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A LOWER BOUND FOR THE $\pi^0 \pi^0$ SCATTERING LENGTH

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A B S T R A C T

Under some additional assumptions we show that the threshold value of the forward scattering amplitude $F(4,0,0)$ is greater than -2.2 pion Compton wavelengths.

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For the sake of simplicity we consider here only the $\pi^0 \pi^0$ scattering amplitude $F(s, t, u)$ where, as usually, the Mandelstam variables s , t and u fulfil the equality $s+t+u=4$. The natural system of units ($\mu = \hbar = c = 1$) is adopted throughout this note. In the centre-of-mass system we have for the s channel $s = 4(q^2 + 1)$ where q denotes the c.m. relative momentum.

As shown in Ref. 1) it is possible to get a more restrictive lower bound for $F(2, 2, 0)$ than that known before in the literature ²⁾. The inequality

$$-7.6 \leq F(2, 2, 0) \quad (1)$$

was obtained under two additional assumptions

$$\begin{aligned} \text{i)} \quad & F(4, 2, -2) < 0, \\ \text{ii)} \quad & \int_{-\infty}^{\infty} \frac{ds}{s |F(s, 2)|} \leq M < \infty. \end{aligned}$$

Inequality (1) leads also in an obvious way to

$$-7.6 \leq F(4, 0, 0) \quad (2)$$

which means that our rather simple method gives even a slightly better lower bound for the S wave scattering length than that obtained by Bonnier and Vinh Mau ³⁾. However, due to the Ineq. (1), the lower bound to $F(4, 0, 0)$ given by these authors can certainly be sharpened since any improvement on the upper bound to $|F(2, 2, 0)|$ leads in their calculations to an automatic improvement for the lower bound to $F(4, 0, 0)$.

In the present note we would like first of all to discuss the assumption i). Then, instead of concentrating our attention on a possible lower bound to $F(2, 2, 0)$ we derive an immediate lower bound for $F(4, 2, -2)$. The results seem to be very interesting and they reveal again the strength of the restrictions imposed by analyticity, unitarity and crossing on the measurable quantities.

Let us start with the partial wave decomposition (in what follows we assume $t=2$)

$$F(s, 2) = \frac{\sqrt{s}}{\ell} \sum_{\substack{l=0 \\ \text{even}}}^{\infty} (2l+1) f_l(s) P_l(x), \quad x = 1 + 4/(s-4). \quad (3)$$

Evaluating $F(s, 2)$ at the threshold ($s=4$) one sums essentially over the scattering lengths

$$F(4, 2) = 2a_0 + \frac{2}{\sqrt{\pi}} \sum_{\substack{l=0 \\ \text{even}}}^{\infty} (2l+1) a_l 2^l \frac{\Gamma(l+1/2)}{\Gamma(l+1)}; \quad (4)$$

here by definition $a_l = \lim_{s \rightarrow 4} f_l(s)/q^{2l+1}$. According to Ref. 4) the higher partial waves contribution to the right-hand side of Eq. (4) is substantially controlled by the D wave:

$$a_{l+2} \leq \frac{1}{16} \frac{(l+2)(l+1)}{(l+5/2)(l+3/2)} a_l, \quad l \geq 2. \quad (5)$$

Taking $a_2 = 1.6 \cdot 10^{-5}$ [cf., Ref. 5)], and using Ineq. (5) in (4) we obtain

$$F(4, 2, -2) \leq 2a_0 + 3.3 \cdot 10^{-3}. \quad (6)$$

Comparing this result to the lower bound (2) or even to the estimate given by Goebel and Shaw⁶⁾

$$-0.5 \leq a_0$$

we see that in any present attempt to improve the lower bound for a_0 the Ineq. (1) must be in fact assumed from the beginning. Therefore, we take it for granted as well as the assumption ii).

