Hilbert Space of Isomorphic Representations of Bosonized Chiral $QCD_2$

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Abstract

We analyse the Hilbert space structure of the isomorphic gauge non-invariant and gauge invariant bosonized formulations of chiral $QCD_2$ for the particular case of the Jackiw-Rajaraman parameter $a = 2$. The BRST subsidiary conditions are found not to provide a sufficient criterion for defining physical states in the Hilbert space and additional superselection rules must be taken into account. We examine the effect of the use of a redundant field algebra in deriving basic properties of the model. We also discuss the constraint structure of the gauge invariant formulation and show that the only primary constraints are of first class.
I. INTRODUCTION

In a series of papers [1–4,20] the Hilbert space and structural properties of bosonized $QCD_2$ in the so-called [1] “local” and “non-local” decoupled formulation have been extensively discussed. In particular it has been shown [4] that the BRST conditions associated with the change from fermionic to bosonic variables play the role of the familiar Lowenstein-Swieca conditions [5] defining the physical Hilbert space in the abelian case. This analysis has recently been extended to chiral $QCD_2$ [6] in the so-called “gauge non-invariant” (GNI) formulation, where it was shown that the Becchi-Rouet-Stora-Tyutin (BRST) conditions play the role of the conditions of Boyanovski et al [7] in the abelian case. In ref. [6] it was further argued, relying on previous considerations in the abelian case [8], that chiral $QCD_2$ for the Jackiw-Rajaraman (JR) parameter $a = 2$ [9,10] is in fact not equivalent to $QCD_2$, contrary to what superficially appears to be the case [11].

In the present paper we complete this demonstration, by examining in detail the physical Hilbert space $H_{\text{phys}}$ of the non-Abelian bosonized formulation, as defined by a set of BRST conditions. As pointed out in Refs. [8,12–14], the use of bosonization techniques raises some delicate questions related to the use of a redundant Bose field algebra. This field algebra contains more degrees of freedom than those needed for the description of the model. Since for an anomalous theory the gauge invariance is spoiled at the quantum level, the physical content of the theory relies strongly and directly on the field algebra intrinsic to the model [8]. Taking proper care in the construction of the Hilbert space associated with the Wightman functions that define the theory, and examining the “charge” content of the Hilbert space, we find that the BRST conditions are not sufficient to define $H_{\text{phys}}$, and that certain superselection rules have also to be respected. We thereby show that the Hilbert space of chiral $QCD_2$ defined for $a = 2$ does not contain the Hilbert space of $QCD_2$ as a physical subspace. This is the content of section 2. In section 3 we then turn to the so-called “gauge invariant” (GI) formulation [15] obtained by embedding the GNI bosonized formulation into a gauge theory, following the standard procedure of refs. [16]. We show in particular that in this case there exist in addition to the above BRST constraints, precisely
one set of (primary) constraints in the sense of Dirac [17]. We show these constraints to be first class, and to be the generators of the gauge symmetries introduced by the gauge invariant embedding, as expected. Some further considerations and technical details relating to sections 2 and 3 are relegated to the appendices A and B, respectively.

**II. GAUGE NON-INVARIANT LOCAL FORMULATION**

In Minkowski space, the generating functional of the GNI formulation of chiral QCD with left-moving fermions coupled to a SU(N) gauge field is given by

\[
Z[\bar{\eta}, \eta, J_\mu] = \int DA_\mu \int D\psi_\ell^o D\psi_\ell^\dagger \int D\psi_r^o D\psi_r^\dagger \exp \left\{ i S[A_\mu, \psi, \bar{\psi}] \right\} \times \exp \left\{ i \int d^2 z \left( \eta_r^\dagger \psi_r^o + \psi_r^o \eta_r + \eta_\ell^\dagger \psi_\ell + \psi_\ell^\dagger \eta_\ell + J_\mu A_\mu \right) \right\},
\]

where

\[
S[A_\mu, \psi, \bar{\psi}] = \int d^2 z \left\{ -\frac{1}{4} tr F^\mu_\nu F^\nu_\mu + \psi_r^o \d_+ \psi_r^o + \psi_\ell^\dagger (i\d_- + eA_-) \psi_\ell \right\}.
\]

Parametrizing \( A_\pm \) in terms of SU(N) matrix valued fields as follows

\[
e A_+ = U^{-1} i\partial_+ U, \quad e A_- = V i\partial_- V^{-1},
\]

making the change of variables \( A_+ \to U, A_- \to V \), as well as the chiral rotation

\[
\psi_\ell^o = V^{-1} \psi_\ell,
\]

and taking due account of the Jacobians in the integration measure [18, 1], we arrive, following the procedure of references [1, 2], at the generating functional [6]

\[
Z[\bar{\eta}, \eta, J_\mu] = Z_{gh}^{\alpha} \tilde{Z}[\bar{\eta}, \eta, J_\mu],
\]

\[^\dagger \text{Our conventions are: } A_\pm = A_0 \pm A_1, \partial_\pm = \partial_0 \pm \partial_1, \epsilon^{01} = 1.\]

\[^\S \text{We follow here the notation of Refs. [2, 3, 6].}\]
where $Z_{gh}^{(0)}$ is the partition function of free ghosts [2] associated with the change of variables (2.3),

$$Z_{gh}^{(0)} = \int D b_+^{(0)} D c_+^{(0)} e^{i \int \operatorname{tr} b_+^{(0)} \partial^- c_+^{(0)}} \int D b_-^{(0)} D c_-^{(0)} e^{i \int \operatorname{tr} b_-^{(0)} \partial^+ c_-^{(0)}} ,$$  

