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TENSOR OPERATORS OF RANK TWO FOR THE HOMOGENEOUS LORENTZ GROUP

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A B S T R A C T

We suggest a formula for the decomposition of a covariant operator which transforms as an arbitrary finite dimensional representation of $SL(2, \mathbb{C})$ into a direct integral of irreducible covariant operators. An earlier proof of this formula for four-vector operators is extended to symmetric and anti-symmetric rank-two tensor operators. The matrix elements of all irreducible rank-two tensor operators are tabulated. Finally we generalize the results for the group $SL(2, \mathbb{C})$ extended by parity.

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1. - INTRODUCTION

Commutators and T products of current density operators are well-known examples of operators which, besides their dependence on a co-ordinate or momentum, behave as Lorentz tensor operators of rank two. The matrix element

$$\int d^4x e^{iqx} \langle p, r_2 | [j_\mu(x), j_\nu(0)] | p, r_1 \rangle \quad (1)$$

of such commutator appears for example in the cross-section for inelastic scattering of electrons off protons when the final states of the baryon are disregarded. In this case p denotes the proton momentum and r its spin components. The matrix element (1) may be regarded as the absorptive part of the Compton scattering amplitude in forward direction of a virtual photon with fixed mass q^2 . In the case of inelastic electron-proton scattering, this mass q^2 is negative. Nevertheless such amplitude for the elastic forward scattering of a virtual photon possesses the right kinematics for a Toller expansion¹⁾. This expansion has somewhat unusual features: not only is the photon space-like, but the amplitude itself is a tensor amplitude instead of a scalar (the photon spin is set equal to zero in this formalism). This forces us to decompose the tensor operator into irreducible tensor operators which in turn are submitted to an analogue of the classical Wigner-Eckart theorem for the group $SU(2)$. The decomposition of an arbitrary covariant operator into irreducible covariant operators is the problem concerning us here in the special case of tensor operators of rank two, both symmetrical and antisymmetrical tensors.

In this article we use heuristic arguments to suggest a decomposition formula for any well behaved covariant operator whose covariance is like a finite dimensional irreducible representation of $SL(2, C)$. This formula coincides with the expansion formula for four-vector operators derived in²⁾, and is proved for tensor operators of rank two now. In particular this amounts to a computation of the normalization coefficients

in this formula, which are not accessible by our heuristic arguments. Our proof uses the methods developed in ^{2),3)}, the reader is referred to these articles for the details of the argument. In fact, we presuppose that the reader is acquainted with the ideas and notations of ^{2),3)}. We make only two minor modifications in the notations. Instead of (m, ϱ) we write (M, λ) with

$$M = \frac{1}{2}m, \quad \lambda = \frac{i}{2}\varrho$$

and we replace $\bar{\chi} = (m, -\varrho)$ by $\chi^* = (M, -\lambda)$, since we want to use (*) systematically for the analytic continuation of the complex conjugate off the principal series (imaginary λ).

The matrix elements of the irreducible tensor operators involved in the expansion formula are explicitly given in a form which is reduced with respect to $SU(2)$. This list of matrix elements can be shortened considerably, if we make use of certain symmetry relations; the symmetry under the replacement $\chi \rightarrow -\chi$ (Weyl symmetry), the symmetry under parity conjugation, and the symmetry under transposition. Use is made of these symmetries also if we extend $SL(2, \mathbb{C})$ by group elements of order two. We consider the parity operation as one example of particular importance.

2. - IRREDUCIBLE COVARIANT OPERATORS

Finite dimensional irreducible representations of $SL(2, \mathbb{C})$ are conventionally classified by two non-negative half integers $[j_1, j_2]$, and can be realized in linear spaces E of polynomials of degree $2j_1, 2j_2$ in the complex variables $z, \bar{z}$, respectively. We introduce the symbol χ by

$$\begin{aligned} \chi &= (M, \lambda) \\ M &= j_2 - j_1, \quad \lambda = j_1 + j_2 + 1 \end{aligned} \quad (2)$$

and denote the space E correspondingly $E_{\chi}(Z)$. In addition we use other linear vector spaces $D_{\chi}(Z)$, $\chi = (M, \lambda)$, where $2M$ is integral and λ arbitrary complex, as the spaces of infinitely differentiable functions $f(z)$ of x and y , $z = x + iy$, with the asymptotic expansion

$$f(z) \cong z^{-M+\lambda-1} \bar{z}^{M+\lambda-1} \sum_{k, l=0}^{\infty} a_{kl} z^{-k} \bar{z}^{-l} \quad (3)$$

at infinity. A stereographic mapping enables us to present $D_{\chi}(Z)$ as a space of infinitely differentiable functions on the unit sphere in three-space, which allows us to introduce a topology in a standard fashion. Obviously the spaces $E_{\chi}(Z)$ are subspaces of special spaces $D_{\chi}(Z)$.

In the space $D_{\chi}(Z)$ we can define the nucleus of a representation of $SL(2, C)$ which we also label by χ , by

$$T_a^{\chi} f(z) = (a_{12}z + a_{22})^{-M+\lambda-1} (\bar{a}_{12}\bar{z} + \bar{a}_{22})^{M+\lambda-1} f(z_a) \\ a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad z_a = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}} \quad (4)$$

In the case when $D_{\chi}(Z)$ possesses a subspace $E_{\chi}(Z)$, this subspace is invariant under (4), and (4) defines the finite dimensional irreducible representation χ in $E_{\chi}(Z)$. In both kinds of spaces E and D we can introduce a "canonical basis" $\{f_Q^J(z)\}$, whose elements were given in ⁴⁾ Eqs. (3.83), (3.84) in the form of Jacobi functions. The labels Q and J run over

$$|Q| \leq \gamma, \quad |M| \leq \gamma \quad \begin{cases} < \infty & \text{in the case of } D_{\chi}(Z) \\ < \lambda & \text{in the case of } E_{\chi}(Z) \end{cases}$$

such that $J - |M|$ and $J - |Q|$ are integers. The $2J+1$ dimensional subspace spanned by the elements $f_Q^J(z)$ with fixed J carries an irreducible representation of $SU(2)$ to the spin J .

We consider now operators $T(z|z')$ which map $D\chi_1(Z)$ into $E\chi(Z')$ with the coefficients of the polynomials in $D\chi_2(Z)$. These coefficients are assumed to depend continuously on the element of $D\chi_1(Z)$. If in addition

$$\begin{aligned} T_{\alpha}^{\chi_2} T(z|z') T_{\alpha}^{\chi_1} &= \\ &= (a_{12}z' + a_{22})^{2j_1} (\bar{a}_{12}\bar{z}' + \bar{a}_{22})^{2j_2} T(z|z'_a) \end{aligned} \quad (5)$$

is satisfied, we call $T(z|z')$ an irreducible covariant operator (in special cases also tensor, vector, or spinor operator) and χ its covariance. Such operators exist only if χ_1 and χ_2 are related as

$$\begin{aligned} \pm\chi_2 &= \chi_1 + \alpha \\ \alpha &= (\Delta M, \Delta \lambda) \end{aligned} \quad (6)$$

where $2\Delta M$ and $2\Delta \lambda$ are integers. We are interested only in the upper sign of (6), since the covariant operator for the lower sign can be obtained from the operator with the upper sign by left multiplication with the intertwining operator $A\chi_2$ which maps $D\chi_2$ into $D-\chi_2$ ^{4),5)}. The covariant operators can be given in the form of differential operators whose form is known from ^{3),6)}

$$\begin{aligned} T_{\nu\mu}(z|z') &= \\ &= (z-z')^{A_1} (\bar{z}-\bar{z}')^{B_1} \left(\frac{\partial}{\partial z}\right)^{\nu} \left(\frac{\partial}{\partial \bar{z}}\right)^{\mu} (z-z')^{A_2} (\bar{z}-\bar{z}')^{B_2} \\ &= \sum_{k=0}^{\nu} \sum_{\ell=0}^{\mu} k! \ell! \binom{\nu}{k} \binom{\mu}{\ell} \binom{A_2}{k} \binom{B_2}{\ell} \\ &\quad \times (z-z')^{2j_1-k} (\bar{z}-\bar{z}')^{2j_2-\ell} \left(\frac{\partial}{\partial z}\right)^{\nu-k} \left(\frac{\partial}{\partial \bar{z}}\right)^{\mu-\ell} \end{aligned} \quad (7)$$

