CM-P00058169

Ref.TH.1144-CERN

ABSOLUTE BOUND ON THE $\pi\pi$ SCATTERING AMPLITUDE

R. Wit ^{*)}
CERN - Geneva

A B S T R A C T

Combining unitarity with the well-known properties of the Herglotz functions an essential improvement on the upper bound of $|F(2,2,0)|$ is obtained under the additional assumption $F(4,2,-2) < 0$.

^{*)} On leave from the Jagellonian University,
Cracow, Poland.

Starting from unitarity for the partial wave amplitudes, crossing symmetry and the twice subtracted dispersion relation, Zukaszuk and Martin ¹⁾ were able to prove the existence of upper bounds for the scattering amplitude inside the analyticity domain. The numerical values they obtained for the bounds were, however, very large

$$|F(3, 2, -1)| < 150, \quad (1a)$$

$$|F(2, 2, 0)| < 37 \quad (1b)$$

and almost useless from the practical point of view. Looking at the derivation of these inequalities one gets the impression that they are far from being the best ones since they have been obtained simultaneously and, therefore, both are likely to be overestimated. It is also evident that the lower bound on $F(4, 0, 0)$ which follows from the relation

$$F(4, 0, 0) = F(2, 2, 0) + \frac{g}{\pi} \int_{\frac{1}{2}}^{\infty} \frac{ds \operatorname{Im} F(s, 0)}{s(s-2)(s-\frac{1}{2})} \geq - |F_{\max}(2, 2, 0)| \quad (2)$$

is certainly not the best one since we neglect here the integral over the imaginary part of the forward scattering amplitude. Indeed, as shown later by Bonnier and Vinh Mau ²⁾ starting even with the crude estimates (1) but using a quite different method, one obtains an essential improvement on the lower bound to $F(4, 0, 0)$

$$-7.5 < F(4, 0, 0). \quad (3)$$

Therefore, any more restrictive bound on $|F(3, 2, -1)|$ and $|F(2, 2, 0)|$, besides being interesting itself, will also, at the same time, improve the lower bound on the scattering length.

One may expect that an independent derivation of the bound for, e.g., $F(2,2,0)$ would be more restrictive than that given by (1b). Indeed, for the unsubtracted dispersion formula we have ¹⁾

$$|F(2,2,0)| < 3.6 \quad (4)$$

but in the case

$$i) \quad \int_{-\infty}^{\infty} \frac{ds}{s |F(s,t)|} \leq M < \infty$$

we can start with a somewhat different approach. In fact, if the condition i) is fulfilled we may try to work with the function $1/F(s,t)$ instead of $F(s,t)$ itself. The only trouble might arise from the zeros of $F(s,t)$. For $0 \leq t \leq 4$ the amplitude $F(s,t)$ is a Herglotz function in the variable $z = (s-2+t/2)^2$ and it is easy to prove that if

$$ii) \quad F(4, t, -t) < 0$$

the amplitude $F(s,t)$ does not have any zero ³⁾. On the other hand, we know that there are no non-trivial lower bounds on the scattering length if it is positive ^{1),2)}. Therefore, the assumption $F(4,0,0) < 0$ is natural for our discussion. For a strong s wave contribution [large and negative $F(4,0,0)$] we expect that the inequality ii) holds by continuity, at least for small values of t .

So we are led to the conditions i) and ii) which are exactly the same for forward scattering as those assumed by Goebel and Shaw ⁴⁾. This fact seems to be rather encouraging since the bound on $F(4,0,0)$ obtained in Ref. 4) from the existing experimental data is an order of magnitude better than (3). Therefore, in the present note, we would like to investigate the compatibility of the general results (mainly based on unitarity) obtained in Ref. 1)

with the assumptions i) and ii), in order to derive (if possible) stronger bounds on $F(s,t)$.

Let us start with the unitarity condition; in the standard notation

$$F(s,t) = \frac{\sqrt{s}}{k} \sum_0^{\infty} (2l+1) f_l(s) P_l \left(1 + \frac{2t}{s-4}\right)$$

it leads to

$$|F(s,0)| = \frac{\sqrt{s}}{k} \left| \sum_0^{\infty} (2l+1) f_l(s) \right| \leq \frac{\sqrt{s}}{k} \sum_0^{\infty} (2l+1) |f_l(s)| \equiv \bar{F}(s), \quad (5a)$$

$$|\bar{H}(s,t)| \equiv \frac{\sqrt{s}}{k} \sum_0^{\infty} (2l+1) |f_l(s)|^2 P_l \left(1 + \frac{2t}{s-4}\right) \leq$$

$$\frac{\sqrt{s}}{k} \sum_0^{\infty} (2l+1) \operatorname{Im} f_l(s) P_l \left(1 + \frac{2t}{s-4}\right) = \operatorname{Im} F(s,t)$$

(5b)

if $0 \leq t \leq 4$ and $4 \leq s$. As proved in Ref. 1), for any fixed value of s and t (positive) and for a given $\bar{F}(s)$ there exists a unique set of partial waves minimizing $\bar{A}(s,t)$. The most important feature of $\bar{A}_{\min}(s,t) \equiv \phi(\bar{F},s,t)$ is that the functions $\phi(\bar{F},s,t)$, $\phi(\bar{F},s,t)/\bar{F}(s)$ and $\partial \phi(\bar{F},s,t)/\partial \bar{F}$ are increasing with $\bar{F}(s)$.