(2.6)

$\hat{Z}$ is given by

$$\hat{Z} [\eta, \tilde{\eta}, J_{\mu}] = \int D \psi_0 D \tilde{\psi}_0 e^{i \int d^2 z \left( \bar{\psi}_0 i \partial \eta_0 + \tilde{\eta}^\dagger V \psi_0 + \psi_0^\dagger V^{-1} \eta + \frac{i}{2} J_- \left[ V^{-1} i \partial_+ U \right] + \frac{i}{2} J_+ \left[ V i \partial_- V^{-1} \right] \right)} ,$$  

(2.7)

and the effective bosonized action is

$$S[U, V] = S_{YM}[\Sigma] - \left( C_V + \frac{a}{2} \right) \Gamma[U, V] + \frac{a}{2} \Gamma[U] + (\frac{a}{2} - 1) \Gamma[V] .$$  

(2.8)

Here

$$S_{YM}[\Sigma] = \frac{1}{4e^2} \int \operatorname{tr} \frac{1}{2} \left[ \partial_+ (\Sigma i \partial_- \Sigma^{-1}) \right]^2$$

(2.9)

$$= \frac{1}{4e^2} \int \operatorname{tr} \frac{1}{2} \left[ \partial_- (\Sigma^{-1} i \partial_+ \Sigma) \right]^2 .$$  

(2.10)

is the Yang-Mills action with $\Sigma$ the gauge-invariant variable $\Sigma = UV$, and $\Gamma[G]$ is the Wess-Zumino-Witten action [19]. Changing variables from $\{U, V\}$ to $\{\Sigma, U\}$, we have [6]

$$S[U, V] \rightarrow S[U, \Sigma] = S_{YM}[\Sigma] - \left( C_V + \frac{a}{2} \right) \Gamma[\Sigma] + \frac{a}{2} \Gamma[U] + (\frac{a}{2} - 1) \Gamma[U^{-1} \Sigma] .$$  

(2.11)

This action provides the starting point for the discussion to follow.

**A. FIELD ALGEBRA AND HILBERT SPACE**

The equations of motion defining the $GNI$ formulation of chiral $QCD_2$ are given in terms of the fundamental set of field operators $\{\psi_\sigma^r, \psi_\ell, A_\mu\}$. Within the context of general principles of Wightman field theory, these field operators constitute the intrinsic mathematical description of the theory and serve as the building material in terms of which the $GNI$ version of the model is formulated [8]. The set of field operators $\{\psi_\sigma^r, \psi_\ell, A_\mu\}$ defines a local
field algebra $\mathfrak{S}$, and the Wightman functions generated from the field algebra $\mathfrak{S}$ identifies a Hilbert space $\mathcal{H} = \mathfrak{S} \Psi_o$ of the GNI formulation of the model.

The bosonization of the model requires the use of a larger field algebra, which includes non-observable Bose fields as well as ghosts. In the local formulation the resulting effective theory is given in terms of the set of fields $\{\psi^\sigma_r, \psi^\sigma_\ell, U, V, gh\}$. These field operators generate an extended Bose-Fermi-ghost field algebra $\mathfrak{S}^e$ which is represented in the Hilbert space $\mathcal{H}^e$. The field algebra $\mathfrak{S}$ is a proper subalgebra of $\mathfrak{S}^e$ and the Hilbert space $\mathcal{H}$ is a subspace of $\mathcal{H}^e$.

The field algebra $\mathfrak{S}^e$ contains elements not intrinsic to the model, which should not be considered as elements of the intrinsic field algebra $\mathfrak{S}$: not all fields $\{\psi^\sigma_r, \psi^\sigma_\ell, U, V, gh\}$ belong to the algebra $\mathfrak{S}$, nor all vectors of $\mathcal{H}^e$ belong to the state space $\mathcal{H}$. Nevertheless, combinations like $V^{-1}\psi^\sigma_\ell$ and $U^{-1}\partial_\mu U, V\partial_V U^{-1}$, do evidently belong to the field algebra $\mathfrak{S}$.

The physical Hilbert space $\mathcal{H}_{phys}$ is a representation of the field algebra $\mathfrak{S}_{phys}$ which satisfies the subsidiary BRST condition

$$[Q_{BRST}, \mathfrak{S}_{phys}] = 0.$$  \hspace{1cm} (2.12)

to be specified later on. Hence

$$Q_{BRST} \mathcal{H}_{phys} = Q_{BRST} \mathfrak{S}_{phys} \Psi_o = 0.$$  \hspace{1cm} (2.13)

These conditions correspond in the abelian case (chiral Schwinger model) [7,4] to the familiar Lowenstein-Swieca [5] conditions requiring that the longitudinal part of the current annihilate the physical states.

