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ABSOLUTE BOUND ON CROSS-SECTIONS AT ALL ENERGIES
AND WITHOUT UNKNOWN CONSTANTS

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A B S T R A C T

We prove that the total cross-sections for $\pi\pi$ scattering are bounded at all energies by

$$\frac{1}{s-4\mu^2} \int_{4\mu^2}^s ds' (s'-4\mu^2) \sigma_{\pi\pi}(s') < \frac{9\pi}{4} \left(\frac{m+1}{m}\right)^2 \frac{s}{\mu^2} \log^2 \frac{s}{\mu^2} +$$

$$+ 12\pi \frac{m+1}{m} \sqrt{\frac{s}{\mu^2}} \log \frac{s}{\mu^2} +$$

$$+ \frac{(320/\sqrt{2})\pi \mu^{-2-1/2}}{m (2m)!} \frac{2^m (m+1)(m!)^2 [2m/(m+1)]^m}{a_2^{\pm} s},$$

valid for all $n \geq 1$, and where a_2^t is the t channel D wave scattering length. At high energies, the right-hand side becomes essentially the Froissart bound without unknown constants, viz.,

$$(9\pi/4) [(m+1)/m\mu]^2 s \log^2 s/\mu^2.$$

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The celebrated Froissart bound ¹⁾ for total cross-sections reads (for, say, $\pi\pi$ scattering)

$$\bar{\sigma}_{\text{tot}}(s) < (\pi/\mu) \log^2(s/s_0), \quad s > S \quad (1)$$

Here μ = pion mass, σ_{tot} is the total cross-section and $\bar{\sigma}_{\text{tot}}$ its average value; s_0 and S are two unknown constants. The importance and great generality ²⁾ of the bound, and the fact that experiment nearly saturates it, makes it desirable to remove its hindrances, namely, to obtain equivalent bounds without the unknown constant s_0 and valid for all energies. In this note, we are able to solve the problem obtaining the absolute bounds, valid for any value of s ,

$$\begin{aligned} & \frac{1}{s-4\mu^2} \int_{4\mu^2}^s ds' (s'-4\mu^2) \sigma_{\text{tot}}(s') < \\ & < \frac{9}{4} \pi \left(\frac{n+1}{n}\right)^2 \frac{s}{\mu^2} \log^2 \frac{s}{\mu^2} + \\ & + 12\pi \frac{n+1}{n} \sqrt{\frac{s}{\mu^2}} \log \frac{s}{\mu} + \\ & + \frac{(320/\sqrt{2}) \pi \mu^{-2-1/2} 2^n (n+1) (n!)^2 [2n/(n+1)]^n}{n(2n)!} a_2^t s, \end{aligned}$$

(2)

where the bound is valid for any value of $n \geq 1$, and a_2^t is the t channel D wave scattering length ³⁾. For large values of s (2) becomes

$$\begin{aligned} \frac{1}{1-4\mu^2} \int_{4\mu^2}^s ds' (s'-4\mu^2) \sigma_{\text{tot}}(s') &\lesssim \\ &\lesssim \frac{9}{4} \pi \left(\frac{n+1}{n}\right)^2 \frac{s}{\mu^2} \log^2 \frac{s}{\mu^2}, \end{aligned} \quad (3)$$

which is equivalent to the Froissart bound. If, furthermore, one assumes that σ_{tot} does not oscillate at infinity ⁴⁾, one also gets, by direct application of the techniques in the second paper of Ref. 4), the local estimate ($n \rightarrow \infty$)

$$\bar{\sigma}_{\text{tot}}(s) \lesssim (\alpha+1) \frac{9}{4\mu^2} \pi \log^2 \frac{s}{\mu^2}, \quad (3')$$

with $\alpha \equiv \lim_{s \rightarrow \infty} \log \sigma_{\text{tot}}(s) / \log(s/\mu^2)$.

To prove our statement, Eq. (2), we first recall that ^{*})

$$\sigma_{\text{tot}}(s) = \frac{32\pi}{1-4} \sum_{\substack{l=0 \\ \text{even}}}^{\infty} (2l+1) \eta_l(s), \quad (4)$$

$$A_s(s, t) = \frac{\sqrt{s}}{2q} \sum_{\substack{l=0 \\ \text{even}}}^{\infty} (2l+1) \eta_l(s) P_l(\cos \theta_s), \quad (5)$$

where $\eta_l(s) = \text{Im } f_l(s)$, $\cos \theta_s = 1 + 2t/(s-4\mu^2)$, and $q = \frac{1}{2}\sqrt{s-4\mu^2}$; A_s is positive when $\cos \theta_s \geq 1$. Furthermore, by using the Froissart-Gribov representation, one can show ⁵⁾ that the t channel D wave scattering length, $a_2^t = \lim_{q^2 \rightarrow 0} f_2(s)/q^5$, is given in terms of A_s by

$$a_2^t = \frac{1}{40\mu} \int_{4\mu^2}^{\infty} ds s^{-3} A_s(s, t=4\mu^2). \quad (6)$$

^{*}) We are neglecting isospin. In fact, the complications due to it are quite inessential.