Now, according to our assumptions i) and ii) we may write

$$-\frac{1}{F(s,t)} = \frac{2}{\pi} \int_4^{\infty} \frac{ds' (s'-2+t/2) \bar{H}(s',t)}{|F(s',t)|^2 (s'-4+s+t)(s'-s)}.$$

Applying unitarity and neglecting $\operatorname{Re} F(s,t)$ in the denominator of the integrand we obtain finally

$$\left| \frac{1}{F(4-t, t, 0)} \right| \leq \frac{2}{\pi} \int_4^{\infty} \frac{ds (s-2+t/2)}{s(s-4+t) \phi(\bar{F}, s, t)} \equiv \alpha(t). \quad (6)$$

On the other hand $u \leftrightarrow t$ crossing and the Cauchy formula for $i \ln F(s, t) / \sqrt{(s-4)(s+t)}$ yield

$$|F(4-t, t, 0)| = \exp \left(\frac{2}{\pi} \sqrt{t(4-t)} \int_4^{\infty} \frac{ds (s-2) \ln |F(s, 0)|}{\sqrt{s(s-4)} (s-4+t)(s-t)} \right); \quad (7)$$

combining this equation with the inequalities (5a) and (6)

$$\frac{1}{\alpha(t)} \leq |F(4-t, t, 0)| = \quad (8a)$$

$$= |F(4-t, 0, t)| \leq \exp \left(\frac{2}{\pi} \sqrt{t(4-t)} \int_4^{\infty} \frac{ds (s-2) \ln \bar{F}(s)}{(s-t)(s-4+t) \sqrt{s(s-4)}} \right). \quad (8b)$$

For fixed $F(4-t, t, 0)$ the value of $\alpha(t)$ cannot be arbitrarily small, otherwise inequality (8a) would be violated. Given $\alpha(t)$ we would like to find the maximum of the right-hand side of inequality (8b); so we have to deal with a well defined variational problem. The extremum condition on $\bar{F}$ reads in this case

$$\frac{(s-2+t/2)(s-t)}{s(s-2)} \sqrt{\frac{s(s-4)}{t(4-t)}} \frac{\bar{F}}{\phi^2} \frac{\partial \phi}{\partial \bar{F}} = \beta(t) \quad (9)$$

where $\beta(t)$ does not depend on s . In order to obtain the values of $\bar{F}(s)$ for $s \geq 4$ we have to solve this equation. The solution may not be unique but for any fixed s , t and given $\beta(t)$ we can find the largest value of $\bar{F}(s)$ which, on the other hand, does not violate unitarity. Both ϕ and $\bar{F}$ depend on $\beta(t)$; we are free to choose it in such a way that inequalities (8a) and (8b) are nearly saturated. If the function $\bar{F} \partial \phi / \partial \bar{F} / \phi^2$ were a decreasing function of $\bar{F}$ there would exist a unique value of β satisfying this requirement. As yet, however, we are not able to present a general proof of such a behaviour of $\bar{F} \partial \phi / \partial \bar{F} / \phi^2$ (except for some simple examples). Nevertheless, it is possible to perform a numerical analysis of our relations and the result we found seems to be rather interesting. Changing $\beta(2)$ in the interval

$$0.05 \leq \beta(2) \leq 60.0$$

we have obtained a saturation of the inequalities (8a) and (8b) for a unique value of $\beta(2)$ equal to 0.30; it leads to the following upper bound on $|F(2,2,0)|$

$$|F(2,2,0)| < 7.6. \quad (10)$$

This bound is much better than (1b) but still double to that given by inequality (4). Note, however, that our result cannot be considered as the best one even within the presented approach since when deriving inequality (6) we neglected the real part. The contributions from $\text{Re } F(s,2)$, especially at low energies, could be quite important. Therefore, we may hope that developing our method in a systematic way one should be able to obtain stronger constraints on $F(s,t)$.

ACKNOWLEDGEMENTS

I am very obliged to Professor A. Martin and to Dr. A. Common for several very useful discussions. The correspondence with Dr. L. Zukaszuk in the early stage of this work is acknowledged. Thanks are due to Dr. G. Winbow for reading the manuscript.

REFERENCES

- 1) L. Zukaszuk and A. Martin, Nuovo Cimento 52, 122 (1967).
- 2) B. Bonnier and R. Vinh Mau, Phys.Rev. 165, 1923 (1967);
cf., A. Common, Nuovo Cimento 63, 451 (1969).
- 3) R. Wit, Acta Phys.Polon. 24, 271 (1963).
- 4) C.J. Goebel and G. Shaw, Phys.Letters 27B, 291 (1968).