ν and μ take on integral values in the intervals

$$\begin{aligned} 0 \leq \nu \leq 2j_1 \\ 0 \leq \mu \leq 2j_2 \end{aligned} \quad (8)$$

A and B are defined as

$$\begin{aligned} A_1 &= -M_2 + \lambda_2 + \nu \\ B_1 &= +M_2 + \lambda_2 + \mu \\ A_2 &= 2j_1 - A_1 \\ B_2 &= 2j_2 - B_1 \end{aligned} \quad (9)$$

For α (6) we obtain

$$\alpha = (\nu - \mu + j_2 - j_1, j_1 + j_2 - \nu - \mu) \quad (10)$$

For one given covariance χ there exist

$$\dim \chi = (2j_1 + 1)(2j_2 + 1)$$

different operators $T_{\nu\mu}(z|z')$. In the Table we give some examples. As (10) shows these operators can be labelled by their α instead of the pair of integers ν, μ . Plotting the possible α 's in the $\Delta M \Delta \lambda$ plane we get the " α diagram" corresponding to the covariance χ . Its general form is depicted in the Figure. Instead of $T_{\nu\mu}$ we shall in the sequel use the notations T_α or $T(\Delta M, \Delta \lambda)$.

For any $\chi = (M, \lambda)$ we use the notation $\chi^* = (M, -\lambda)$ and $\hat{\chi} = -\chi^* = (-M, \lambda)$. For reasons which become clear in Section 5, we call a pair of representations $\chi, \hat{\chi}$ parity conjugate. The diagrams for irreducible covariant operators with parity conjugate covariances are mapped onto each other by a reflection at the $\Delta \lambda$ axis. We call such pair of α diagrams also parity conjugate.

Covariance χ $= [j_1, j_2] = (M, \lambda)$	Name of representation	Number of operators with covariance χ
$[0, 0] = (0, 1)$	Scalar	1
$[\frac{1}{2}, \frac{1}{2}] = (0, 2)$	Four-vector	4
$[1, 0] = (-1, 2)$	Self-dual antisymmetric tensor of rank two	3
$[0, 1] = (+1, 2)$	Antiself-dual antisymmetric tensor of rank two	3
$[1, 1] = (0, 3)$	Symmetric traceless tensor of rank two	9

- Irreducible covariant operators for tensors of low rank -

3. - COVARIANT OPERATORS AND THEIR DECOMPOSITION

We consider the space $C^\infty(SL(2, C))$ of infinitely differentiable functions $f(a)$ on $SL(2, C)$ which have compact support. A covariant operator can be defined as a linear map $T(z)$ from the tensor product space $C_1^\infty \times C_2^\infty$ into the space $E_\chi(z)$ so that under all right translations in C^∞

$$\begin{aligned}
 T(z) (T_a^r f_1 \times T_a^r f_2) &= \\
 &= (a_{11}z + a_{21})^{2j_1} (\bar{a}_{11}\bar{z} + \bar{a}_{21})^{2j_2} T(z_a)(f_1 \times f_2)
 \end{aligned} \tag{11}$$

In addition we require continuity of $T(z)$ in both arguments f_1 and f_2 . Such a covariant operator can be presented in the form

$$T(z)(f_1 \times f_2) = \int \Gamma(z; a_2, a_1) f_2(a_2) f_1(a_1) \times d\mu(a_2) d\mu(a_1) \quad (12)$$

where Γ is a polynomial in z and $\bar{z}$ with distributions as coefficients. $d\mu(a)$ is the Haar measure on $SL(2, \mathbb{C})$ in our standard normalization ⁴⁾. If we use the canonical basis in $E_\chi(Z)$ to expand Γ ,

$$\Gamma(z; a_2, a_1) = \sum_{j, Q} \Gamma_Q^j(a_2, a_1) f_Q^j(z) \quad (13)$$

we can express the covariance condition (11) as

$$\Gamma_Q^j(a_2 \bar{a}'_1, a_1 \bar{a}'_2) = \sum_{j', Q'} D_{j'Q'jQ}^\chi(a) \Gamma_{Q'}^{j'}(a_2, a_1) \quad (14)$$

The functions $D_{j_1 Q_1 j_2 Q_2}^\chi(a)$ are defined as

$$T_a^\chi f_Q^j(z) = \sum_{j', Q'} D_{j'Q'jQ}^\chi(a) f_{Q'}^{j'}(z) \quad (15)$$

and reduce on $SU(2)$ to

$$D_{jQj'Q'}^\chi(u) = \delta_{jj'} D_{QQ'}^j(u), \quad j \geq 1/2 \quad (16)$$

In this article we are in fact only interested in a class of "smooth" covariant operators, which are generated by infinitely differentiable functions $\Gamma_Q^j(a_2, a_1)$. We assume further that the functions $\Gamma_Q^j(a, e)$ fall off faster than any power of $|a|$

$$|a|^2 = T\pi(aa^\dagger) \quad (17)$$

for $|a| \rightarrow \infty$. Extending the decomposition of this restricted class to the class of "distribution valued" covariant operators is a problem closely related with the extension of harmonic analysis on $SL(2, \mathbb{C})$ from C^∞ to spaces of distributions. It lies beyond the scope of this article.

We restrict the class of smooth covariant operators further.

We define

$$\begin{aligned} \Gamma_Q^J(a_2, a_1)_{q_2}^{S_2 \tau_2}, \quad S_1 \tau_1, \quad q_1 &= \\ = \int D_{q_2 \tau_2}^{S_2}(u_2^{-1}) D_{\tau_1 q_1}^{S_1}(u_1) \Gamma_Q^J(u_2 a_2, u_1 a_1) d\mu(u_2) d\mu(u_1) \end{aligned} \quad (18)$$

with the normalized Haar measure $d\mu(u)$ on $SU(2)$ and the functions $D_{q_1 q_2}^S(u)$ as in (16). The representation property of these functions implies

$$\begin{aligned} \Gamma_Q^J(u_2 a_2, u_1 a_1)_{q_2}^{S_2 \tau_2}, \quad S_1 \tau_1, \quad q_1 &= \sum_{q'_1 q'_2} D_{q_2 q'_2}^{S_2}(u_2) \\ &\times D_{q'_1 q_1}^{S_1}(u_1^{-1}) \Gamma_Q^J(a_2, a_1)_{q'_2}^{S_2 \tau_2}, \quad S_1 \tau_1, \quad q'_1 \end{aligned} \quad (19)$$

As the inversion of (18) we have the expansion

$$\begin{aligned} \Gamma_Q^J(a_2, a_1) &= \\ = \sum_{\substack{S_1, S_2 \\ q_1, q_2}} (2S_2+1)(2S_1+1) \Gamma_Q^J(a_2, a_1)_{q_2}^{S_2 q_2}, \quad S_1 q_1 &\quad (20) \end{aligned}$$

Any covariant operator can therefore be decomposed into a sum of covariant operators which are generated by a set of functions or distributions $\Gamma_Q^J(a_2, a_1)_{q_2 q_1}$ with the property (19). Consequently we restrict our investigation to such projections.