As stressed in Refs. [8,6], since the theory has lost the local gauge invariance at the quantum level, it is a peculiarity of the anomalous chiral theory that $\mathfrak{S}_{phys} \equiv \mathfrak{S}$; all operators belonging to the intrinsic local field algebra $\mathfrak{S}$ commute with the BRST charges and thus represent physical observables [6]. Hence $\mathcal{H} = \mathcal{H}_{phys}$. The Hilbert space is thus isomorphic to $\mathcal{H} = \mathcal{H}_{\psi_r} \otimes \mathcal{H}_{\psi_\ell} \otimes \mathcal{H}_{A_\mu}$, contrary to what happens in a genuine gauge theory like QED and QCD, in which the physical Hilbert subspace is defined by equivalence classes corresponding to gauge invariant states. In terms of the bosonic field variables $\{U, V\}$, or
{\Sigma, U}, we can write \( \mathcal{H} = \mathcal{H}_{\psi^\alpha} \otimes \mathcal{H}_{\Sigma, U, \psi^\alpha, \phi^h} \). The BRST conditions (2.13) impose non-trivial restrictions on further decompositions of the closure of the space \( \mathcal{H}_{\Sigma, U, \psi^\alpha, \phi^h} \).

The field algebra can be further enlarged by introducing the bosonized partition function of free \((U(N))\) Fermi fields \(\psi^\alpha\) in the factorized form

\[
\int D\psi^\alpha D\overline{\psi} e^{i \int d^2 z \overline{\psi} i \phi \psi} = Z_{U(1)}^{(0)} \times \int D\psi e^{i \Gamma[G]},
\]

where, like \(U\) and \(V\), \(G\) is a \(SU(N)\) matrix valued field. The factorized \(U(1)\) partition function is given by

\[
Z_{U(1)}^{(0)} = \int D\phi e^{i \int d^2 z (\partial_\mu \phi)^2},
\]

where the massless scalar field \(\Phi\) acts as potential for the conserved \(U(1)\) current

\[
\mathcal{J}^\mu = -\frac{1}{\sqrt{\pi}} \partial^\mu \Phi.
\]

In this way, we obtain a field algebra \(S^E \supset S^r\), generated from the set of field operators \(\{U, V, G, \Phi, gh\}\), and represented in the enlarged Hilbert space \(H^E \supset H^r\).

**B. THE CASE \(a = 2\).**

For \(a = 2\), the effective bosonized action (2.11) decouples as [6]

\[
S[U, \Sigma] = S_{QCD_2}[\Sigma] + \Gamma[U].
\]

Except for the “decoupled” WZW action \(\Gamma[U]\), which appears to play merely a spectator role, this is the action of \(QCD_2\) in the decoupled local formulation [1–3,6,20]. The partition function thus factorizes as follows:

\[
Z[0] = Z_{WZW}^U \left( Z_{gh}^{(0)} Z_{U(1)}^{(0)} Z_{WZW}^G Z_{QCD_2}^\Sigma \right).
\]

However, the factorization does not apply to the Hilbert space since the fields \(U, \Sigma\) and \(\psi^\alpha\) are coupled by the BRST conditions (2.13) [6] with

\[
Q_{BRST} = Q_\pm = \int dx^1 trc^{(0)}_\pm \left( \Omega_\pm - \frac{1}{2} \{b_\pm^{(0)}, c_\pm^{(0)}\} \right),
\]
where

\[
\Omega_+ = \frac{1}{4e^2} \Sigma^{-1} \left[ \partial_+^2 (\Sigma i \partial_+ \Sigma^{-1}) \right] \Sigma
- \frac{1 + C_V}{4\pi} \Sigma^{-1} i \partial_+ \Sigma + \psi_i \psi_i^\dagger + \{ b_+^{(0)}, c_+^{(0)} \}
\]

(2.20)

\[
\Omega_- = \frac{1}{4e^2} \Sigma [ \partial_-^2 (\Sigma^{-1} i \partial_- \Sigma) ] \Sigma^{-1}
- \frac{1 + C_V}{4\pi} \Sigma i \partial_- \Sigma^{-1} + \frac{1}{4\pi} U i \partial_- U^{-1} + \{ b_-^{(0)}, c_-^{(0)} \}.
\]

(2.21)

These conditions must be implemented on the Hilbert space, and require the physical Hilbert space to be neutral with respect to the BRST charges.** In other words: the field U is not a BRST invariant. We can nevertheless arrive at a factorization in terms of BRST invariant sectors as follows.

Since U and G are WZW fields describing non-interacting fermions, we may factorize them into right and left moving parts, \( U = U_r U_\ell, G = G_r G_\ell \). The Polyakov-Wiegmann formula [21]

\[
\Gamma[AB] = \Gamma[A] + \Gamma[B] + \frac{1}{4\pi} \int d^2 x \text{tr} (A^{-1} \partial_+ A)(B \partial_- B^{-1})
\]

(2.22)

allows us to write (recall (2.18))

\[
Z_f^{(0)} \int D U e^{i \Gamma[U]} = Z_{U(1)}^{(0)} \int D G e^{i \Gamma[G]} \int D U e^{i \Gamma[U]}
= Z_{U(1)}^{(0)} \int D \tilde{U} e^{i \Gamma[\tilde{U}]} \int D \tilde{G} e^{i \Gamma[\tilde{G}]}
= \tilde{Z}_{U}^{(0)} \int D \tilde{U} e^{i \Gamma[\tilde{U}]}
\]

(2.23)

where \( Z_{U(1)}^{(0)} \) carries the \( U(1) \) degrees of freedom of the original free fermions \( \psi^a, \tilde{U} = G_r U_\ell \tilde{G} \), and where we have taken account of the right-(left-) moving property of \( G_\ell, U_\ell \) in the decomposition \( G = G_r G_\ell, U = U_r U_\ell \). We see from (2.20) and (2.21) that the new field \( \tilde{U} \) is BRST neutral, \( [Q_{BRST}, \tilde{U}] = 0 \).

