MASSIVE FIELDS AND THE 2D STRING

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The first massive level of closed bosonic string theory is studied. Free-field equations are derived by imposing Weyl invariance on the world sheet. A two-parameter solution to the equation of motion and constraints is found in two dimensions with a flat linear-dilaton background. One-to-one tachyon scattering is studied in this background. The results support Dhar, Mandal and Wadia's proposal that 2D critical string theory corresponds to the $c = 1$ matrix model in which both sides of the Fermi sea are excited.

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1. Introduction

It is well known that the requirement of Weyl and reparameterisation invariance implies constraints on the spacetime fields of critical string theory (see Green, Schwarz and Witten\textsuperscript{1} and the references contained therein). These constraints govern the dynamics of the spacetime fields and so can be used to predict the outcome of various scattering experiments.

For the bosonic string in an empty flat background, cancellation of the Weyl anomaly implies a critical dimension of 26, a fact which was first elucidated by Polyakov\textsuperscript{2}. Weyl invariance with a background graviton field was first studied by Friedan\textsuperscript{3}. One of the achievements of his thesis was the calculation the graviton’s beta-function to two-loops using a normal coordinate expansion in the partition function. Setting the beta-function to zero, up to a curl, is equivalent to demanding Weyl invariance. The result is that the spacetime metric must be Ricci (to one loop, up to a curl) in order that the theory have vanishing Weyl anomaly. Callan et. al \textsuperscript{4} found the coupled field equations for all massless fields in the bosonic string by setting the beta functions equal to zero. Their calculation has since been extended beyond one loop. The tachyon field can also be added without ruining the renormalisability of the theory. However, Das and Sathiapalan\textsuperscript{5} noticed that with the inclusion of the tachyon, there were contributions to the Weyl anomaly that were invisible to any finite order in the loop-expansion. These contributions can be obtained by using a weak-field expansion instead of the loop expansion. Since the goal of this paper is to study tachyon scattering, a weak field expansion will have to be employed.

A popular method of implementing the weak-field expansion is the “Wilson renormalisation group” approach which was pioneered by Banks and Martinec\textsuperscript{6} and developed by Hughes et al.\textsuperscript{7}. All the massive levels of the string are included which makes the theory non-renormalisable. The theory is regulated with a short-distance cutoff, and scale invariance is imposed by asserting that the couplings must be at a fixed point of their Wilson renormalisation-group flows. This paper studies the dynamics of the first massive level within the Wilson renormalisation group framework. The free field equations are obtained at linear order in the expansion. At the quadratic order, the situation becomes vastly more complicated as, generically, every field will contribute to every equation of motion. Therefore, it must be assumed that it is consistent to study a finite subset of string modes by taking the other modes to be higher-order in this expansion. This point will be returned to when the coupled field equations are derived in Sec. 2.

String theory in two target-space dimensions is both non-trivial and solvable. With a linear dilaton background there is only one propagating particle, it is massless and is conventionally called the ‘tachyon’. The higher ‘discrete modes’ are non-propagating. The first such is a blackhole solution for the 2D metric-dilaton system. Blackhole physics is naturally non-perturbative and thus beyond the reach of the purely perturbative description of string theory discussed so far. 2D string theory is exciting because it has a well-known non-perturbative description through its identification with the non-critical $c = 1$ string. This non-perturbative description is the double-scaled $c = 1$ matrix model (see the reviews by Ginsparg and Moore\textsuperscript{8} and Polchinski\textsuperscript{9} and the references therein). In principle then, here is a toy model in which blackhole evolution and the stringy effects on black holes can be studied.

In the double scaling limit, the $c = 1$ matrix model consists simply of nonrelativistic, free fermions living in an inverted harmonic oscillator potential. The Fermi surface lies just below the top of the
potential. Once an identification between excitations in the matrix model and the spacetime fields is found, the free-fermion picture can be used to calculate the result of any spacetime scattering experiment. It has been shown\(^1\) that the spacetime tachyons are related through the ‘leg-pole transform’ to the bosonised fluctuations of the Fermi surface. This identification was made possible by comparing bulk scattering amplitudes of Louisville field theory\(^1\) with small-pulse scattering in the matrix model\(^2\). The leg-pole transform is nonlocal and it gives rise to the spacetime gravitational physics which is absent in the free-fermion picture. The discrete modes were not identified until relatively recently.

Dhar, Mandal and Wadia\(^{13,14}\) (DMW) considered fluctuations of the Fermi surface on both sides of the potential. Until their paper, working with both sides of the Fermi sea was deemed unnecessary for small pulses in the semi-classical region, since tunnelling is a non-perturbative occurrence. However, considering both sides gave two scalars, the average and the difference of the bosonised fluctuations on each side of the sea. The authors compared scattering in this model with the effective tachyon-graviton theory (with a linear dilaton) obtained by imposing Weyl invariance. It was shown that the tachyon was the leg-pole transform of the average, while the mass of the blackhole was the energy associated with the difference. They postulated that the other discrete modes corresponded to higher moments of the difference variable. This could not be checked since the effective theory including the higher modes had not yet been worked out.