As the first step in our heuristic approach to the decomposition theorem we construct an invariant bilinear functional (f, g) on $E_{\chi}(Z)$. We give it in the canonical basis since this is the form we need. We expand $f(z) \in E_{\chi}(Z)$ as

$$f(z) = \sum_{j,Q} A_Q^j f_Q^j(z)^x \quad (21)$$

and $g(z)$ correspondingly. The ansatz

$$(f,g) = \sum_{j,Q} (-1)^{j+Q} G_j A_Q^j B_{-Q}^j \quad (22)$$

guarantees invariance under $SU(2)$ for arbitrary G_j . If we set

$$G_j = \frac{(j_1+j_2+j+1)!(j_1+j_2-j)!}{(2 \max(j_1,j_2)+1)(2j_1)!(2j_2)!} \quad (23)$$

the form (22) is invariant under the whole group $SL(2,C)$ and is normalized by

$$G_{|j_1-j_2|} = 1 \quad (24)$$

In a similar fashion we obtain a linear-antilinear functional for

$f(z) \in E_{\chi}(Z)$ and $g(z) \in E_{\hat{\chi}}(Z)$. In fact, if $g(z) \in E_{\hat{\chi}}(Z)$, then $\overline{g(z)} \in E_{\chi}(Z)$ and [see ⁴⁾, Eqs. (3.83), (3.84)]

$$g(z) = \sum_{j,Q} B_Q^j f_Q^j(z)^{\hat{\chi}}, \quad \overline{g(z)} = \sum_{j,Q} B'^j_Q f_Q^j(z)^x \quad (25)$$

$$B'^j_Q = (-1)^{Q-M} \overline{B_{-Q}^j}$$

Instead of (22) we have then

$$(f,\bar{g}) = \sum_{j,Q} (-1)^{j-M} G_j A_Q^j \overline{B_Q^j} \quad (26)$$

As a second tool we need matrix elements of the irreducible covariant operators which possess the properties (14), (19). If χ_2 belongs to the principal series and A is an operator from $D\chi_1$ into $D\chi_2$, the scalar product enables us to define matrix elements of A in the respective canonical bases by

$$\begin{aligned} \langle \chi_2, \gamma_2 Q_2 | A | \chi_1, \gamma_1 Q_1 \rangle &= \\ &= \int \overline{f_{Q_2}^{\gamma_2}(z)^{\chi_2}} A f_{Q_1}^{\gamma_1}(z)^{\chi_1} dz \\ dz &= dx dy, \quad z = x + iy, \quad -\infty < x, y < \infty \end{aligned} \quad (27)$$

With the help of the relation

$$\overline{f_Q^{\gamma}(z)^{\chi_2}} = (-1)^{Q-M} f_{-Q}^{\gamma}(z)^{-\chi_2} \quad (28)$$

this definition can be extended from the principal series to all χ_2 by analytic continuation in λ_2 . In particular we get the orthogonality relation

$$\langle \chi_2, \gamma_2 Q_2 | \chi_1, \gamma_1 Q_1 \rangle = \delta_{\gamma_1, \gamma_2} \delta_{Q_1, Q_2} \quad (29)$$

and the expansion

$$\begin{aligned} A f_{Q_1}^{\gamma_1}(z)^{\chi_1} &= \\ &= \sum_{\gamma_2 Q_2} \langle \chi_2, \gamma_2 Q_2 | A | \chi_1, \gamma_1 Q_1 \rangle f_{Q_2}^{\gamma_2}(z)^{\chi_2} \end{aligned} \quad (30)$$

converging in the topology of $D\chi_2$.

Such matrix elements exist also for the operators $T_Q^J(\alpha)$ obtained by expanding the irreducible covariant operator $T_\alpha(z|z')$ in the canonical basis of $E\chi(z')$,

$$T_{\alpha}(z|z') = \sum_{JQ} T_Q^J(z|\alpha) f_Q^J(z')^{\chi} \quad (31)$$

On $SU(2)$ these matrix elements can be reduced as

$$\begin{aligned} \langle \chi_2 j_2 Q_2 | T_Q^J(\alpha) | \chi_1 j_1 Q_1 \rangle &= (-1)^{2j_1} (2j_1 + 1)^{-\frac{1}{2}} \\ &\times (j_2 Q_2 j_1 Q_1 | j_1 Q_1) \langle \chi_2 j_2 | T^J(\alpha) | \chi_1 j_1 \rangle \end{aligned} \quad (32)$$

where $(j_2 Q_2 j_1 Q_1 | j_1 Q_1)$ is the Clebsch-Gordan coefficient of $SU(2)$ in the phase conventions of Edmonds ⁷⁾.

If we compare the matrix elements

$$\langle \chi_2 j_2 S_2 q_2 | T_{a_2}^{\chi_2} T_Q^J(\alpha) T_{a_1}^{\chi_1} | \chi_1 j_1 S_1 q_1 \rangle \quad (33)$$

where the operators $T_Q^J(\alpha)$ have covariance χ' and S_1, S_2 are fixed, with the functions $\Gamma_Q^J(a_2, a_1)$ which generate a projection (18) with fixed labels S_1, S_2, r_1, r_2 of a covariant operator with the same covariance χ' , we realize that both behave identically under (14) and (19). This behaviour is obviously independent of χ_2 and α . We suggest therefore that the following expansion holds

$$\begin{aligned} \Gamma_Q^J(a_2, a_1) q_2 q_1 &= i \int_{-i\infty}^{+i\infty} d\lambda \sum_{\mu} \sum_{\alpha} F_{\alpha}(\mu, \lambda) \\ &\times \langle \chi_2 j_2 S_2 q_2 | T_{a_2}^{\chi_2} T_Q^J(\alpha) T_{a_1}^{\chi_1} | \chi_1 - \alpha; S_1 q_1 \rangle \end{aligned} \quad (34)$$

The summation over α extends over the whole α diagram of χ' , and the summation over M is restricted to the finite set

$$|M| \leq S_2, \quad |M - \Delta_\alpha M| \leq S_1 \quad (35)$$

The expansion formula (34) is not yet complete and has therefore only a formal meaning. The rigorous formula is postponed till the end of this Section [formula (40)]. We have next to invert (34), namely, to express F by Γ .

We guess the general form of the inversion formula for χ in the principal series as

$$F_\alpha(\chi) = C_\alpha(\chi) \int d\mu(a_2 a_1^{-1}) \sum_{\gamma Q} (-1)^{J-M'} G_\gamma \Gamma_Q^\gamma(a_2, a_1) q_2 q_1 \\ \times \langle \chi; S_2 q_2 | T_{a_2}^\chi T_Q^\gamma(\alpha^*) T_{a_1^{-1}}^{\chi-\alpha^*} | \chi-\alpha^*; S_1 q_1 \rangle \quad (36)$$

α^* belongs to the parity conjugate α diagram corresponding to the covariance $\hat{\chi}'$, when the covariance of Γ is denoted $\chi' = (M', \lambda')$. $C_\alpha(\chi)$ are rational functions of λ , which we denote the "normalization coefficients". If we replace a_2 and a_1 in the integral (36) by $a_2 a^{-1}$, $a_1 a^{-1}$ respectively, the dependence on a cancels because of (5), (14), and the orthogonality relation

$$\sum_{\gamma Q} (-1)^{J-M'} G_\gamma D_{\gamma Q \gamma' Q'}^{\chi'}(a) D_{\gamma Q \gamma'' Q''}^{\hat{\chi}'}(a) \\ = \delta_{\gamma' \gamma''} \delta_{Q' Q''} (-1)^{J'-M'} G_{\gamma'} \quad (37)$$

following from the invariance of the functional (26). If we replace a_2, a_1 by $u_2 a_2, u_1 a_1$ respectively, we can verify the independence of the integral (36) on q_2, q_1 . This proves the consistency of the ansatz (36). In the case where we worked out the proof of (36), i.e., for the covariances listed in the Table, the normalization coefficients have been computed. They are given in the subsequent Section.