**Within the context of the generating functional this is insured by coupling \textit{ab initio} the sources to the set of intrinsic fields defining the model, as done in (2.7). In this way we ensure to reproduce all Wightman functions defining the model.
The partition function (2.18) can thus be factorized into two partition functions that are separately neutral with respect to the BRST charges:

\[ Z = \left( \int \mathcal{D} \tilde{U} e^{i\Gamma[\tilde{U}]} \right) \left( Z_{\text{sh}}^{(0)} \tilde{Z}_{\text{r}}^{(0)} Z^{\Sigma}_{\text{QCD}_2} \right). \] (2.24)

The above factorization of the partition function suggests that the Hilbert space of the anomalous chiral theory can be factorized as \( \mathcal{H} = \mathcal{H}_{\tilde{U}} \otimes \mathcal{H}_{\text{QCD}_2} \), implying that \( \mathcal{H} \) contains \( \mathcal{H}_{\text{QCD}_2} \) as a physical subspace. However, as we now show, this still is an improper factorization of the Hilbert space since violates certain superselection rules. This is contained in the following proposition [13,8]:

Let \( Q \) a local charge operator satisfying

\[ [Q, \mathfrak{S}] = 0, \] (2.25)

and which is trivialized in the restriction from \( \mathcal{H}^{\Sigma} \) to \( \mathcal{H} \subset \mathcal{H}^{\Sigma} \), i. e.,

\[ Q \mathcal{H}^{\Sigma} \neq 0, \quad Q \mathcal{H} = 0. \] (2.26)

Let \( A \) be an operator satisfying \([Q_{\text{BRST}}, A] = 0\), but carrying the charge \( Q \) such that \([Q, A] \neq 0\). Then the operator \( A \) does not belong to the intrinsic local field algebra \( \mathfrak{S} \) and cannot be defined as a solution of the BRST condition in \( \mathcal{H} \), i. e., \( A \) is not an element of \( \mathcal{H} \).

We now show that the BRST invariant field \( \tilde{U} \) carries a charge that is trivialized in the restriction from \( \mathcal{H}^{\Sigma} \) to \( \mathcal{H} \). To this end consider the left and right WZW currents

\[ j_{\ell}^\mu = -\frac{1}{4\pi} U^{-1} i(\partial^\mu + \tilde{\partial}^\mu) U, \]
\[ j_{r}^\mu = -\frac{1}{4\pi} U i(\partial^\mu - \tilde{\partial}^\mu) U^{-1}. \] (2.27)

The conserved vector and axial vector currents are respectively defined by

\[ j^\mu = \frac{1}{2} (j_{\ell}^\mu + j_{r}^\mu), \]
\[ \tilde{j}^\mu = \frac{1}{2} (j_{\ell}^\mu - j_{r}^\mu) = \epsilon^{\mu\nu} j_{\nu}. \] (2.28)
Denoting by $Q$ and $Q^5$ the respective charges, we have

$$[Q^5, \mathfrak{S}] = 0, \quad Q^5 \mathcal{H} = 0.$$  \hfill (2.29)

since $[Q^5, U] = 0$. The field $U$ is however not BRST invariant, and therefore does belong to the field algebra $\mathfrak{S}$. This shows that $\mathcal{H}$ cannot be factorized as $\mathcal{H} = \mathcal{H}_U \otimes \mathcal{H}_{QCD_2}$. On the other hand, $\tilde{U}$ is BRST invariant, but

$$[Q^5, \tilde{U}] = \tilde{U}.$$  \hfill (2.30)

The BRST invariant field $\tilde{U}$ thus carries the charge $Q^5$, and therefore again does not belong to the field algebra $\mathfrak{S}$ according to the above proposition. We conclude that the physical Hilbert space cannot be factorized in the form $\mathcal{H}_U \otimes \mathcal{H}_{QCD_2}$, either; i.e., just as in the abelian case [8], the chiral $QCD_2$ for the JR parameter $a = 2$ is not equivalent to $QCD_2$ plus a “decoupled” free massless field. In Appendix A we illustrate by an explicit example the effect of the use of an external field algebra in deriving basic physical properties of the model and show that the improper factorization of the Hilbert space leads to misleading conclusions about the physical content of the anomalous chiral theory.