Now we proceed to the unitarity condition; in particular

$$\bar{H}(s, 2) \equiv \frac{\sqrt{s}}{\ell} \sum_{l=0}^{\infty} (2l+1) |f_l(s)|^2 P_l(x) \leq \text{Im } F(s, 2).$$

By definition

$$F(s, 0) \leq \frac{\sqrt{s}}{\ell} \sum_{\ell=0}^{\infty} (2\ell+1) |f_{\ell}(s)| \equiv \bar{F}(s).$$

As proved in Ref. 2) the unique set of partial waves $f_{\ell}(s)$ which [for a given $\bar{F}(s)$] minimizes $\bar{A}(s, 2)$, takes the form

$$|f_{\ell}(s)| = 1, \quad \ell = 0, \dots, 2L$$

$$|f_{\ell}(s)| = c(s)/P_{\ell}(x), \quad \ell = 2L+2, \dots$$

$$P_{2L}(x) < c(s) \leq P_{2L+2}(x).$$

For this choice of $|f_{\ell}(s)|$ we put $\bar{A}(s, 2) \equiv \phi(\bar{F}, s, 2)$. Essential to our final results is the remark that whatever the sign of $F(4, 2, -2)$ the function $F(s, 2) - F(4, 2)$ does not have any zero ⁷⁾ (except for the threshold branch point). Therefore, one can write

$$-\frac{1}{F(s, 2) - F(4, 2)} = \frac{2}{\pi} \int_4^{\infty} \frac{ds' (s'-1) \operatorname{Im} F(s', 2)}{(s'-s)(s'-2+s) \left[(\operatorname{Re} F(s', 2) - F(4, 2))^2 + (\operatorname{Im} F(s', 2))^2 \right]}$$

or, neglecting the real part contribution, taking into account i), making use of unitarity and putting $s=2$

$$\frac{1}{\operatorname{INT}} \leq |F(2, 2, 0)| - |F(4, 2, -2)| \leq \operatorname{EXP} - |F(4, 2, -2)|, \quad (7)$$

where

$$\operatorname{INT} = \frac{2}{\pi} \int_4^{\infty} \frac{ds (s-1)}{(s-2) s \phi(\bar{F}, s, 2)}, \quad \operatorname{EXP} = \exp \left(\frac{4}{\pi} \int_4^{\infty} \frac{ds \ln \bar{F}(s)}{\sqrt{s(s-4)} (s-2)} \right).$$

What is the maximum of EXP for a given INT and fixed $|F(4,2,-2)|$? The extremum condition on $\bar{F}(s)$ yields in this case

$$\sqrt{s(s-1)} \frac{s-1}{s} \frac{\bar{F} \frac{\partial \phi}{\partial \bar{F}}}{\phi^2} = 2\beta \quad (8)$$

Given β we solve this equation for $s \gg 4$ and find INT, EXT as well as their product PR. What we observe in a widely changed range of β is that as β increases so do INT and PR but EXP decreases. For the lowest possible value of β , for which $PR=1$, the quantity $|F(4,2,-2)|$ must be zero, otherwise Ineq. (7) would be violated. Then, for greater values of β we have more room for $|F(4,2,-2)|$ but in this scheme there exists a natural upper bound for it since Ineq. (7) must hold.

The most interesting part of our numerical analysis is presented in the Table; we display the general correlation between EXP and $|F_{\max}(4,2,-2;\beta)|$ in the Figure. By inspection one gets

$$\text{Max } |F_{\max}(4,2,-2;\beta)| \leq 2.2 \quad (9)$$

with a corresponding bound on $|F(2,2,0)|$

$$|F(2,2,0)| \leq 4.20.$$

Since, as we have shown above, the higher partial waves contribution to $F(s,2)$ at threshold is almost negligible we can conclude

$$-2.2 \leq F(1,0,0).$$

This bound, representing a remarkable improvement on (2) is, however, still not very restrictive from the phenomenological point of view. Therefore, any further improvement of our result would be very useful.

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β	INT	EXP	$ F_{\max}(4,2,-2) $
0.3	0.132	7.519	0.00
0.5	0.236	5.771	1.54
0.7	0.343	4.822	1.91
0.8	0.395	4.498	1.97
0.9	0.447	4.234	2.00
1.0	0.497	4.014	2.00
1.1	0.548	3.827	2.00
1.2	0.597	3.664	1.99
1.3	0.647	3.521	1.98
1.4	0.697	3.393	1.96
1.5	0.747	3.279	1.94

Numerical values of INT, EXP and $|F_{\max}(4,2,-2)|$
as functions of β .

R E F E R E N C E S

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Mutual dependence of EXP and $|F_{\max}(4, 2, -2; \beta)|$
as functions of the parameter β .