As in ²⁾, Eq. (30) we may set

$$a_1 = u_1 d(\eta_1)^{-1} a$$

$$a_2 = u_2 d(\eta_2) a$$

$$d(\eta_i) = e^{\frac{1}{2} \eta_i \sigma_3}, \quad i = 1, 2$$

$$\eta_i \geq 0, \quad \eta = \eta_1 + \eta_2$$

$$d\mu(a_2 a_1^{-1}) = (4\pi)^{-1} \sinh^2 \eta d\eta d\mu(u_1) d\mu(u_2) \quad (38)$$

$$D_{j_1 q_1 j_2 q_2}^x(d(\eta)) = \delta_{q_1 q_2} d_{j_1 j_2 q_1}^x(\eta)$$

and obtain

$$\begin{aligned} F_\alpha(x) = & C_\alpha(x) [4\pi(2S_1+1)(2S_2+1)]^{-1} \\ & \times \sum_{jQ} (-1)^{j-M'} G_j \sum_{j_1 j_2 q_1 q_2} \langle x^*_{j_2 q_2} | T_Q^j(\alpha^*) | x^*_{j_1 q_1} \rangle \\ & \times \int_0^\infty d\eta \sinh^2 \eta d_{S_2 j_2 q_2}^{x^*}(\eta_2) d_{j_1 S_1 q_1}^{x^* - \alpha^*}(\eta_1) \\ & \times \Gamma_Q^j(d(\eta_2), d(\eta_1)^{-1}) q_2 q_1 \end{aligned} \quad (39)$$

In the step from (36) to (39) we made use of the fact that the matrix elements of the irreducible covariant operators are real functions (polynomials) of λ .

In the form (39) the analytic continuation of $F_{\alpha}(\lambda)$ is straightforward. It results that $F_{\alpha}(\lambda)[C_{\alpha}(\lambda)]^{-1}$ is an entire function of λ . Parallel to the imaginary axis this function drops off faster than any power of $|\lambda|$. In the special cases of scalar or four-vector covariance $F_{\alpha}(\lambda)$ is entire itself, but it may have poles on the real λ axis in general (see the expression for the normalization coefficients given in the subsequent Section). The Weyl symmetry (44), (46) enables us to introduce functions of the second kind (see ²⁾ and ⁴⁾, Section 4) into (34). These functions have a finite number of poles on the real λ axis. We define a contour $C_{+}(C_{-})$ in the λ plane, which consists of two upwards oriented infinite intervals on the imaginary axis and of a finite piece which goes around all the poles on the real axis in the positive (negative) sense. We replace then (34) by the integral

$$\Gamma_Q^J(d(\eta), e)_{q_2 q_1} = 2i \int_{C_{\pm}} d\lambda \sum_M \sum_{\alpha} F_{\alpha}(M, \lambda) \quad (40)$$

$$\times \sum_{j_2} e^{x_{S_2 j_2 q_2}}(\eta) \langle x; j_2 q_2 | T_Q^J(\alpha) | x - \alpha; S_1 q_1 \rangle$$

This integral converges rapidly and yields the correct asymptotic behaviour of Γ_Q^J at $|\eta| \rightarrow \infty$. If we deform the contours $C_{\pm}$ into the imaginary axis, in addition to (34) we get contributions from the poles. In analogy with the Plancherel theorem for $SU(1,1)$ we may call them contributions of the "discrete series".

The Fourier expansion (36) and its inversion (40) have been proved (apart from the contributions of the "discrete series" for the representations contained in the Table. Our proof is based on the Plancherel theorem, from which we derive an expansion theorem for covariant operators whose covariance lies in the principal series ²⁾. From the principal series we reach the finite dimensional representations by analytic continuation. Explicitly we have to prove the limit

$$\lim_{x'} \sum_{q_1 q_2 Q} (j_2 q_2; j_Q | j_1 q_1) (j_2 q_2; j'_Q | j_1 q_1)$$

$$\times G(x_1^*, x_2, x' | j_1 j_1 q_1; j_2 j_2 q_2; j j'_Q) =$$

$$\begin{aligned}
 &= -\frac{\pi}{4} i \sum_{\alpha} C_{\alpha}(\chi_2) [(M_1^2 - \lambda_1^2)(M_2^2 - \lambda_2^2)]^{-1} (-1)^{j_1 - M_1'} G_{j_1'} \\
 &\quad \times [(2j_1 + 1)(2j_1 + 1)]^{\frac{1}{2}} \langle \chi_{2j_2}^* | j_2 \| T^{j_1'}(\alpha^*) \| \chi_{2j_2}^* - \alpha^* | j_1 \rangle \\
 &\quad \times \langle \chi_{2j_2} | j_2 \| T^{j_1}(\alpha) \| \chi_{2j_2} - \alpha | j_1 \rangle \delta(\lambda_1, \lambda_2 - \Delta_{\alpha} \lambda) \delta_{M_1, M_2 - \Delta_{\alpha} M} \\
 &\hspace{15em} (41)
 \end{aligned}$$

where the delta functionals on the right side appear through pairs of poles pinching the integration contour in the λ_1 plane, and these functionals are understood to be applied to Weyl symmetric (see next Section) functions in χ_1 . This kind of proof is too complicated to establish the expansion theorem (36), (40) in general. The function G whose explicit functional form is used in the proof, is related with the Clebsch-Gordan coefficients of $SL(2, \mathbb{C})$.

4. - MATRIX ELEMENTS OF IRREDUCIBLE COVARIANT OPERATORS

The labour of computing matrix elements is considerably reduced by three symmetries exhibited by these matrix elements. We consider first the "Weyl symmetry", which reflects the equivalence of the representations χ and $-\chi$ provided the spaces $D_{\pm\chi}$ are irreducible. This equivalence means that an intertwining operator $A_{\chi}^{(4),5)}$ exists which establishes a bicontinuous map

$$D_{\chi} \xrightarrow{A_{\chi}} D_{-\chi} \xrightarrow{(A_{\chi})^{-1}} D_{\chi}$$

and intertwines the representations

$$T_a^{-\chi} A_{\chi} = A_{\chi} T_a^{\chi} \quad \text{for all } a \in SL(2, \mathbb{C}) \quad (42)$$

This intertwining operator is unique up to a constant factor which may depend on χ . We normalize A_χ such that on the canonical basis [Ref. 4), Eqs. (3.93), (3.94)]

$$\begin{aligned} A_\chi f_Q^j(z)^\chi &= \beta^j(\lambda) f_Q^j(z)^{-\chi} \\ \beta^j(\lambda) &= \Gamma(j+\lambda+1) [\Gamma(j-\lambda+1)]^{-1} \end{aligned} \quad (43)$$

Explicit calculation gives for the irreducible covariant operator (7) with covariance $\chi' = [j_1, j_2]$

$$\begin{aligned} \langle -\chi; j_2 \| T^j(-\alpha) \| -\chi + \alpha; j_1 \rangle &= \\ = (-1)^{2j_2} \beta^{j_2}(\lambda) \beta^{j_1}(-\lambda + \Delta_\alpha \lambda) & \quad (44) \\ \times \langle \chi; j_2 \| T^j(\alpha) \| \chi - \alpha; j_1 \rangle \end{aligned}$$

The operators on both sides exhibit the same covariance. From the corresponding symmetry ³⁾, Eqs. (1.11), (1.32), of the function G and from (41), (44) we derive

$$C_{-\alpha}(-\chi) = (-1)^{2(j_2-j_1)} C_\alpha(\chi) \quad (45)$$

If we insert (44), (45) into (39) we obtain the Weyl symmetry of the function $F_\alpha(\chi)$

$$\begin{aligned} F_{-\alpha}(-\chi) &= \\ = (-1)^{2j_2} \beta^{s_2}(-\lambda) \beta^{s_1}(\lambda - \Delta_\alpha \lambda) F_\alpha(\chi) & \quad (46) \end{aligned}$$

In order to derive (46) no additional property of the functions $\Gamma_Q^j(a_2, a_1)_{q_2 q_1}$ has been used. This is different in the case of the two other symmetries.