The aim of this paper is to check the proposition of DMW by considering the first massive mode of the string. Sec. 2 presents the method that is used to impose Weyl invariance of the theory in arbitrary spacetime dimensions. By way of example, the tachyonic and massless levels are examined. In Sec. 3 the first massive level at linear order is studied. The non-propagating solution to the equation of motion and constraints in two dimensions with a flat linear-dilaton background is presented. Finally, the effective theory of tachyons interacting with the massive background is derived. This is compared with DMW’s prediction in Sec. 4.

### 2. Method and an Example

Consider the closed bosonic string theory defined by the partition function

\[
Z = \int \frac{[dg_{ab}d\xi]}{V_{\text{Weyl}}} e^{-S[g_{ab}, \xi]},
\]

where \(\xi^\mu\) are a collection of \(D\) scalar fields living on the string worldsheet which has the topology of a sphere and metric \(g_{ab}\). Latin letters \(a, b, \ldots = 1, 2\) are used to indicate worldsheet indices, while the spacetime indices will always be denoted by Greek letters, \(\mu = 0, \ldots, D - 1\). Worldsheet reparameterisation invariance will be kept manifest throughout the calculation, so after fixing the conformal gauge \(g_{ab} = \delta_{ab}\) in two patches on the sphere, the partition function reads

\[
Z = \int \frac{[d\sigma]}{V_{\text{Weyl}}} \left\{ \int [d^D \xi] \exp \left( \frac{D - 26}{24\pi} \int \sigma \Box \sigma - S[\xi, g] \right) \right\},
\]

with \(\Box = \delta^{ab} \partial_a \partial_b\). Weyl invariance of the theory means that arbitrary correlation functions \(\langle \prod_i \xi^{\mu_i} \rangle\) calculated with the path integral contained in the curly parentheses are be independent of \(\sigma\). Then the measure \(\int [d\sigma]/V_{\text{Weyl}}\) can be set to unity. In the Wilson renormalisation group approach, the equations
of motion are usually obtained as an operator statement inside the path integral. In contrast, this paper utilises a source $J^\mu$ and explicitly calculates the generating functional

$$Z[J] = e^{\frac{D}{16\pi^2} \int d^2 \xi \sqrt{|g|} e^{-S[\xi]} + \int J \cdot f(\xi)} ,$$

by employing a weak-field expansion and using a short-distance cutoff. The coupling $\int J \cdot f(\xi)$ will be explained soon. This approach is closely related to the one used by Brustein, Nemeschansky and Yankielowicz.\textsuperscript{15}

2.1. The linearised field equations for the tachyon and the massless fields

There are a number of subtleties inherent in this method and these are most easily illustrated by considering the familiar scenario of a string living in a background of three massless spacetime fields, the graviton $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton $\Phi$, and one tachyonic field $T$. The action is given by

$$S[\xi,\sigma] = \frac{1}{4\pi} \int_{\mathcal{M}} \left( \frac{1}{2} \sqrt{|g|} G_{\mu\nu}(\xi) \partial_\alpha \xi^\mu \partial_\beta \xi^\nu g^{ab} + \frac{1}{2} B_{\mu\nu}(\xi) \partial_\alpha \xi^\mu \partial_\beta \xi^\nu g^{ab} + \sqrt{|g|} R \Phi(\xi) + \sqrt{|g|} T(\xi) \right) ,$$

in which string tension $\alpha'$ has been set to 2 and $g = \det g_{ab}$. It is well known from beta-function results that strings can consistently propagate in a flat linear-dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu} , \quad \Phi(\xi) = Q \xi , \quad \text{and} \quad B_{\mu\nu} = 0 ,$$

where $3Q^2 = 26 - D$. The weak-field expansion used in this paper is about this background,

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad \Phi = Q \xi + \phi ,$$

so that the fields $h_{\mu\nu}, \phi$ and $B_{\mu\nu}$ are considered to be $O(\lambda)$ where, formally, $\lambda$ is a small parameter.

The guiding principle in writing down Eq. (4) is that it should be the most general action with at most two derivatives that is reparameterisation invariant both on the worldsheet and in spacetime. Therefore, the term $\frac{1}{16\pi} \int \sqrt{|g|} \tilde{A}_\mu D^2 \xi^\mu$ must also be considered. Here $D^2$ is the covariant Laplacian

$$D^2 \xi^\mu = g^{ab} D_a \partial_b \xi^\mu = g^{ab} \left( \partial_\alpha \partial_b \xi^\mu - \Gamma^\mu_{ab} g^{\alpha\nu} \partial_b \xi^\nu + \Gamma^\mu_{b\lambda} G_{\nu\lambda} \partial_\alpha \xi^\nu \partial_b \xi^\lambda \right) .$$

