma)_q^S &= i \int d\lambda \sum_{\xi, \Delta_\alpha \lambda} \\ &\times \left\{ \sum_{M \geq 0} F(-M, \Delta_\alpha \lambda; 01)(\lambda, \xi) \right. \\ &\times \langle \lambda, +1; 050 | T_{q_2}^{\lambda, +1} T_Q^j(-M, \Delta_\alpha \lambda; 01) T_{q_1^{-1}}^{M, \lambda - \Delta_\alpha \lambda} | \\ &\times | M, \lambda - \Delta_\alpha \lambda; S q \pi_1 \rangle \\ &+ F(0, \Delta_\alpha \lambda; 00)(\lambda, \xi) \\ &\times \langle \lambda, +1; 050 | T_{q_2}^{\lambda, +1} T_Q^j(0, \Delta_\alpha \lambda; 00) T_{q_1^{-1}}^{\lambda - \Delta_\alpha \lambda, \sigma_1} | \\ &\times | \lambda - \Delta_\alpha \lambda, \sigma_1; S q \rangle \left. \right\} \end{aligned}$$

(43)

The matrix elements are as usual defined by analytic continuation from the pseudobasis. The correct covariance of the ansatz (43) can be verified if we take into account

$$\langle \lambda \sigma_2; j' \xi q | T_{S'}^{\lambda \sigma_2} = \sigma_2 \langle \lambda \sigma_2; j' \xi, -q | \quad (44)$$

which can be proved by expanding both sides in the canonical basis and using the integral representation ⁴⁾, Eq. (7.31). $F(0, \Delta_\alpha \lambda; 00)$ is non-zero only if $\sigma_1 = \sigma$. Restricting the ansatz (43) to $SL(2, \mathbb{C})$ and comparing with (37) gives

$$2^{-\frac{1}{2}} F_{(-M, \Delta_\alpha \lambda; 01)}(\lambda, \xi) = F_{(-M, \Delta_\alpha \lambda)}(\lambda, \xi) \\ = \sigma \sigma_1 (-1)^M F_{(M, \Delta_\alpha \lambda)}(\lambda, \xi)$$

$$\delta_{\sigma_1, \sigma} F_{(0, \Delta_\alpha \lambda; 00)}(\lambda, \xi) = F_{(0, \Delta_\alpha \lambda)}(\lambda, \xi)$$

(45)

The two representations $\chi' = [1, 0]$ and $\chi' = [0, 1]$ of $SL(2, \mathbb{C})$ yield one representation of G labelled $(M', \lambda') = (1, 2)$, $M' > 0$ by definition. We take over the definition ²⁾, (76) and postulate the extension of Γ_Q^J on G

$$\Gamma_{Q\pi}^J(a_2, a_1 | M', \lambda')_9^S = \Gamma_{Q\pi}^J(s' a_2, a_1 | M', \lambda')_9^S \\ = \pi_1 \Gamma_{Q\pi}^J(a_2, s a_1 | M', \lambda')_9^S \\ = \pi^{-1} \Gamma_{Q\pi}^J(a_2 s^{-1}, a_1 s^{-1} | M', \lambda')_9^S$$

(46)

These conditions are compatible if and only if

$$\Gamma_Q^J(d(\eta), e | M', \lambda')_Q^S = \sigma_1 (-1)^{M'} \Gamma_{-Q}^J(d(\eta), e | -M', \lambda')_{-Q}^S$$

(47)

With the covariant operators on G ²⁾, (87), (88) we make the ansatz

$$\Gamma_{Q\tau}^J(q_2, q_1 | M', \lambda')_9^S = i \int d\lambda \sum_{\xi, \Delta_\alpha \lambda} \\ \times \left\{ \sum_{M' > 0, \epsilon = \pm 1} F_{(-\epsilon M, \Delta_\alpha \lambda; 0\epsilon)}(\lambda, \xi) \right.$$

$$\begin{aligned}
 & \times \langle \lambda, +1; 050 | T_{g_2}^{\lambda, +1} T_{Q\tau}^{\tau} (-\epsilon M, \Delta_\alpha \lambda; 0\epsilon) \\
 & \quad T_{g_1^{-1}}^{M, \lambda - \Delta_\alpha \lambda} | M, \lambda - \Delta_\alpha \lambda; S q \pi_1 \rangle \\
 & + F_{(0, \Delta_\alpha \lambda; 00)}(\lambda, 5) \\
 & \times \langle \lambda, +1; 050 | T_{g_2}^{\lambda, +1} T_{Q\tau}^{\tau} (0, \Delta_\alpha \lambda; 00) \\
 & \quad T_{g_1^{-1}}^{\lambda - \Delta_\alpha \lambda, \sigma_1} | \lambda - \Delta_\alpha \lambda, \sigma_1; S q \rangle \}
 \end{aligned}$$