The differential operators (7), (31) for a pair of parity conjugate covariances are related by

$$T_Q^j(\bar{z}|\alpha) D_{\chi-\alpha} \rightarrow D_\chi = (-1)^{Q-M'} T_{-Q}^j(z|\hat{\alpha}) D_{\hat{\chi}-\hat{\alpha}} \rightarrow D_{\hat{\chi}} \quad (47)$$

the covariances being χ ' on the left and $\hat{\chi}$ ' on the right-hand side. Similarly as (28) we have

$$f_Q^j(\bar{z})^\chi = (-1)^{Q-M} f_{-Q}^j(z)^{\hat{\chi}} \quad (48)$$

Both (47), (48) imply the symmetry under "parity conjugation"

$$\begin{aligned} \langle \chi; j_2 \| T^j(\alpha) \| \chi-\alpha; j_1 \rangle &= \\ &= (-1)^{j_2+j-j_1+M'+\Delta_\alpha M} \langle \hat{\chi}; j_2 \| T^j(\hat{\alpha}) \| \hat{\chi}-\hat{\alpha}; j_1 \rangle \end{aligned} \quad (49)$$

Inserting (49) into (41) gives

$$C_\alpha(\chi) = C_{\hat{\alpha}}(\hat{\chi}) \quad (50)$$

A symmetry relation for the function $F_\alpha(\chi)$ follows, however, only if the covariant operator to be decomposed is also covariant under the group $SL(2, \mathbb{C})$ extended by the parity operation (Section 5).

Transposing the differential operators (7), (31) after their insertion into (27) leads us to the symmetry under transposition

$$\begin{aligned} \langle \chi; j_2 \| T^j(\alpha) \| \chi - \alpha; j_1 \rangle = \\ = (-1)^{j-M'} \langle -\chi + \alpha; j_1 \| T^j(\alpha) \| -\chi; j_2 \rangle \end{aligned} \quad (51)$$

From (41) and (51) we deduce

$$C_\alpha(\chi) = (-1)^{2(j_2-j_1)} C_\alpha(-\chi + \alpha) \quad (52)$$

Again there is in general no corresponding symmetry of the functions $F_\alpha(\chi)$. In the case of the applications treated in ²⁾, this symmetry is related with time reversal invariance, but for the Toller expansion of the matrix element (1) it is something different.

In certain special cases the symmetries (44), (49), (51) may conspire and cause a matrix element to vanish. We mention two such cases. If $\alpha = (0,0)$ is in the centre of the α diagram (this is possible only if j_1 and j_2 are integers) and if $J_1 = J_2$, (44) and (51) imply

$$\langle \chi; j_1 \| T^j(0,0) \| \chi; j_1 \rangle = 0 \quad (53)$$

if $j_1 + j_2 + J$ is odd.

If $M = \Delta_\alpha M = 0$ and $M' = 0$ (i.e., $\chi' = \hat{\chi}'$) (49) alone implies

$$\langle 0, \lambda; j_2 \| T^j(0, \Delta_\alpha \lambda) \| 0, \lambda - \Delta_\alpha \lambda; j_1 \rangle = 0 \quad (54)$$

if $J_2 + J - J_1$ is odd.

Now we turn to an explicit evaluation of $SU(2)$ reduced matrix elements. A closed formula for the matrix elements of a general irreducible covariant operator is not given here. Only four-vector and rank-two tensor covariances are of major physical interest. For the purpose of checking our general symmetry formulae it suffices to know the matrix element

$$\begin{aligned} & \langle \chi, j+j_1+j_2 \| T^{j_1+j_2}(\alpha) \| \chi-\alpha; j \rangle = \\ & = (-1)^{2(j_1+j_2)-j} \frac{\Gamma(\lambda-j-\Delta_\alpha \lambda)}{\Gamma(\lambda-j-j_1-j_2)} \end{aligned} \quad (55)$$

$$\times \left[\pi \frac{(2j_1)!(2j_2)!(2j+1)!(j+j_1+j_2+M)!(j+j_1+j_2-M)!}{(2j_1+2j_2+1)!(2j+2j_1+2j_2)!(j+M-\Delta_\alpha M)!(j-M+\Delta_\alpha M)!} \right]^{\frac{1}{2}}$$

Lists of the matrix elements of the four-vector operators were given in ²⁾. In order to adjust their normalization to (7), (31), (32) we must multiply them with the factor $\frac{1}{2}\pi^{\frac{1}{2}}$. The normalization coefficient is $+\frac{8}{\pi}$ for all four four-vector operators.

A. - Antisymmetric tensors of rank two

The two irreducible representations $\chi' = [1,0]$ and $\chi' = [0,1]$ contained in antisymmetric tensors of rank two are parity conjugate. The irreducible covariant operators belonging to $\alpha = (0,0)$ are generators of $SL(2,C)$ (see ⁴⁾, Chapter 3). The independent reduced matrix elements for $\chi' = [1,0]$ are :

$$\begin{aligned} & \underline{\alpha = (-1,1)} \\ & \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi, j+1 \| T^1 \| \chi-\alpha; j \rangle = + \left[\frac{(j-M)(j-M+1)}{2(j+1)} \right]^{\frac{1}{2}} \\ & \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi, j \| T^1 \| \chi-\alpha; j \rangle = - \left[\frac{(2j+1)(j-M)(j+M+1)}{2j(j+1)} \right]^{\frac{1}{2}} \end{aligned} \quad (56)$$

which can be summarized into a Clebsch-Gordan coefficient of $SU(2)$ (see ⁷⁾, Table 5.2)

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j_2 \| T^1 \| \chi - \alpha; j_1 \rangle = (2j_2 + 1)^{\frac{1}{2}} (j_2 M; 11 | j_1, M+1) \quad (57)$$

$$\alpha = (00)$$

$$\begin{aligned} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^1 \| \chi; j \rangle &= + (j+1-\lambda) \left[\frac{(j+1)^2 - M^2}{2(j+1)} \right]^{\frac{1}{2}} \\ \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^1 \| \chi; j \rangle &= \\ &= - [j(j+1) - M\lambda] \left[\frac{2j+1}{2j(j+1)} \right]^{\frac{1}{2}} \end{aligned} \quad (58)$$

The normalization coefficients for the same covariance are

$$\begin{aligned} C_{(-1,1)}(\chi) &= - \frac{12}{\pi} \frac{M+\lambda}{M-\lambda+1} \\ C_{(0,0)}(\chi) &= - \frac{24}{\pi} \frac{M^2 - \lambda^2}{(M-\lambda)^2 - 1} \end{aligned} \quad (59)$$

B. - Symmetric traceless tensors of rank two

We choose to compute the four operators in the upper right quadrangle of the α diagram. For the computation we may refer to the known matrix elements of the vector operators, namely using the relation

$$T_{\alpha_1 + \alpha_2}^t(z|z') = T_{\alpha_1}^v(z|z') T_{\alpha_2}^v(z|z') \quad (60)$$

The superscripts t and v denote tensor and vector, respectively.

$$\alpha = (0, 2)$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^0 \| \chi - \alpha; j \rangle = +\frac{1}{3} (2j+1)^{\frac{1}{2}}$$