### III. EXTENDED GAUGE INVARIANT LOCAL FORMULATION

Our starting point is the partition function (3.4) in the local $GNI$ formulation. The action $S[U, V]$ is not invariant under the gauge transformation $U \to Ug, V \to g^{-1}V$:

$$S[U, V] \to S[Ug, g^{-1}V] = S[U, V] + S_{WZ}[U, V, g],$$  \hfill (3.1)

with $S_{WZ}[U, V, g]$ the Wess-Zumino-Witten action [19]

$$S_{WZ}[U, V, g] = \frac{a}{2} \Gamma[g] + \left(\frac{a}{2} - 1\right) \Gamma[g^{-1}]$$

$$+ \frac{a}{2} \frac{1}{4\pi} \int d^2x \text{ tr } \left(U^{-1}i\partial_+ Ug i\partial_- g^{-1}\right)$$

$$+ \left(\frac{a}{2} - 1\right) \frac{1}{4\pi} \int d^2x \text{ tr } \left(gi\partial_+ g^{-1} V_i \partial_- V^{-1}\right).$$  \hfill (3.2)
We proceed now to embed the anomalous theory into a gauge theory, following the standard procedure in configuration space [16]. To this end we introduce in the generating functional $Z[\bar{\eta}, \eta, J_\mu]$, given by (2.7), the identity
\[ \Delta_F[UV] \int Dg \delta[F(Ug^{-1}, gV)] = 1. \] (3.3)
The Faddeev-Popov determinant $\Delta_F[UV]$ is gauge-invariant. Setting $Ug^{-1} = U$, $gV = V$, using the invariance of the Haar measure, we obtain from (2.7) the generating functional of the GI formulation
\[ Z_{GI}[\bar{\eta}, \eta, J_\mu] = Z_F^o \int Dg \int DUDV \Delta_F[UV] \delta[F(U, V)] e^{iS[UV]} \times \epsilon^{i} \int d^2z \{ \eta_{\bar{\mu}}^\dagger \psi_{\bar{\mu}}^g \psi_{\bar{\mu}}^\dagger V \eta + \frac{1}{2} J_{-} \left( (Ug)^{-1} \partial_\mu (Ug) \right) + \frac{1}{2} J_{+} \left( (g^{-1}V) \partial_\mu (g^{-1}V)^{-1} \right) \}. \] (3.4)
Repeating this procedure one can easily generalize this expression to more general gauges. Dropping “bars”, etc., the generalization reads
\[ Z_{GI}[\bar{\eta}, \eta, J_\mu] = Z_F^o \int Dg \int DUDV \Delta_F[Ug, gV] \delta[F(U, V, g)] \times e^{iS[UV]} \times \epsilon^{i} \int d^2z \{ \eta_{\bar{\mu}}^\dagger \psi_{\bar{\mu}}^g \psi_{\bar{\mu}}^\dagger V \eta + \frac{1}{2} J_{-} \left( (Ug)^{-1} \partial_\mu (Ug) \right) + \frac{1}{2} J_{+} \left( (g^{-1}V) \partial_\mu (g^{-1}V)^{-1} \right) \}, \] (3.5)
where
\[ S[Ug, g^{-1}V] = S_{YM}[UV] - \left( C_v + \frac{a}{2} \right) \Gamma[UV] + \frac{a}{2} \Gamma[Ug] + \left( \frac{a}{2} - 1 \right) \Gamma[g^{-1}V], \] (3.7)
and which also can be written in terms of the WZ action as (3.1).

In the unitary gauge, $F(U, V, g) = g - 1$, and we recover the GNI formulation. However, as exhaustively discussed in Refs. [8,15,23], the isomorphism between these two formulations is valid in an arbitrary gauge.

A. CONSTRAINT STRUCTURE

In the GNI bosonized decoupled formulation, the action $S[U, V]$ describes from the Dirac point of view an unconstrained system. The second-class constraints of the original quantum
fermionic formulation have now been replaced by BRST constraints restricting the bosonic Hilbert space of the present formulation to the physical Hilbert space $\mathcal{H}$ generated by the intrinsic set of field operators $\{\psi^c, \psi, \mathcal{A}_\mu\}$ which defines the original theory. On the other hand, in the GI bosonized formulation, we expect $S[Ug, g^{-1}V]$ to describe a Hamiltonian system with only one first-class constraint generating the extended gauge transformation $U \to UG, \; V \to G^{-1}V, g \to G^{-1}g$. In the following we demonstrate this. Because of the non-abelian nature of the problem, the demonstration involves some technicalities.

Our starting point is the GI effective bosonized action

$$S[Ug, g^{-1}V] = S_{YM}[\Sigma] - \left(\frac{a}{2} + C_V\right) \Gamma[\Sigma] + \frac{a}{2} \Gamma[Ug] + \left(\frac{a}{2} - 1\right) \Gamma[g^{-1}V], \quad (3.8)$$

obtained by performing a gauge transformation in the bosonized action (2.11).

The general form of the constraint can be displayed as follows: use that $V = U^{-1}\Sigma$, and express the action (3.8) in terms of the gauge invariant field variables $\Sigma$ and $\overline{U} = Ug$: $S[Ug, g^{-1}V] \equiv S[\Sigma, \overline{U}]$. Thus we have for the canonical momenta

$$\Pi^{(U)}_{ij} = \Pi_{il}^{(\overline{U})} g_{jl}, \quad \Pi^{(g)}_{ij} = \Pi_{lj}^{(\overline{U})} U_{li}, \quad (3.9)$$

with

$$\Pi_{il}^{(\overline{U})} = \frac{\delta S[\Sigma, \overline{U}]}{\delta (\partial_0 U)^{il}}. \quad (3.10)$$