If we restrict (48) to $SL(2, \mathbb{C})$ and compare with (37), we get the relations (48)

$$\begin{aligned}
 2^{-\frac{1}{2}} F_{(-\epsilon M, \Delta_\alpha \lambda; 0\epsilon)}(\lambda, 5) &= F_{(-\epsilon M, \Delta_\alpha \lambda)}^{(M', \lambda')}(\lambda, 5) \\
 &= \sigma_1 (-1)^M F_{(\epsilon M, \Delta_\alpha \lambda)}^{(-M', \lambda')}(\lambda, 5) \\
 F_{(0, \Delta_\alpha \lambda; 00)}(\lambda, 5) &= F_{(0, \Delta_\alpha \lambda)}^{(M', \lambda')}(\lambda, 5) \\
 &= \sigma_1 F_{(0, \Delta_\alpha \lambda)}^{(-M', \lambda')}(\lambda, 5)
 \end{aligned}$$

(49)

Finally, we investigate the consequences of crossing symmetry. We split the amplitude (1) into two parts $M^{(\pm)}$ which are symmetric, respectively antisymmetric, under the transposition $ab \rightarrow ba$ of the superscripts

$$M_{\mu\nu}^{ab}(k, p) q_2 q_1 = M_{\mu\nu}^{(+)}(k, p) q_2 q_1 + M_{\mu\nu}^{(-)}(k, p) q_2 q_1$$

(50)

The analysis, up to this point, can be performed equally for both parts. Crossing symmetry implies

$$M_{\mu\nu}^{(\pm)}(k, p) q_2 q_1 = \mp M_{\nu\mu}^{(\pm)}(-k, p) \quad (51)$$

If $a(k)$ is a boost for the momentum k , $i\sigma_2 a(k)$ is a boost for the momentum $-k$ [use (5) and the definition, Ref. 1), (A.4)].

Therefore (51) entails

$$\Gamma_Q^{(\pm)}(a_2, a_1 | \chi')_Q^S = \mp (-1)^{M'} \Gamma_Q^{(\pm)}(i\sigma_2 a_2, a_1 | \chi')_Q^S \quad (52)$$

From (52) we obtain as a special case

$$\begin{aligned} \Gamma_Q^{(\pm)}(d(\eta), e | \chi')_Q^S &= \mp (-1)^{M'} \Gamma_Q^{(\pm)}(i\sigma_2 d(\eta), e | \chi')_Q^S \\ &= \mp (-1)^{J-S+M'} \Gamma_{-Q}^{(\pm)}(d(-\eta), e | \chi')_{-Q}^S \end{aligned} \quad (53)$$

Inserting (34) and (53) into (36) yields

$$F_\alpha^{(\pm)}(\lambda, \zeta) = \mp (-1)^{M'} F_\alpha^{(\pm)}(\lambda, -\zeta) \quad (54)$$

In the following Sections (apart from the Appendix) the alternative $(\pm)$ refers to the even and odd part (50) with the single exception of formula (60).

5. THE TIMELIKE PARTICLE WITH SPIN $\frac{1}{2}$ AND EQUAL CURRENT PARITIES

We specialize to the case where the timelike particle has spin $\frac{1}{2}$ and the currents have the same parity, namely both are vectors or both are axial vectors. Taking into account time reversal invariance we can decompose each part (50) of the amplitude (1) into seven Dirac covariants, each one accompanied by an invariant function $\rho_\alpha(v, \mu^2)$, $\alpha = 1, 2, 3, \dots, 7$ with

$$v = \frac{k \cdot p}{M} = -\mu \sinh \eta \quad (55)$$

With the conventions

$$\epsilon_{0123} = +1, \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \bar{u}_{q_2}(p) u_{q_1}(p) = \delta_{q_1 q_2}$$

and the standard low-energy representation of the Dirac matrices, we define ⁵⁾

$$\begin{aligned} M_{\mu\nu}(k, p)_{q_2 q_1} &= \\ &= \bar{u}_{q_2}(p) \left[p_\mu p_\nu \rho_1 + k_\mu k_\nu \rho_2 + \frac{1}{2} (k_\mu p_\nu + k_\nu p_\mu) \rho_3 \right. \\ &\quad + g_{\mu\nu} \rho_4 + \epsilon_{\mu\nu\sigma\tau} p^\sigma \gamma_5 \gamma^\tau \rho_5 + \epsilon_{\mu\nu\sigma\tau} k^\sigma \gamma_5 \gamma^\tau \rho_6 \\ &\quad \left. + (\epsilon_{\nu\lambda\sigma\tau} k_\mu - \epsilon_{\mu\lambda\sigma\tau} k_\nu) p^\sigma k^\lambda \gamma_5 \gamma^\tau \rho_7 \right] u_{q_1}(p) \end{aligned}$$

(56)

Crossing symmetry (51) implies

$$\rho_{\alpha}^{(\pm)}(\nu, \mu^2) = \mp \varepsilon_{\alpha} \rho_{\alpha}^{(\pm)}(-\nu, \mu^2)$$

$$\varepsilon_{\alpha} = 1 \quad \text{for} \quad \alpha = 1, 2, 4, 6$$

$$\varepsilon_{\alpha} = -1 \quad \text{for} \quad \alpha = 3, 5, 7$$

(57)

and current conservation yields

$$\begin{aligned} 2M\nu \rho_1 - \mu^2 \rho_3 &= 0 \\ 2\mu^2 \rho_2 - M\nu \rho_3 - 2\rho_4 &= 0 \\ \rho_5 + \mu^2 \rho_7 &= 0 \end{aligned}$$