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^1 \| \chi - \alpha; j \rangle = -\frac{1}{2} M \left[\frac{2j+1}{j(j+1)} \right]^{\frac{1}{2}}$$

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^1 \| \chi - \alpha; j \rangle = +\frac{1}{2} \left[\frac{(j+1)^2 - M^2}{j+1} \right]^{\frac{1}{2}}$$

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^2 \| \chi - \alpha; j \rangle = +\frac{1}{6} [3M^2 - j(j+1)] \times \left[\frac{2j+1}{j(j+1)(2j-1)(2j+3)} \right]^{\frac{1}{2}}$$

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^2 \| \chi - \alpha; j \rangle = -\frac{1}{2} M \left[\frac{(j+1)^2 - M^2}{3j(j+1)(j+2)} \right]^{\frac{1}{2}}$$

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+2 \| T^2 \| \chi - \alpha; j \rangle = +\frac{1}{2} \left[\frac{[(j+1)^2 - M^2][(j+2)^2 - M^2]}{6(j+1)(j+2)(2j+3)} \right]^{\frac{1}{2}}$$

(61)

$$\alpha = (1, 1)$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^0 \| \chi - \alpha; j \rangle = +\frac{1}{3} [(2j+1)(j+M)(j-M+1)]^{\frac{1}{2}}$$

$$\begin{aligned} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^1 \| \chi - \alpha; j \rangle &= \\ &= -\frac{1}{4} (M+\lambda-1) \left[\frac{2j+1}{j(j+1)} (j+M)(j-M+1) \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^1 \| \chi - \alpha; j \rangle &= \\ &= +\frac{1}{4} (2j+3-M-\lambda) \left[\frac{(j+M)(j+M+1)}{j+1} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^2 \| \chi - \alpha; j \rangle &= \\ &= -\frac{1}{4} \left[\frac{2}{3} j(j+1) - 2M\lambda + M + \lambda - 1 \right] \left[\frac{(2j+1)(j+M)(j-M+1)}{j(j+1)(2j-1)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^2 \| \chi - \alpha; j \rangle &= \\ &= -\frac{1}{4} [(j+2)(M+\lambda-1) - 2M\lambda] \left[\frac{(j+M)(j+M+1)}{3j(j+1)(j+2)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+2 \| T^2 \| \chi - \alpha; j \rangle = \\ = +\frac{1}{2} (j-\lambda+2) \left[\frac{(j+M)(j+M+1)(j+M+2)(j-M+2)}{6(j+1)(j+2)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

(62)

$$\underline{\alpha = (2, 0)}$$

$$\begin{aligned} \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^0 \| \chi - \alpha; j \rangle = \\ = +\frac{1}{3} [(2j+1)(j+M-1)(j+M)(j-M+1)(j-M+2)]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^1 \| \chi - \alpha; j \rangle = \\ = -\frac{1}{2} \lambda \left[\frac{2j+1}{j(j+1)} (j+M-1)(j+M)(j-M+1)(j-M+2) \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^1 \| \chi - \alpha; j \rangle = \\ = +\frac{1}{2} (j-\lambda+1) \left[\frac{1}{j+1} (j+M-1)(j+M)(j+M+1)(j-M+2) \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^2 \| \chi - \alpha; j \rangle = -\frac{1}{2} \left[\frac{1}{3} j(j+1) - \lambda^2 \right] \\ \times \left[\frac{(2j+1)(j+M-1)(j+M)(j-M+1)(j-M+2)}{j(j+1)(2j-1)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^2 \| \chi - \alpha; j \rangle = -\frac{1}{2} \lambda (j-\lambda+1) \\ \times \left[\frac{(j+M-1)(j+M)(j+M+1)(j-M+2)}{3j(j+1)(j+2)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+2 \| T^2 \| \chi - \alpha; j \rangle = +\frac{1}{2} (j-\lambda+1)(j-\lambda+2) \\ \times \left[\frac{(j+M-1)(j+M)(j+M+1)(j+M+2)}{6(j+1)(j+2)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

(63)

$$\alpha = (0,0)$$

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^0 \| \chi; j \rangle = -\frac{1}{6} [2j(j+1) - M^2 - \lambda^2 + 1] (2j+1)^{\frac{1}{2}}$$

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^1 \| \chi; j \rangle = 0 \quad [\text{due to the conspiracy (53)}]$$

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^1 \| \chi; j \rangle = -\frac{1}{2} (j-\lambda+1) [(j+1)(j+1)^2 - M^2]^{\frac{1}{2}}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j \| T^2 \| \chi; j \rangle = & +\frac{1}{6} [j^2(j+1)^2 + j(j+1)(M^2 + \lambda^2 - 1) \\ & - 3M^2 \lambda^2] \left[\frac{2j+1}{j(j+1)(2j-1)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+1 \| T^2 \| \chi; j \rangle = & -\frac{1}{2} M \lambda (j-\lambda+1) \\ & \times \left[\frac{(j+1)^2 - M^2}{3j(j+1)(j+2)} \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \left(\frac{5}{\pi}\right)^{\frac{1}{2}} \langle \chi; j+2 \| T^2 \| \chi; j \rangle = & -\frac{1}{2} (j-\lambda+1)(j-\lambda+2) \\ & \times \left[\frac{(j+M+1)(j+M+2)(j-M+1)(j-M+2)}{6(j+1)(j+2)(2j+3)} \right]^{\frac{1}{2}} \end{aligned}$$

(64)

Finally we list the normalization coefficients.

$$C_{(0,2)}(\chi) = -\frac{12}{\pi} \cdot \frac{1}{(M+\lambda-1)(M-\lambda+1)}$$

$$C_{(1,1)}(\chi) = -\frac{24}{\pi} \cdot \frac{M-\lambda}{(M+\lambda-1)(M-\lambda-1)(M-\lambda+1)}$$

$$C_{(2,0)}(\chi) = -\frac{12}{\pi} \cdot \frac{1}{(M+\lambda-1)(M-\lambda-1)}$$

$$C_{(0,0)}(\chi) = -\frac{48}{\pi} \cdot \frac{(M+\lambda)(M-\lambda)}{(M+\lambda-1)(M+\lambda+1)(M-\lambda-1)(M-\lambda+1)}$$

(65)

It is easily verified that these normalization coefficients satisfy the symmetry constraint (52).

5. - EXTENSION OF $SL(2, \mathbb{C})$ BY PARITY

We extend $SL(2, \mathbb{C})$ by a parity operation s which we define by

$$ss = e^{i\pi\sigma_3}$$

$$sas^{-1} = a'$$

$$\left. \begin{aligned} a &= u_1 d(\eta) u_2 \\ a' &= u_1 d(-\eta) u_2 \end{aligned} \right\} u_{1,2} \in SU(2)$$

(66)

We call the extended group G and its elements g . Since ⁸⁾

$$D_{j_1 q_1 j_2 q_2}^{\chi}(a') = (-1)^{j_2 - j_1} D_{j_1 q_1 j_2 q_2}^{\hat{\chi}}(a)$$

(67)

$\hat{\chi}$ is denoted the parity conjugate representation of χ . If $\chi = \hat{\chi}$, the representation χ can be extended to a pair of parity associate representations of G labelled by two numbers λ and σ , $\sigma^2 = 1$,

$$D_{j_1 q_1 j_2 q_2}^{\lambda, \sigma}(a) = D_{j_1 q_1 j_2 q_2}^{(0, \lambda)}(a) \quad a \in SL(2, C) \quad (68)$$

$$D_{j_1 q_1 j_2 q_2}^{\lambda, \sigma}(s) = \sigma (-1)^{j_1} \delta_{j_1 j_2} \delta_{q_1 q_2}$$

We call the representations $\sigma = +1$ "true" and those with $\sigma = -1$ "pseudo", as for example "symmetric pseudo tensor". The conventional notation "axial vector" means the same as "pseudo four-vector".