Note that the product of the Bose field variables $U, \; V$, and the corresponding “transposed” momenta is a gauge invariant quantity

$$\tilde{\Pi}^{(U)} U = \tilde{\Pi}^{(\overline{U})}\overline{U}, \quad (3.11)$$

where the “tilde” stands for “transpose”. From (3.9) and (3.11) we read off the general form of the primary constraint

$$\Omega := \tilde{\Pi}^{(U)} U - g \tilde{\Pi}^{(g)} = 0, \quad (3.12)$$

The general conclusions of this section are valid for arbitrary JR parameter $a$. In order to simplify the discussion we choose $a = 2$. In this case (3.8) reduces to
\[ S[Ug, g^{-1}V] = SYM[\Sigma] - (1 + C_V)\Gamma[\Sigma] + \Gamma[Ug]. \tag{3.13} \]

The canonical quantization can proceed in two ways:

i) We make use of the Polyakov-Wiegmann formula [21]

\[ \Gamma[Ug] = \Gamma[U] + \Gamma[g] + \frac{1}{4\pi} \int d^2x \, tr[(U^{-1}\partial_+ U)(g\partial_- g^{-1})], \tag{3.14} \]

and note that \( \Gamma[G] \) is of the form [4]

\[ \Gamma[G] = \frac{1}{8\pi} \int d^2x \, tr(\partial_\mu G \partial^\mu G^{-1}) + \frac{1}{4\pi} \int d^2x \, tr(A(G)\partial_0 G). \tag{3.15} \]

We then obtain for the momenta canonically conjugate to \( U \) and \( g \)

\[ \pi_{ij}^{(U)} = \frac{1}{4\pi} \left\{ \partial_0 U^{-1} + A(U) + [(g\partial_- g^{-1})U^{-1}] \right\}_{ji}, \]
\[ \pi_{ij}^{(g)} = \frac{1}{4\pi} \left\{ \partial_0 g^{-1} + A(g) + [g^{-1}(U^{-1}\partial_+ U)] \right\}_{ji}. \tag{3.16} \]

ii) We leave (3.13) as it stands and compute the momenta conjugate to \( U \) and \( g \). Making use of (3.15) we then obtain

\[ \Pi_{ij}^{(U)} = \frac{1}{4\pi} \left\{ \partial_0 U^{-1} + [(g\partial_0 g^{-1})U^{-1} + gA(Ug)] \right\}_{ji}, \]
\[ \Pi_{ij}^{(g)} = \frac{1}{4\pi} \left\{ \partial_0 g^{-1} - [g^{-1}(U^{-1}\partial_0 U) + A(Ug)U] \right\}_{ji}. \tag{3.17} \]

Making use of fundamental property [4]

\[ \frac{\partial A(G)_{ij}}{\partial G_{kl}} - \frac{\partial A(G)_{ik}}{\partial G_{ji}} = \partial_0 G^{-1}_{ik} G^{-1}_{lj} - G^{-1}_{ik} \partial_0 G^{-1}_{lj}, \tag{3.18} \]

it is straightforward to show that the two sets of canonical momenta are related by a canonical transformation. We leave the demonstration of this to the appendix B.

In the following it turns out to be more convenient to work with the canonical momenta (3.17). Defining \( \Omega^a = tr(t^a \Omega) \), with \[ [t^a, t^b] = if^{abc} t^c \], a simple calculation shows that

\[ \{\Omega^a(x), \Omega^b(y)\} = -f^{abc} \Omega^c(x)\delta(x^1 - y^1). \tag{3.19} \]

Hence the primary constraints (3.12) are first class [17]. It remains to show that there are no further constraints. To see this we need to compute \( \{\Omega^a, H_T\} \), where \( H_T \) is the total
Hamiltonian, \( H_T = H_c + \int d^2 x v^a \Omega^a \). It is a straightforward matter to show that \( H_c \) is weakly equivalent to

\[
H_c \approx \int dy^1 tr \left\{ -\tilde{\Pi}(E) \partial_1 E + \tilde{\Pi}(\Sigma) \partial_1 \Sigma + 2ie\tilde{\Pi}(\Sigma) \tilde{\Pi}(E) + \frac{1}{2} E^2 \\
+ \frac{(1 + CV)}{2\pi} \epsilon^2 (\tilde{\Pi}(E))^2 - 4\pi(\tilde{\Pi}(U)g\tilde{\Pi}(g)) \\
- \frac{1 + CV}{4\pi} \partial_1 \Sigma \partial_1 \Sigma^{-1} + \frac{1}{8\pi} \partial_1 (Ug) \partial_1 (Ug)^{-1} \right\}, \tag{3.20}
\]

where (see also [22])

\[
\tilde{\Pi}(U) = \tilde{\Pi}(U) - \frac{1}{4\pi} gA(Ug), \\
\tilde{\Pi}(g) = \tilde{\Pi}(g) - \frac{1}{4\pi} A(Ug)U. \tag{3.21}
\]

For the computation of \( \{ \Omega^a, H_c \} \) it is useful to observe that \( \int d^2 x v^a \Omega^a \) is the generator of the gauge transformation \( U \rightarrow UG, \ V \rightarrow G^{-1}V, \ g \rightarrow G^{-1}g \):

\[
\{ U_{ij}(x), \Omega^a(y) \} = (Ut^a)_{ij} \delta(x^1 - y^1), \\
\{ g_{ij}(x), \Omega^a(y) \} = -(t^a g)_{ij} \delta(x^1 - y^1). \tag{3.22}
\]

As a consequence we have for any functional of \( Ug \),

\[
\{ f[Ug], \Omega^a(x) \} = 0. \tag{3.23}
\]