(58)

With the notation (35) the relations between the $\Gamma_Q^J(\chi')$ and the ρ_{α} can be given in the form

$$\begin{aligned} \Gamma_0^2(1,1) &= - \left(\frac{2\pi}{5}\right)^{\frac{1}{2}} \frac{2}{3} \kappa^2 \eta \mu^2 \rho_2 \\ \Gamma_0^1(1,1) &= + \left(\frac{2\pi}{3}\right)^{\frac{1}{2}} [-2\kappa \eta \kappa \eta \mu^2 \rho_2 + \kappa \eta M \mu \rho_3] \\ \Gamma_0^0(1,1) &= - (2\pi)^{\frac{1}{2}} [M^2 \rho_1 + (\kappa \mu^2 \eta + \frac{1}{3} \kappa^2 \eta) \mu^2 \rho_2 \\ &\quad - \kappa \eta M \mu \rho_3] \end{aligned}$$

(59)

$$\begin{aligned}
 \Gamma_{\pm 1}^1(1,0) &= -\left(\frac{\pi}{3}\right)^{\frac{1}{2}} [M p_5 \pm e^{\mp \eta} \mu p_6 \\
 &\quad + \kappa \eta e^{\mp \eta} M \mu^2 p_7] \\
 \Gamma_0^1(1,0) &= -\left(\frac{\pi}{3}\right)^{\frac{1}{2}} [M p_5 - \kappa \eta \mu p_6] \\
 \Gamma_Q^1(0,1) &= \Gamma_{-Q}^1(1,0) \\
 \Gamma_0^0(0,0) &= -(2\pi)^{\frac{1}{2}} [M^2 p_1 - \mu^2 p_2 - \kappa \eta M \mu p_3 + 4 p_4]
 \end{aligned}
 \tag{60}$$

All other functions $\Gamma_Q^J(\chi')$ vanish. Time reversal invariance restricts the spin to $S=0$ for the symmetric tensors and $S=1$ for the antisymmetric tensors. Comparing (59) and (61) with the relation of parity invariance (42) gives for the symmetric tensors ($\sigma=+1$)

$$\sigma_1 = +1, \pi_1 = +1
 \tag{62}$$

whereas (60) compared with (47) yields for the antisymmetric tensors ($\sigma=+1$)

$$\sigma_1 = -1, \pi_1 = +1
 \tag{63}$$

If the currents are conserved, another definition of the invariant amplitudes often used in the literature is

$$\begin{aligned}
 2\pi W_1 &= \kappa^2 \eta M^2 p_1 - \mu^2 p_2 \\
 2\pi W_2 &= M^2 p_1
 \end{aligned}
 \tag{64}$$

The Fourier transforms $F_\alpha(\lambda, \xi)$ are defined as the following integrals, which assume a particularly compact form in terms of the p_α 's.

$$\underline{\chi' = [1, 1]}$$

$$F_{(0,0)}(\lambda, \zeta) = + \frac{1}{2^{\frac{1}{2}} \pi} \int_{-\infty}^{+\infty} d\eta \, \cosh \eta \, e^{\zeta \eta \lambda} \frac{\lambda}{\lambda^2 - 1} \\ \times \left\{ -2M^2 \varrho_1 + 2\mu^2 \varrho_2 + 5M\mu \varrho_3 \left(\frac{\lambda-1}{\lambda+1} e^{\zeta \eta} - \frac{\lambda+1}{\lambda-1} e^{-\zeta \eta} \right) \right\}$$

$$F_{(0,2)}(\lambda, \zeta) = - \frac{1}{2^{\frac{1}{2}} \pi} \int_{-\infty}^{+\infty} d\eta \, \cosh \eta \, e^{\zeta \eta \lambda} \lambda(\lambda-2) \\ \times \left\{ M^2 \varrho_1 \frac{1}{\lambda} + \mu^2 \varrho_2 \frac{e^{-2\zeta \eta}}{\lambda-2} + 5M\mu \varrho_3 \frac{e^{-\zeta \eta}}{\lambda-1} \right\}$$