From (6), (5), we obtain the chain of inequalities

$$\begin{aligned}
 a_2^L &> \frac{\sqrt{s}}{80\mu q} \int_{4\mu^2}^s ds' s'^{-3} \sum (2l+1) \eta_l(s') P_l\left(1 + \frac{8\mu^2}{s-4\mu^2}\right) > \\
 &> \frac{\sqrt{s}}{80\mu q} (2l+1) \int_{4\mu^2}^s ds' s'^{-3} P_l\left(1 + \frac{8\mu^2}{s-4\mu^2}\right) \eta_l(s') > \\
 &> \frac{s^{-2-1/2}}{80\mu q} (2l+1) P_l\left(1 + \frac{8\mu^2}{s-4\mu^2}\right) \int_{4\mu^2}^s ds' \eta_l(s').
 \end{aligned}$$

(7)

If we split the sum in (4) into two pieces, one from 0 to $L = N\sqrt{s/\mu^2} \log(s/\mu^2)$, and the rest (N will be fixed in a moment), multiply by $s-4\mu^2$ and integrate from $4\mu^2$ to s , we shall get

$$\begin{aligned}
 &\frac{1}{s-4\mu^2} \int_{4\mu^2}^s ds' (s-4\mu^2) \sigma_{\text{tot}}(s') = \\
 &= 32\pi \sum_{\substack{l=0 \\ \text{even}}}^L (2l+1) \frac{1}{s-4\mu^2} \int_{4\mu^2}^s ds' \eta_l(s') + \\
 &+ \frac{32\pi}{s-4\mu^2} \sum_{\substack{l=L+2 \\ \text{even}}}^{\infty} (2l+1) \int_{4\mu^2}^s ds' \eta_l(s').
 \end{aligned}$$

Using the unitarity bound, $\eta_l(s) \leq 1$ for $l \leq L$, and (7) for $l > L$, we obtain

$$\begin{aligned}
 &\frac{1}{s-4\mu^2} \int_{4\mu^2}^s ds' (s-4\mu^2) \sigma_{\text{tot}}(s') < \\
 &< 32\pi N^2 \frac{s}{2\mu^2} \log^2 \frac{s}{\mu^2} + 64\pi N \sqrt{\frac{s}{4\mu^2}} \log \frac{s}{\mu^2} + (\text{Rest}),
 \end{aligned}$$

(8a)

where the (Rest) equals

$$(Rest) \leq \frac{32\pi}{s-4\mu^2} 80\mu q s^{2+1/2} \sum_{\substack{L+2 \\ \text{even}}}^{\infty} \frac{1}{P_L(1+8\mu^2/[s-4\mu^2])}. \quad (8b)$$

Now, one has the bounds ^{*)}, valid for all $n \geq 1$, $x > 1$,

$$P_L(x) > \frac{(2m)!}{2^m (m!)^2 \left(\frac{2m}{m+1}\right)^m} \left[x + \frac{m}{m+1} \sqrt{x^2-1} \right]^L,$$

so that, with $x = 1+8\mu^2/(s-4\mu^2)$, the above sum is bounded by

$$\begin{aligned} & \frac{(2m)!}{2^m (m!)^2 \left(\frac{2m}{m+1}\right)^m} \cdot \frac{x + [m/(m+1)] \sqrt{x^2-1}}{x + [m/(m+1)] \sqrt{x^2-1} - 1} \times \\ & \times \left[x + \frac{m}{m+1} \sqrt{x^2-1} \right]^{-N\sqrt{s/\mu^2} \log(s/\mu^2)} \end{aligned}$$

(9)

With a little bit of work, we then see that, in order to match the powers of s that appear in (8b) with the term (9), it is sufficient that

$$N \geq \frac{3}{8} \frac{m+1}{m} \sqrt{\frac{s-4\mu^2}{s}}.$$

^{*)} These bounds are due to A. Martin.

Since the bound (2) is anyway not very good for s near $4\mu^2$, we can just take $N=3(n+1)/8n$. Plugging this back into (9) and the result into (8b), we obtain the bound

$$(R_{\text{tot}}) < (320/\sqrt{2}) \pi \mu^{-2-1/2} \frac{2^n (n+1) (n!)^2 [2n/(n+1)]^n}{n (2n)!} a_2^{\pm 1},$$

whence, substituting into (8a), the desired result, Eq. (2), follows.

It is interesting to ask whether this result could be generalized to other processes, say πN , KN or NN scattering where experimental results exist with which we could compare (2). This is certainly impossible at the moment if one insists in using only results provable from, say, axiomatic field theory, as the analyticity necessary for obtaining (6) has not yet been established. However, with some extra assumptions (essentially analyticity in domains corresponding to what field theory yields for $\pi\pi$), which are certainly fulfilled if Mandelstam analyticity holds, the analysis goes through exactly as before, and corresponding results obtain; the details will be presented in a separate paper.

It is also to be remarked that in our derivation we have implicitly assumed that Martin's pathology ⁶⁾ does not occur, and a_2^t is finite. Of course, the finiteness of a_2^t has not yet been proved from "fundamental principles", but it seems quite safe to assume it. Finally, let me comment a little on the quantitative value of the bound. If we take, e.g., (3') (with $\alpha=0$) for scattering in the range 1.5-10 GeV/c, the bound is of the order of ~ 2000 mb. Clearly, this is due to the fact that, in deriving the bound, we have taken the $\pi\pi$ dynamics to be governed by 2π exchange, whereas experimentally it seems to be governed by the ρ . In fact, for $\rho\rho$ scattering, and neglecting low-lying pion states, (3') holds with 2μ substituted by m_ρ . This gives a depressing factor of $\sim 1/100$, thus bringing the bound to the order of a few dozens of mb, i.e., the same magnitude as experimentally measured

total cross-sections. It is also a priori clear that, say, (3') cannot be improved too much as the constant factor is $9\pi/4\mu^2$, only about above the geometrical limit of $2\pi r^2$, $r \sim 1/(2\mu)^2$.

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R E F E R E N C E S

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$$\int_{4\mu^2}^s ds' s'^{-3} \exp \left[2 \log \frac{s}{\mu^2} \sqrt{\mu^2 \sigma_{\text{tot}}(s') / 4\pi \log^2 \frac{s}{\mu^2}} - 1 \right]$$
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- 5) See, for example,
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