Furthermore

\[
\{ \tilde{\Pi}^{(U)}(x), \Omega^a(y) \} = -(t^a \tilde{\Pi}^{(U)})_{ij} \delta(x^1 - y^1), \\
\{ \tilde{\Pi}^{(g)}(x), \Omega^a(y) \} = (\tilde{\Pi}^{(g)} t^a)_{ij} \delta(x^1 - y^1). \tag{3.24}
\]

\( \xi \)From here it follows that

\[
\left\{ \Omega^a, tr \left( \tilde{\Pi}(U)g\tilde{\Pi}(g) \right) \right\} = 0. \tag{3.25}
\]

This result, together with (3.23) then implies \( \{ \Omega^a, H_c \} = 0 \), which shows that \( \Omega^a = 0 \) are the only constraints.
IV. CONCLUSIONS AND FINAL REMARKS

The bosonization of chiral $QCD_2$ in terms of group valued fields involves a Hilbert space $\mathcal{H}^B$ which is much larger than the physical Hilbert space $\mathcal{H}$. This required a careful analysis based on the construction of the physical Hilbert space as a representation of the intrinsic field algebra, in order to avoid misleading conclusions. We have thereby shown that the BRST conditions to be imposed on $\mathcal{H}^B$ are in general not sufficient to define $\mathcal{H}$, and that certain superselection rules also have to be taken into account. This observation proved crucial for establishing the non-equivalence of chiral $QCD_2$ with JR-parameter $a = 2$ to $QCD_2$.

The above analysis was carried out in the GNI formulation, where the bosonic degrees of freedom are not restricted by Dirac-type constraints [17]. In the GNI formulation, we have an unconstrained system in the sense of Dirac and the BRST conditions replace the second class constraints of the original formulation in terms of gauge and fermion fields. When embedding the theory into a gauge theory by suitable addition of a WZW-term, following the procedure of [16], we showed that one arrives at a constrained system in the sense of Dirac, with one set of first class constraints. This was to be expected, since the embedding has left us with a formulation exhibiting a local symmetry. As we showed, these first class constraints are indeed the generators of gauge transformations in the bosonic formulation.

An open question refers to the generalization of the bosonization technique for obtaining also a decoupled formulation of chiral $QCD_2$ in the case $a > 1$. It appears questionable at present, whether a complete factorization of the partition function can also be achieved in the general case.
In this Appendix we examine the effect of the use of a redundant field algebra in deriving basic physical properties of the model. In analogy with the abelian case \[11,8\], this will be illustrated by an example in which the Bose field \( \tilde{U} \) is improperly introduced in the physical field algebra.

To begin with, consider the chiral density operator (normal ordering and point-splitting are presumed)

\[
\mathcal{M} = \text{tr}(\psi_\ell \psi^\dagger_{\ell}) \in \mathcal{S},
\]

(A.1)

\[
[Q_{BRST}, \mathcal{M}] = 0.
\]

(A.2)

In terms of the bosonic variables we can write

\[
\mathcal{M} = \text{tr}(V \psi_\ell \psi^\dagger_{\ell}) = \text{tr}(V G^{-1}) = \text{tr}(U^{-1} \Sigma G^{-1}).
\]

(A.3)

where \( G \) is a WZW field.

In order to decouple the chiral operator into two BRST-neutral pieces, consider the operator

\[
\mathcal{M} = (G_r U_\ell)^{-1} G_r U_\ell^{-1} \Sigma G^{-1} = \tilde{U}^{-1} (G_r U_\ell^{-1} \Sigma G^{-1}) = \tilde{U}^{-1} (\tilde{G}^{-1} \Sigma).
\]

(A.4)

where \( \tilde{U} = G_r U_\ell \). Defining a “new” chiral density operator by extracting the \( \tilde{U} \) dependence in (A.4),

\[
\tilde{\mathcal{M}} = \text{tr} (\Sigma \tilde{G}^{-1}),
\]

(A.5)

with \( \tilde{G} = U_r G_\ell \). The chiral density (A.5) corresponds to the mass operator of QCD2:

\[
\tilde{\mathcal{M}} = \text{tr} (\Sigma \tilde{G}^{-1}).
\]

The definition of the operator \( \tilde{\mathcal{M}} \) relies on the incorrect assumption that the field \( \tilde{U} \) can be introduced in the intrinsic field algebra \( \mathcal{S} \). Since the operator \( \tilde{U} \) carries the charge \( Q^5 \), that is trivialized in the restriction from \( \mathcal{H}^B \) to \( \mathcal{H} \), one cannot define the operator \( \tilde{U} \) as a solution of the subsidiary condition on \( \mathcal{H} \) and therefore the operator \( \tilde{\mathcal{M}} \) also cannot be defined on \( \mathcal{H} \).
APPENDIX B

We show here that the two definitions (3.16) and (3.17) of the momenta conjugate to $U$ and $g$ are canonically equivalent by showing that the canonical commutation relations of $\Pi_{ij}^{(U)}, U_{ij}, \Pi_{ij}^{(g)}, g_{ij}$ imply canonical commutation relations of $\pi_{ij}^{(U)}, U_{ij}, \pi_{ij}^{(g)}, g_{ij}$. From (3.16) and (3.17) we have

\begin{equation}
\tilde{\pi}^{(U)} = \tilde{\Pi}^{(U)} + \frac{1}{4\pi}(A(U) - gA(Ug)) - \frac{1}{4\pi}(g\partial_1 g^{-1}) U^{-1}, \tag{B.1}
\end{equation}

\begin{equation}
\tilde{\pi}^{(g)} = \tilde{\Pi}^{(g)} + \frac{1}{4\pi}(A(g) - A(Ug)U) - \frac{1}{4\pi}g^{-1}(U^{-1}\partial_1 U).
\end{equation}