(65)

$$\underline{\chi' = [1, 0]}$$

$$2^{\frac{1}{2}} F_{(0,0)}(\lambda, \zeta) = \frac{1}{2 \cdot 3^{\frac{1}{2}} \pi} \int_{-\infty}^{+\infty} d\eta \, \cosh \eta \, e^{\zeta \eta \lambda} \frac{\lambda}{\lambda+1} \\ \times \left\{ -2M \varrho_5 + 5\mu \varrho_6 \left(\frac{\lambda-1}{\lambda+1} e^{\zeta \eta} - \frac{\lambda+1}{\lambda-1} e^{-\zeta \eta} \right) \right\}$$

$$F_{(-1,1)}(\lambda, \zeta) = \frac{1}{4 \cdot 3^{\frac{1}{2}} \pi} \int_{-\infty}^{+\infty} d\eta \, \cosh \eta \, e^{\zeta \eta \lambda} \\ \times \left\{ -2(\lambda-2)M \varrho_5 - 2 \frac{\lambda(\lambda-2)}{\lambda-1} 5\mu \varrho_6 e^{-\zeta \eta} \right. \\ \left. + M\mu^2 \varrho_7 (-\lambda+2 - \lambda e^{-2\zeta \eta}) \right\}$$

(66)

$$\begin{aligned} \underline{\chi' = [0,0]} \\ F(\lambda, \eta) = -\frac{1}{2^{\frac{1}{2}}\pi} \int_{-\infty}^{+\infty} d\eta \, \cosh \eta \, e^{5\eta\lambda} \lambda \\ \times \{ M^2 \rho_1 - \mu^2 \rho_2 - \sinh \eta \, M \mu \rho_3 + 4 \rho_4 \} \end{aligned} \quad (67)$$

We remind the reader that we are still dealing with the formal problem where the functions ρ_α are smooth and rapidly decreasing in $\sinh \eta$. The convergence of the integral transforms (65) to (67) hence presents no problem. Moreover, the functions F can be expanded in Taylor or Laurent series whose coefficients are certain integrals over the ρ_α 's. Those non-trivial Fourier transforms which are not listed above can either be obtained by the Weyl symmetry (38) or by the parity symmetry (49) (namely for $\chi' = [0, 1]$). Crossing symmetry (54) can be verified easily.

At first sight it might be puzzling for the reader who is unexperienced with Fourier transforms on Lie groups that seven "kinematically independent" functions ρ_α possess only five independent analytic functions as Fourier transforms. In fact, the functions in the two half planes $\operatorname{Re} \lambda \gtrless 0$ or equivalently the members of a pair of functions related by the Weyl symmetry should be counted as independent. This can be seen from the formulae for the inverse Fourier transformations which we give later. It is also supported by the following arguments. If we extend the space of functions under consideration from smooth, rapidly decreasing functions to L^2 functions, the Fourier transformation turns into an isometric mapping of Hilbert spaces and the dimension of a linear manifold in the Hilbert space sense is preserved. During the extension the Fourier transforms show singularities appearing in the two half planes which approach the principal series and finally yield a set of L^2 limit functions. One can verify indeed that the dimension of the linear span of these functions is equal to the dimension of the original manifold. But it is certainly not necessary to extend as far as

to Hilbert spaces to reconcile the physicists' intuition. We shall find in Section 6 that there exist as many independent residues of Lorentz poles as there are independent functions ϑ_α .

The inverse Fourier transformation assumes its most compact form if we use the functions Γ_Q^J .

$$\underline{\chi' = [1, 1]}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \Gamma_0^2(1, 1) &= \frac{1}{3(2\cosh\eta)^3} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta} \\ &\times \left[\frac{\lambda+2}{\lambda-1} e^{2\eta} + 2 \frac{\lambda^2-4}{\lambda^2-1} + \frac{\lambda-2}{\lambda+1} e^{-2\eta} \right] \{ (\lambda^2-1) F_{(0,0)}(\lambda, +) \\ &- \frac{\lambda+1}{\lambda-2} F_{(0,2)}(\lambda, +) \mp \frac{\lambda-1}{\lambda+2} F_{(0,2)}(-\lambda, +) \} \end{aligned}$$

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \Gamma_0^1(1, 1) = \frac{1}{(2\cosh\eta)^2} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta} \left[\frac{e^\eta}{\lambda-1} + \frac{e^{-\eta}}{\lambda+1} \right]$$

$$\times \{ (\lambda^2-1) F_{(0,0)}(\lambda, +) + (\lambda+1) F_{(0,2)}(\lambda, +) \mp (\lambda-1) F_{(0,2)}(-\lambda, +) \}$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \Gamma_0^0(1, 1) = \frac{-1}{3(2\cosh\eta)} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta}$$

$$\times \{ (\lambda^2-1) F_{(0,0)}(\lambda, +) + 2 F_{(0,2)}(\lambda, +) \pm 2 F_{(0,2)}(-\lambda, +) \}$$

$$\underline{\chi' = [1, 0]}$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \Gamma'_1(1, 0) = \frac{-2}{(2ch\eta)^3} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta}$$

$$\times \left\{ 2^{\frac{1}{2}} F_{(0,0)}(\lambda, +) \left[\frac{\lambda+1}{\lambda-1} + e^{-2\eta} \right] \right. \\ \left. + F_{(-1,1)}(\lambda, +) \left[\frac{\lambda}{\lambda-2} e^{2\eta} + 2 \frac{\lambda}{\lambda-1} + e^{-2\eta} \right] \right. \\ \left. + F_{(-1,1)}(-\lambda, +) \frac{2}{(\lambda+1)(\lambda+2)} e^{-2\eta} \right\}$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \Gamma'_0(1, 0) = \frac{1}{(2ch\eta)^3} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta}$$