If $\chi \neq \hat{\chi}$ we choose χ to have $M > 0$. Both representations of $SL(2, C)$ lead to one parity associate representation of G

$$D_{j_1 q_1 \tau_1, j_2 q_2 \tau_2}^{M, \lambda}(a) = \delta_{\tau_1 \tau_2} D_{j_1 q_1 j_2 q_2}^{(\tau M, \lambda)}(a)$$

$$D_{j_1 q_1 \tau_1, j_2 q_2 \tau_2}^{M, \lambda}(s) = (-1)^{j_1 - \tau_1 M} \delta_{\tau_1, -\tau_2} \delta_{j_1 j_2} \delta_{q_1 q_2} \quad (69)$$

$$\tau = \pm 1$$

In the case of (68) each vector of the canonical basis is an eigenvector of the parity operator with eigenvalue

$$\pi = \sigma (-1)^{j_1} \quad (70)$$

In the case (69) such eigenvectors can be formed as linear combinations of vectors of the canonical basis in each $SL(2, C)$ invariant subspace (in the ket-formalism)

$$|M, \lambda; \tau Q \pi\rangle =$$

$$= 2^{-\frac{1}{2}} \{ |M, \lambda; \tau Q, \tau = +1\rangle + \omega |M, \lambda; \tau Q, \tau = -1\rangle \} \quad (71)$$

with

$$\begin{aligned}\omega &= \pi (-1)^{J-M} \\ \pi &= \sigma i^{2J} \\ \sigma^2 &= 1\end{aligned}\tag{72}$$

such that the eigenvalue is π .

We have first to investigate under which conditions an operator which is covariant under $SL(2, C)$ can be made covariant under G . The definition of an operator covariant under G is a straightforward generalization of the definition given in Section 3. In the case of a covariance $\chi' = (0, \lambda')$ we define

$$\Gamma_Q^J(a_2, a_1 | \lambda', \sigma)_{q_2 q_1} = \Gamma_Q^J(a_2, a_1 | 0, \lambda')_{q_2 q_1}\tag{73}$$

and require

$$\begin{aligned}\Gamma_Q^J(s a_2, a_1 | \lambda', \sigma)_{q_2 q_1} &= \pi_2 \Gamma_Q^J(a_2, a_1 | \lambda', \sigma)_{q_2 q_1} \\ \Gamma_Q^J(a_2, s a_1 | \lambda', \sigma)_{q_2 q_1} &= \pi_1^{-1} \Gamma_Q^J(a_2, a_1 | \lambda', \sigma)_{q_2 q_1} \\ \Gamma_Q^J(a_2 \bar{s}^1, a_1 \bar{s}^1 | \lambda', \sigma)_{q_2 q_1} &= \pi \Gamma_Q^J(a_2, a_1 | \lambda', \sigma)_{q_2 q_1}\end{aligned}\tag{74}$$

with

$$\pi_i^2 = (-1)^{2S_i}, \quad i = 1, 2\tag{75}$$

and π as defined by (70). For a pair of parity conjugate covariant operators we define ($M' > 0$)

$$\begin{aligned}
 \Gamma_{Q\tau}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} &= \Gamma_Q^J(a_2, a_1 | \tau M', \lambda')_{q_2 q_1} \\
 \Gamma_{Q\pi}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} &= \\
 &= 2^{-\frac{1}{2}} \{ \Gamma_{Q,+}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} + \omega^{-1} \Gamma_{Q,-}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} \}
 \end{aligned} \tag{76}$$

and require

$$\begin{aligned}
 \Gamma_{Q\pi}^J(s a_2, a_1 | M', \lambda')_{q_2 q_1} &= \pi_2 \Gamma_{Q\pi}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} \\
 \Gamma_{Q\pi}^J(a_2, s a_1 | M', \lambda')_{q_2 q_1} &= \pi_1^{-1} \Gamma_{Q\pi}^J(a_2, a_1 | M', \lambda')_{q_2 q_1} \\
 \Gamma_{Q\pi}^J(a_2 \bar{s}', a_1 \bar{s}' | M', \lambda')_{q_2 q_1} &= \pi \Gamma_{Q\pi}^J(a_2, a_1 | M', \lambda')_{q_2 q_1}
 \end{aligned} \tag{77}$$

These requirements can be fulfilled if and only if the relation

$$\begin{aligned}
 \Gamma_Q^J(d(\eta_2), d(\eta_1)^{-1} | 0, \lambda')_{q_2 q_1} &= \\
 &= \pi_2 \pi_1^{-1} \sigma (-1)^{s_2 - s_1} \Gamma_{-Q}^J(d(\eta_2), d(\eta_1)^{-1} | 0, \lambda')_{-q_2, -q_1}
 \end{aligned} \tag{78}$$

respectively

$$\begin{aligned}
 \Gamma_Q^J(d(\eta_2), d(\eta_1)^{-1} | M', \lambda')_{q_2 q_1} &= \\
 &= \pi_2 \pi_1^{-1} (-1)^{s_2 - s_1 + M'} \Gamma_{-Q}^J(d(\eta_2), d(\eta_1)^{-1} | -M', \lambda')_{-q_2, -q_1}
 \end{aligned} \tag{79}$$

is true. For the Fourier transforms $F_\alpha(X)$ we obtain from (78), (79) the following relations

$$F_\alpha(X) = (-1)^{s_2 - s_1 + \Delta_\alpha M} \pi_2 \pi_1^{-1} \sigma F_{\hat{\alpha}}(\hat{X}) \tag{80}$$

respectively

$$F_{\alpha}(X) = (-1)^{S_2 - S_1 + \Delta_{\alpha} M} \pi_2 \pi_1^{-1} F_{\hat{\alpha}}(\hat{X}) \quad (81)$$

Next we construct irreducible G covariant operators. We consider the two classes of covariances separately.

A. - Covariance M', λ'

We assume $M' > 0$ as always in this Section (if $M' \neq 0$). The covariant operator $T_Q^J \tau$ consists of one part $\tau = +1$ with $SL(2, C)$ covariance $\chi' = (+M', \lambda')$ and of another part $\tau = -1$ with $SL(2, C)$ covariance $\hat{\chi}'$. In order to characterize the operator it is sufficient to give the α for the first part. We can distinguish between nine different types of such covariant operators, each one labelled by a pair ϵ_2, ϵ_1 , $\epsilon = +1, 0, -1$. The covariance condition for these operators is

$$\begin{aligned} & T_{g^{-1}} T_{Q\tau}^J (\alpha; \epsilon_2 \epsilon_1) T_g \\ &= \sum_{J' Q' \tau'} D_{J Q \tau, J' Q' \tau'}^{M', \lambda'}(g) T_{Q' \tau'}^{J'} (\alpha; \epsilon_2 \epsilon_1) \end{aligned} \quad (82)$$

and has the form

$$\begin{aligned} \alpha &= (\Delta_{\alpha} M, \Delta_{\alpha} \lambda) \\ &= (\epsilon_2 M_2 - \epsilon_1 M_1, \lambda_2 - \lambda_1) \end{aligned} \quad (83)$$

With the short hand

$$\zeta_{\epsilon_2 \epsilon_1} = [\sigma_2 i^{2M_2}]^{\frac{1}{2} \epsilon_2 (\epsilon_2 - 1)} [\sigma_1 i^{2M_1}]^{\frac{1}{2} \epsilon_1 (\epsilon_1 - 1)} \quad (84)$$