It is clear that (B.1) implies

\begin{align*}
\{U_{ij}(x)\Pi_{kl}^{(U)}(y)\} &= \{U_{ij}(x)\Pi_{kl}^{(U)}(y)\} = \delta_{ik}\delta_{jl}\delta(x^1 - y^1), \\
\{g_{ij}(x)\Pi_{kl}^{(g)}(y)\} &= \{g_{ij}(x)\Pi_{kl}^{(g)}(y)\} = \delta_{ik}\delta_{jl}\delta(x^1 - y^1), \\
\{U_{ij}(x)\Pi_{kl}^{(g)}(y)\} &= \{g_{ij}(x)\Pi_{kl}^{(U)}(y)\} = 0. \tag{B.2}
\end{align*}

We now compute the remaining commutators.

i) For $\{\pi_{ij}^{(U)}, \pi_{kl}^{(U)}\}$ one finds, using (B.1),

\begin{align*}
4\pi\{\pi_{ij}^{(U)}(x), \pi_{kl}^{(U)}(y)\} &= \left\{-\left(\frac{\partial A_{lk}(U)}{\partial U_{ij}} - \frac{\partial A_{ji}(U)}{\partial U_{kl}}\right)\right. \\
&\quad + \left( g_{lr} \frac{\partial A(Ug)_{rk}}{\partial U_{ij}} - g_{jr} \frac{\partial A(Ug)_{ri}}{\partial U_{kl}} \right) \\
&\quad - \left[ (g\partial_1 g^{-1}) U^{-1}_{ji} U^{-1}_{lk} \right] \\
&\quad + \left[ (g\partial_1 g^{-1}) U^{-1}_{jk} U^{-1}_{li} \right] \delta(x^1 - y^1). \tag{B.3}
\end{align*}

Noting that $(\bar{U} = Ug)$

\begin{equation}
g_{lr} \frac{\partial A(\bar{U})_{rk}}{\partial U_{ij}} = g_{lr} \frac{\partial A(\bar{U})_{rk}}{\partial U_{is}} g_{js}, \tag{B.4}
\end{equation}

we find

\begin{align*}
g_{lr} \frac{\partial A(\bar{U})_{rk}}{\partial U_{ij}} - g_{jr} \frac{\partial A(\bar{U})_{ri}}{\partial U_{kl}} &= \left[ \frac{\partial A(U)_{lk}}{\partial U_{ij}} - \frac{\partial A(U)_{ji}}{\partial U_{kl}} \right] \\
&\quad + \left[ (g\partial_1 g^{-1}) U^{-1}_{ji} U^{-1}_{lk} \right] \\
&\quad + \left[ (g\partial_1 g^{-1}) U^{-1}_{jk} U^{-1}_{li} \right] \delta(x^1 - y^1). \tag{B.5}
\end{align*}
Making further use of (B.3), one finds that the r.h.s. of (B.3) vanishes identically. Similarly
one verifies that \( \{ \pi^{(g)}_{ij}(x), \pi^{(g)}_{kl}(y) \} = 0. \)

ii) The computation of \( \{ \pi^{(U)}_{ij}(x), \pi^{(g)}_{kl}(y) \} = 0 \) follows along similar lines. We find

\[
4\pi \{ \pi^{(U)}_{ij}(x), \pi^{(g)}_{kl}(y) \} = \left\{ \left( \frac{\partial A(\bar{U})_{lr}}{\partial U_{ij}} U_{rk} - g_{jr} \frac{\partial A(\bar{U})}{\partial g_{kl}} \right) \right.
\]
\[
- (g^{-1}U^{-1})_{li}(U^{-1}\partial_{1}U)_{jk} + (g\partial_{1}g^{-1})_{jk}(g^{-1}U^{-1})_{li} \}
\]
\[
- (g^{-1}(y)U^{-1}(y))_{rl}\delta_{jk}\delta(\delta_{1} - y_{1})
\]
\[
+ (g^{-1}(x)U^{-1}(x))_{li}\delta_{jk}\delta(\delta_{1} - y_{1}). \]  

(B.6)

Note that the sum of the last two terms in (B.6) can be written as \(-\delta_{jk}\partial_{1}(Ug)^{-1}\cdot\delta(\delta_{1} - y_{1}).\)

Noting further that \((\bar{U} = Ug)\)

\[
\frac{\partial A(\bar{U})_{er}}{\partial U_{ij}} U_{rk} - g_{jr} \frac{\partial A(\bar{U})}{\partial g_{kl}} = g_{js} \left( \frac{\partial A(\bar{U})_{lr}}{\partial U_{is}} - \frac{\partial A(\bar{U})_{si}}{\partial U_{rl}} \right) U_{rk}, \]  

(B.7)

and using (3.18), we find that the r.h.s. of (B.5) vanishes identically. This concludes the proof.

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