$$\times \left\{ -2^{\frac{1}{2}} F_{(0,0)}(\lambda, +) \left[\frac{(\lambda+1)^2}{\lambda-1} e^{2\eta} + 2 \frac{\lambda^2-3}{\lambda-1} + (\lambda-1) e^{-2\eta} \right] \right. \\ \left. + 4 F_{(-1,1)}(\lambda, +) \left[\frac{\lambda}{(\lambda-1)(\lambda-2)} e^{2\eta} + \frac{1}{\lambda-1} \right] \right. \\ \left. + 4 F_{(-1,1)}(-\lambda, +) \left[\frac{1}{\lambda+1} + \frac{\lambda}{(\lambda+1)(\lambda+2)} e^{-2\eta} \right] \right\}$$

(69)

$$\underline{\chi' = [0, 0]}$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \Gamma'_0(0, 0) = \frac{-2}{2ch\eta} i \int_{C_-} \frac{d\lambda}{\lambda} e^{-\lambda\eta} F(\lambda, +)$$

(70)

The function $\Gamma_{-1}^1(1,0)$ can be obtained from $\Gamma_{+1}^1(1,0)$ by crossing symmetry (53), a compensating replacement $\lambda \rightarrow -\lambda$ in the integral, and by Weyl symmetry (38) and crossing symmetry (54). In the integrals (68) to (70) the integrands are holomorphic in the half plane $\text{Re } \lambda < 0$ with possible exception of the points $\lambda = -1$ and $\lambda = -2$. At these points they may possess poles of first order. The integrands are continuous on the boundary of the half plane, in particular at $\lambda = 0$. For the contour C_- we may always take the path depicted in the Figure. If we deform the contour C_- into the imaginary axis, we obtain the terms of the discrete series in addition to the integral. We list them in Appendix 2. We have checked explicitly that the integrals over $C_+ - C_-$ vanish, so that C_+ and C_- are equivalent contours.

6. LORENTZ POLES

The functions ρ_α which we meet in physics are neither smooth nor rapidly decreasing in general. We may cut them off, however, and regularize them so that they exhibit these properties used in our formalism so far. What happens if the regularization and the cut-off is removed? If we give up the regularization, the rapid fall-off along the principal series and parallel to it ceases to hold. We can only make the hypothesis that this fall-off stays good enough for displacements of C_- to the left and right to be possible without getting contributions from the contour at infinity. A condition for this to hold can easily be deduced from the integral transforms (65) to (67) and the Riemann-Lebesgue lemma: As long as the ρ_α are cut off at infinity it suffices if they are locally integrable over η . When we remove the cut-off in an appropriate smooth fashion, the functions F_α show up singularities first at $\text{Re } \lambda = \pm \omega$, which move across the λ plane and may finally impinge on our contour. We assume that these singularities are such that a deformation of the contour saves our integral representations (68) to (70). We permit each F_α to obtain its own contour.

The position of a singularity of a function $F_{\alpha}(\lambda, \zeta)$ can be identified with a certain representation of G [that was $SL(2, C)$ extended by parity]. The physical interpretation of this is that an object transforming as such a representation under the Poincaré group with little group G has been exchanged between the proton and the photon. This interpretation is, however, ambiguous. We may first expand the scattering amplitude as in (43) and (48) with $g_1 = e$. In this case we have the proton spin fitted into a representation $\lambda - \Delta_{\alpha}\lambda, \sigma_1$ respectively $M, \lambda - \Delta_{\alpha}\lambda$ of G . This representation is coupled with the tensor to give the representation $\lambda, +1$ of G which is finally propagated to the photon side. On the other hand we may set $g_2 = e$ in (43) and (48) such that the tensor appears to be coupled to the photon side and the representations $\lambda - \Delta_{\alpha}\lambda, \sigma_1$ and $M, \lambda - \Delta_{\alpha}\lambda$ are propagated from one side to the other. Though these two interpretations are mathematically equivalent, they lead to different results if we introduce the concept of Lorentz poles. In this case we concentrate our interest on the latter alternative and denote it as "natural exchange".

The physical image of the Lorentz poles and in general the interpretation of any singularity involves a principle of universality: we expect a Fourier transform to exhibit a singularity of a specific kind if a corresponding object can be exchanged between photon and proton. A Lorentz pole of class I is defined to have the invariants λ_I, σ_1 with $\sigma_1 = +1$. A natural exchange of this Lorentz pole induces a first order pole in $F_{(0,0)}^{[1,1]}(\lambda, +)$, $F_{(0,2)}^{[1,1]}(\lambda, +)$, $F_{(0,-2)}^{[1,1]}(\lambda, +)$, and $F_{(0,0)}^{[0,0]}(\lambda, +)$ [see (62)] at the positions

$$\lambda_I = \lambda - \Delta_{\alpha}\lambda$$

(whereas with unnatural exchange we would have $\lambda_I = \lambda$). In the neighbourhood of these poles we set

$$\begin{aligned}
 F_{(0,0)}^{[1,1]}(\lambda,+) &\approx \frac{A}{\lambda-\lambda_I} \\
 F_{(0,2)}^{[1,1]}(\lambda,+) &\approx \frac{B}{\lambda-\lambda_I-2} \\
 F_{(0,2)}^{[1,1]}(-\lambda,+) &\approx \frac{C}{\lambda-\lambda_I+2} \\
 F_{(0,0)}^{[0,0]}(\lambda,+) &\approx \frac{D}{\lambda-\lambda_I}
 \end{aligned}$$