The nine types of irreducible covariant operators can be given in the form

$\varepsilon_1 \neq 0, \varepsilon_2 \neq 0$:

$$\begin{aligned} & \langle M_2 \lambda_2; j_2 Q_2 \tau_2 | T_{Q\tau}^j(\alpha; \varepsilon_2 \varepsilon_1) | M_1 \lambda_1; j_1 Q_1 \tau_1 \rangle = \\ & = \zeta_{\varepsilon_2 \varepsilon_1} \tau^{j+j_2-j_1+M'+\Delta_\alpha M} \delta_{\tau, \varepsilon_2 \tau_2} \delta_{\tau, \varepsilon_1 \tau_1} \\ & \quad \times \langle \varepsilon_2 M_2, \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | \varepsilon_1 M_1, \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (85)$$

$\varepsilon_1 = 0, \varepsilon_2 \neq 0$:

$$\begin{aligned} & \langle M_2 \lambda_2; j_2 Q_2 \tau_2 | T_{Q\tau}^j(\alpha; \varepsilon_2 0) | \lambda_1 \sigma_1; j_1 Q_1 \rangle = \\ & = \zeta_{\varepsilon_2 0} \tau^{j+j_2-j_1+M'+\Delta_\alpha M} \sigma_1^{\frac{1}{2}(1-\tau)} \delta_{\tau, \varepsilon_2 \tau_2} \\ & \quad \times \langle \varepsilon_2 M_2, \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | 0 \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (86)$$

$\varepsilon_1 \neq 0, \varepsilon_2 = 0$:

$$\begin{aligned} & \langle \lambda_2 \sigma_2; j_2 Q_2 | T_{Q\tau}^j(\alpha; 0 \varepsilon_1) | M_1 \lambda_1; j_1 Q_1 \tau_1 \rangle = \\ & = \zeta_{0 \varepsilon_1} \tau^{j+j_2-j_1+M'+\Delta_\alpha M} \sigma_2^{\frac{1}{2}(1-\tau)} \delta_{\tau, \varepsilon_1 \tau_1} \\ & \quad \times \langle 0 \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | \varepsilon_1 M_1, \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (87)$$

$\varepsilon_1 = \varepsilon_2 = 0$:

$$\begin{aligned} & \langle \lambda_2 \sigma_2; j_2 Q_2 | T_{Q\tau}^j(\alpha; 0 0) | \lambda_1 \sigma_1; j_1 Q_1 \rangle = \\ & = \tau^{j+j_2-j_1+M'} [\sigma_1 \sigma_2]^{\frac{1}{2}(1-\tau)} \\ & \quad \times \langle 0 \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | 0 \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (88)$$

If we introduce parity eigenvectors (71), the operator

$$T_{Q\pi}^j = 2^{-\frac{1}{2}} \{ T_{Q, \tau=+1}^j + \omega^{-1} T_{Q, \tau=-1}^j \} \quad (89)$$

gives the matrix elements

$$\begin{aligned} & \langle \dots \sigma_2 | T_{Q\pi}^j(\alpha; \epsilon_2 \epsilon_1) | \dots \sigma_1 \rangle = \\ & = 2^{-\frac{1}{2}} (1 + |\epsilon_1| + |\epsilon_2|) (1 + \sigma_1 \sigma_2 \sigma(-1))^{j+j_2-j_1} \\ & \times \langle \epsilon_2 M_2, \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | \epsilon_1 M_1, \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (90)$$

in all nine cases.

B. - Covariance λ', σ

The classification of covariant operators is similar as in A., but because of $\chi' = \hat{\chi}'$ the types ϵ_2, ϵ_1 and $-\epsilon_2, -\epsilon_1$ coincide. In particular we can use the notation (84). The covariant operator is characterized by an α defined as in (83). We obtain

$$\begin{aligned} & \epsilon_1 \neq 0, \epsilon_2 = 1 \\ & \langle M_2 \lambda_2; j_2 Q_2 \tau_2 | T_Q^j(\alpha; 1 \epsilon_1) | M_1 \lambda_1; j_1 Q_1 \tau_1 \rangle = \\ & = \delta_{1 \epsilon_1} \tau_2^{j+j_2-j_1+\Delta_\alpha M} \sigma^{\frac{1}{2}(1-\tau_2)} \delta_{\tau_2, \epsilon_1 \tau_1} \\ & \times \langle M_2 \lambda_2; j_2 Q_2 | T_Q^j(\alpha) | \epsilon_1 M_1, \lambda_1; j_1 Q_1 \rangle \end{aligned} \quad (91)$$

$$\varepsilon_1 = 0, \varepsilon_2 = 1:$$

$$\begin{aligned} & \langle M_2 \lambda_2; \gamma_2 Q_2 \tau_2 | T_Q^J(\alpha; 10) | \lambda_1 \sigma_1; \gamma_1 Q_1 \rangle = \\ & = \tau_2^{\gamma_1 + \gamma_2 - \gamma_1 + M_2} [\sigma \sigma_1]^{\frac{1}{2}(1 - \tau_2)} \\ & \times \langle M_2 \lambda_2; \gamma_2 Q_2 | T_Q^J(\alpha) | 0 \lambda_1; \gamma_1 Q_1 \rangle \end{aligned} \quad (92)$$

$$\varepsilon_1 = 1, \varepsilon_2 = 0:$$

$$\begin{aligned} & \langle \lambda_2 \sigma_2; \gamma_2 Q_2 | T_Q^J(\alpha; 01) | M_1 \lambda_1; \gamma_1 Q_1 \tau_1 \rangle = \\ & = \tau_1^{\gamma_1 + \gamma_2 - \gamma_1 - M_1} [\sigma \sigma_2]^{\frac{1}{2}(1 - \tau_1)} \\ & \times \langle 0 \lambda_2; \gamma_2 Q_2 | T_Q^J(\alpha) | M_1 \lambda_1; \gamma_1 Q_1 \rangle \end{aligned} \quad (93)$$

$\varepsilon_1 = \varepsilon_2 = 0$: due to the conspiracy (54) a covariant operator exists only if $\sigma_1 \sigma_2 \sigma = +1$ and is given by

$$\begin{aligned} & \langle \lambda_2 \sigma_2; \gamma_2 Q_2 | T_Q^J(\alpha; 00) | \lambda_1 \sigma_1; \gamma_1 Q_1 \rangle \\ & = \langle 0 \lambda_2; \gamma_2 Q_2 | T_Q^J(\alpha) | 0 \lambda_1; \gamma_1 Q_1 \rangle \end{aligned} \quad (94)$$

In all these cases the matrix element of the covariant operator between eigenvectors of the parity operator (71) is

$$\begin{aligned} & \langle \dots \sigma_2 | T_Q^J(\alpha; \varepsilon_2 \varepsilon_1) | \dots \sigma_1 \rangle = \\ & = 2^{-\frac{1}{2}(|\varepsilon_1| + |\varepsilon_2|)} (1 + \sigma_1 \sigma_2 \sigma (-1)^{\gamma_1 + \gamma_2 - \gamma_1}) \\ & \times \langle \varepsilon_2 M_2, \lambda_2; \gamma_2 Q_2 | T_Q^J(\alpha) | \varepsilon_1 M_1, \lambda_1; \gamma_1 Q_1 \rangle \end{aligned} \quad (95)$$

The issue to decompose a G covariant operator into G covariant irreducible operators is easily reduced to the corresponding problem for $SL(2, \mathbb{C})$. We make an ansatz of the type (34) involving Fourier transforms $G_{\alpha}(M, \lambda)$ and $G_{\alpha}(\lambda, \sigma)$. Restricting the functions Γ to $SL(2, \mathbb{C})$ allows us to express the G_{α} 's by the F_{α} 's in two different fashions such that the constraints (80), (81) follow. With the help of the relations (78), (79) we can finally transform the integral (36) into an integral over G .

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FIGURE CAPTION

The α diagram for a covariance $\chi' = [5/2, 11/2]$.