(71)

These functions possess additional poles ("mirror poles") at positions obtained by the replacement $\lambda \rightarrow -\lambda$ due to Weyl symmetry and crossing symmetry. A Lorentz pole of class II has quantum numbers λ_{II} , σ_1 with $\sigma_1 = -1$, and may appear solely in $F_{(0,0)}^{[1,0]}(\lambda,+)$

$$2^{\frac{1}{2}} F_{(0,0)}^{[1,0]}(\lambda,+) \approx \frac{E}{\lambda-\lambda_{II}}$$

(72)

A Lorentz pole of class III with quantum numbers M, λ_{III} with $M=1$ contributes to $F_{(-1,1)}^{[1,0]}(\lambda,+)$ and $F_{(1,-1)}^{[1,0]}(\lambda,+)$. We set

$$F_{(-1,1)}^{[1,0]}(\lambda,+) \approx \frac{F}{\lambda-\lambda_{III}-1}, \quad F_{(1,-1)}^{[1,0]}(-\lambda,+) \approx \frac{G}{\lambda-\lambda_{III}+1}$$

(73)

Use has not been made of the constraints (58) of current conservation so far. In the case of smooth and rapidly decreasing functions φ_α one can verify that these constraints are equivalent with

$$(\lambda^2 - 1) F_{(0,0)}^{[1,1]}(\lambda, +) + (\lambda - 1) F_{(0,2)}^{[1,1]}(\lambda + 2, +) \\ \mp (\lambda + 1) F_{(0,2)}^{[1,1]}(-\lambda + 2, +) = 0$$

$$2 F_{(0,0)}^{[0,0]}(\lambda, +) - (\lambda + 1) F_{(0,2)}^{[1,1]}(\lambda + 2, +) \\ \pm (\lambda - 1) F_{(0,2)}^{[1,1]}(-\lambda + 2, +) = 0$$

$$F_{(-1,1)}^{[1,0]}(\lambda + 1, +) \mp F_{(-1,1)}^{[1,0]}(-\lambda + 1, +) = 0$$

(74)

We assume that these relations are still valid as identities between analytic functions when the cut-off is removed. The arguments of the functions in (74) correspond to the invariant of the same exchanged object, which is of class I in the first two equations and of class III in the last equation. If we insert the Lorentz poles (71) and (73) into (74), each term in one equation (74) possesses a pole at the same position and (74) reduces simply to three constraints on the residues. If we give a sequence of Lorentz poles with $\text{Re } \lambda$ ordered into a falling sequence, we may shift the contour C_- to the left over these poles and obtain an expansion in decreasing powers of e^η , namely an asymptotic expansion for $\eta \rightarrow -\infty$ or $\nu \rightarrow +\infty$. If the residues of one Lorentz pole satisfy the constraints following from (74), the whole contribution (and not only the highest power of ν) satisfies the constraints (58). We shall later make the observation [see (75), (78)] that each term in one of the constraints (58) appears with the same highest power. This fact can be used to summarize our findings on the asymptotic behaviour of the ρ_α : for $\nu \rightarrow +\infty$ ρ_1 falls off one power of ν faster than ρ_3 and two powers faster than ρ_2 and $\rho_4 \cdot \rho_5$ and ρ_7 decrease with the same power. Only ρ_6 does not occur in the constraints (58). It goes down more slowly than ρ_5 by one power

for a class II pole and faster than ρ_5 by one power for a class III pole. This simple characterization of the asymptotic behaviour of the invariant amplitude by means of the constraints of current conservation (which is also possible if current conservation is not fulfilled) depends obviously on the choice of the invariant amplitudes.

Finally, we give the detailed results for the leading powers in ν of the Lorentz poles (71) to (73), assuming that the Lorentz pole in each F_α lies to the left of the deformed contour C_- .

Class I :

$$\begin{aligned}
 M^2 \rho_1 &\approx -2\pi\sqrt{2} \frac{1}{\lambda_I} \left\{ A (\lambda_I - 1)(\lambda_I - 2) \right. \\
 &\quad \left. + B \frac{(\lambda_I - 1)(\lambda_I - 2)}{(\lambda_I + 1)(\lambda_I + 2)} \pm C \right\} \left(\frac{2\nu}{\mu}\right)^{\lambda_I - 3} \\
 \mu^2 \rho_2 &\approx -2\pi\sqrt{2} B \frac{1}{\lambda_I + 2} \left(\frac{2\nu}{\mu}\right)^{\lambda_I - 1} \\
 M\mu \rho_3 &\approx -2\pi\sqrt{2} \left\{ A \frac{(\lambda_I - 1)^2}{\lambda_I} + 2B \frac{\lambda_I - 1}{(\lambda_I + 1)(\lambda_I + 2)} \right\} \\
 &\quad \times \left(\frac{2\nu}{\mu}\right)^{\lambda_I - 2} \\
 4\rho_4 &\approx -2\pi\sqrt{2} \left\{ -A \frac{(\lambda_I - 1)^2}{2\lambda_I} + 2B \frac{1}{(\lambda_I + 1)(\lambda_I + 2)} \right. \\
 &\quad \left. + D \frac{1}{\lambda_I} \right\} \left(\frac{2\nu}{\mu}\right)^{\lambda_I - 1}
 \end{aligned} \tag{75}$$

Current conservation implies

$$\begin{aligned}
 \pm C &= A (\lambda_I - 1) + B \frac{\lambda_I - 1}{\lambda_I + 1} \\
 D &= -\frac{1}{2} A (\lambda_I - 1)^2 + 2B \frac{\lambda_I}{\lambda_I + 1}
 \end{aligned} \tag{76}$$

Class II :

$$\begin{aligned} M\rho_5 &\simeq -8\pi\sqrt{3} E \frac{\lambda_{II}-2}{\lambda_{II}} \left(\frac{2\nu}{\mu}\right)^{\lambda_{II}-3} \\ \mu\rho_6 &\simeq -4\pi\sqrt{3} E \frac{\lambda_{II}-1}{\lambda_{II}} \left(\frac{2\nu}{\mu}\right)^{\lambda_{II}-2} \\ M\mu^2\rho_7 &\simeq +8\pi\sqrt{3} E \frac{\lambda_{II}-2}{\lambda_{II}} \left(\frac{2\nu}{\mu}\right)^{\lambda_{II}-3} \end{aligned}$$

(77)

The current is automatically conserved.

Class III :

$$\begin{aligned} M\rho_5 &\simeq +4\pi\sqrt{3} (F \pm G) \frac{1}{\lambda_{III}+1} \left(\frac{2\nu}{\mu}\right)^{\lambda_{III}-2} \\ \mu\rho_6 &\simeq -4\pi\sqrt{3} (F \pm G) \frac{2(\lambda_{III}-2)}{\lambda_{III}(\lambda_{III}+1)} \left(\frac{2\nu}{\mu}\right)^{\lambda_{III}-3} \\ M\mu^2\rho_7 &\simeq -8\pi\sqrt{3} F \frac{1}{\lambda_{III}+1} \left(\frac{2\nu}{\mu}\right)^{\lambda_{III}-2} \end{aligned}$$

(78)

Current conservation implies

$$F = \pm G$$

(79)

The crossing relation (57) can be used to deduce the asymptotic behaviour at $\nu \rightarrow -\infty$. The intercept of the parent trajectory and the pole positions are as usual connected by

$$\lambda_{I,II,III} = \alpha(0) + 1$$

(80)

The result (75) in the case of current conservation coincides with a known result ⁷⁾ obtained by an intuitive method based on helicity amplitudes.

A P P E N D I X

1. THE CANONICAL BASIS FOR TENSORS OF RANK TWO

A tensor of rank two can be decomposed into four irreducible representations $\chi' = [j_1, j_2]$ with $j_{1,2}$ equal to 0 or 1. We want to identify them with the symmetric traceless, the antisymmetric self-dual and anti-selfdual, and the trace part. We start with the definition

$$\begin{aligned} A_{\mu\nu} &= A_{\mu\nu}^S + A_{\mu\nu}^Q + A_{\mu\nu}^T \\ A_{\mu\nu}^S &= \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{4} g_{\mu\nu} \sum_{\lambda,\sigma} A_{\lambda\sigma} g^{\lambda\sigma} \\ A_{\mu\nu}^Q &= \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu}) \\ A_{\mu\nu}^T &= \frac{1}{4} g_{\mu\nu} \sum_{\lambda,\sigma} A_{\lambda\sigma} g^{\lambda\sigma} \end{aligned} \tag{A.1}$$

In the polynomial space $E_{\chi'}$ for $\chi' = [1, 1]$ we introduce the tensor basis $f^{\mu\nu}(z)$ as

$$f^{\mu\nu}(z) = f^{\mu}(z) f^{\nu}(z) \tag{A.2}$$

where $f^{\mu}(z)$ is the vector basis [Ref. 1), Eq. (A.24)]. We relate it with the canonical basis [Ref. 4), (3.83), (3.84)] by

$$\sum_{\mu\nu} A_{\mu\nu}^S f^{\mu\nu}(z) = \sum_{\lambda, Q} A_Q^{\lambda} f_Q^{\lambda}(z) \tag{A.3}$$

We find

$$\begin{aligned}
 A_0^0 &= \pi^{\frac{1}{2}} \frac{4}{3} A_{00}^S \\
 A_0^1 &= -\left(\frac{\pi}{3}\right)^{\frac{1}{2}} 2 A_{03}^S, \quad A_{\pm 1}^1 = \pm \left(\frac{2\pi}{3}\right)^{\frac{1}{2}} (A_{01}^S \mp i A_{02}^S) \\
 A_0^2 &= +\left(\frac{\pi}{5}\right)^{\frac{1}{2}} (A_{33}^S - \frac{1}{3} A_{00}^S) \\
 A_{\pm 1}^2 &= \mp \left(\frac{2\pi}{15}\right)^{\frac{1}{2}} (A_{31}^S \mp i A_{32}^S) \\
 A_{\pm 2}^2 &= +\left(\frac{\pi}{30}\right)^{\frac{1}{2}} (A_{11}^S - A_{22}^S \mp 2i A_{12}^S)
 \end{aligned}$$

(A.4)

The antisymmetric part $A_{\mu\nu}^a$ decomposes into the representations $\chi' = [1, 0]$ and $\chi' = [0, 1]$. We define for

$$\underline{\chi' = [1, 0]}$$

$$\begin{aligned}
 A_{\pm 1}^1 &= +\left(\frac{\pi}{3}\right)^{\frac{1}{2}} \frac{1}{2} [\pm A_{23}^a - i A_{31}^a \mp i A_{01}^a - A_{02}^a] \\
 A_0^1 &= -\left(\frac{\pi}{3}\right)^{\frac{1}{2}} 2^{-\frac{1}{2}} [A_{12}^a - i A_{03}^a]
 \end{aligned}$$

(A.5)

$$\underline{\chi' = [0, 1]}$$

$$\begin{aligned}
 A_{\pm 1}^1 &= +\left(\frac{\pi}{3}\right)^{\frac{1}{2}} \frac{1}{2} [\pm A_{23}^a - i A_{31}^a \pm i A_{01}^a + A_{02}^a] \\
 A_0^1 &= -\left(\frac{\pi}{3}\right)^{\frac{1}{2}} 2^{-\frac{1}{2}} [A_{12}^a + i A_{03}^a]
 \end{aligned}$$

(A.6)

The trace part gives the trivial representation $\chi' = [0, 0]$

$$A_0^0 = \pi^{\frac{1}{2}} \sum_{\mu\nu} A_{\mu\nu} g^{\mu\nu} \quad (\text{A.7})$$

2. THE DISCRETE SERIES

If we deform the contour C_- in (68) to (70) into the imaginary axis we obtain contributions from the poles, which we ascribe to a discrete series of irreducible tensor operators. We find for

$$\underline{\chi' = [1, 1]}$$

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}} (+) \Gamma_0^2(1, 1) \text{ from the pole at } \lambda = -1$$

$$- \frac{8 \cdot 2^{\frac{1}{2}} \sinh \eta}{(2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \sinh \eta' \mu^2 \rho_2^{(+)}$$

(A.8)

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}} (-) \Gamma_0^2(1, 1) \text{ from the pole at } \lambda = -2$$

$$- \frac{4 \cdot 2^{\frac{1}{2}}}{(2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \mu^2 \rho_2^{(-)}$$

(A.9)

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} (+) \Gamma_0^1(1, 1) \text{ from the pole at } \lambda = -1$$

$$+ \frac{2 \cdot 2^{\frac{1}{2}}}{(2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' [\rho_1 \mu \rho_3^{(+)} - 2 \sinh \eta' \mu^2 \rho_2^{(+)}]$$

(A.10)

$$\underline{\chi' = [1,0]}$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} (+) \Gamma_{\pm 1}^1(1,0) \quad \text{from the pole at} \quad \lambda = -2$$

$$- \frac{2}{3^{\frac{1}{2}} (2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \Gamma \mu^2 \rho_7^{(+)}$$

(A.11)

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} (-) \Gamma_{\pm 1}^1(1,0) \quad \text{from the pole at} \quad \lambda = -1$$

$$\pm \frac{4}{3^{\frac{1}{2}} (2 \cosh \eta)^3} \left\{ - \cosh \eta \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \mu \rho_6^{(-)} + e^{\mp \eta} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \sinh \eta' \Gamma \mu^2 \rho_7^{(-)} \right\} \quad (\text{A.12})$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} (+) \Gamma_0^1(1,0) \quad \text{from the pole at} \quad \lambda = -2$$

$$+ \frac{4}{3^{\frac{1}{2}} (2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \Gamma \mu^2 \rho_7^{(+)}$$

(A.13)

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} (-) \Gamma_0^1(1,0) \quad \text{from the pole at} \quad \lambda = -1$$

$$+ \frac{8 \sinh \eta}{3^{\frac{1}{2}} (2 \cosh \eta)^3} \int_{-\infty}^{+\infty} d\eta' \cosh \eta' \sinh \eta' \Gamma \mu^2 \rho_7^{(-)}$$

(A.14)

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