The field $\tilde{A}_\mu$ makes no contribution to $S$-matrix elements since, to $O(\lambda)$, the on-shell condition is $D^2 \xi^\mu = 0$. In the language of this paper, the equivalent statement is that $\tilde{A}_\mu$ can be soaked up by a field redefinition of $\xi^\mu$. This is because

$$S[\xi^\mu] + \frac{1}{4\pi} \int \tilde{A}_\mu D^2 \xi^\mu = S[\xi^\mu - \tilde{A}^\mu(\xi)] + \frac{1}{4\pi} \int \sqrt{|g|} R Q_\mu \tilde{A}^\mu ,$$

to first order in $\lambda$. Then, with the definitions

$$\xi'^\mu = \xi^\mu - \tilde{A}^\mu(\xi) \quad \text{and} \quad \phi' = \phi + Q \tilde{A} ,$$

and the use of the ‘covariant’ measure\textsuperscript{a}

$$\left[ d^D \xi \right]_{\text{cov}} = \frac{d^D \xi \sqrt{\det \left( \frac{1}{2} G_{\mu\nu} - \nabla(\mu \tilde{A}_\nu) \right)}}{} ,$$

\textsuperscript{a}Covariant measures will be used frequently throughout this work. They need to be covariant under spacetime diffeomorphisms, and they must also be local worldsheet scalars.
the partition function reads

$$Z = \int [d^D \xi] e^{-S[\phi, \xi]} \frac{1}{\lambda} \int \dot{A}_\mu D^2 \xi = \int \left[ d^D \xi' \sqrt{\det \frac{1}{2} G_{\mu\nu}(\xi')} \right] e^{-S[\phi', \xi']}, \quad (11)$$

to $O(\lambda)$. (In the following calculations the prime on the dilaton will be dropped.) This procedure would have also worked if $\eta_{\mu\nu}$, instead of $G_{\mu\nu}$, had been used in the covariant measure.

It is also useful to note that $\tilde{A}^\mu$ could have been absorbed directly into the metric since the action is invariant under the transformations

$$\delta \tilde{A}_\mu = \Lambda_{\mu} \quad \text{and} \quad \delta G_{\mu\nu} = \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu, \quad (12)$$

where $\nabla_\mu$ is the covariant spacetime derivative. This makes $\tilde{A}^\mu$ look like a ‘Stückelberg’ field — a field which is introduced in order that a massive field theory have a gauge invariance. $\tilde{A}^\mu$ is not a Stückelberg field in the true sense of the term since $G_{\mu\nu}$ is massless, however. Such fields will be encountered at the first massive level and their corresponding gauges will be fixed by setting the Stückelberg fields to zero.

The source $J^\mu$ can be coupled to any worldsheet scalar and the theory will remain invariant under reparameterisations of the worldsheet. All choices will break spacetime reparameterisation invariance so the equations derived by imposing Weyl invariance will be gauge fixed. Different couplings $\int J \cdot f$ will correspond to different gauges. The physics of the theory should not depend on the gauge choice, but for the purposes of this paper it is convenient to choose the source term to be

$$\int J_\mu (\xi^\mu - \sigma Q^\mu). \quad (13)$$

There are two reasons for this particular form. Firstly, it handles contributions from the linear part of the dilaton field exactly. Secondly, using the usual coupling $\int J \cdot \xi$ and demanding Weyl-invariance of the one-point function results in the gauge condition

$$0 = \partial_\mu \Phi + \frac{1}{2} \partial_\mu \partial_\nu \xi^\mu h_{\nu\nu} - \partial_\nu h_{\mu\nu}, \quad (14)$$

to $O(\lambda)$. A flat linear-dilaton background is obviously inconsistent with this gauge condition. Of course this background can be rotated to be compatible with the gauge and, since the other equations of motion are covariant, S-matrix elements will be unaffected. However, this is an unnecessary nuisance. Eq. (13) is not really all that exotic since it is well known from spontaneously broken theories that expanding around different points in configuration space can be advantageous. Using this analogy, the choice $\int J(\xi + \sigma Q)$ is equivalent to expanding around the true vacuum, while the coupling $\int J \cdot \xi$ corresponds to expanding around the unstable maxima — here there is a non-zero tadpole that runs away into the vacuum.

Completing the square, the generating functional can be written as

$$Z[J] = P[\sigma] \int [d^D \xi] \exp \left( -\frac{1}{8\sigma} \int \partial_\mu \xi^\mu \partial_\nu \xi^\nu \eta_{\mu\nu} + \frac{1}{4\sigma} a_0 \left( J_0 - \frac{20}{\sqrt{3}} \right) - S_{\text{int}}(\xi + X) \right), \quad (15)$$

where

$$P[\sigma] = \exp \left( \frac{1}{2} \int J \Delta J - \left( Q^2 + \frac{D-20}{3} \right) \frac{1}{8\sigma} \int \sigma \Delta \sigma \right), \quad (16)$$
Finally, the symmetry that allows the absorption of momentum conservation and in this formula determines Gaussian integrals. These integrals are regulated using a short-distance cut-off $\epsilon$. The unregulated propagator, in stereographic coordinates on a sphere with constant curvature, is

$$\Delta(z, w) = -\log \left( \frac{|z - w|^2}{(1 + z\bar{z})(1 + w\bar{w})} \right).$$

The denominator in the logarithm accounts for the zero mode on the sphere. The zero mode enforces momentum conservation but seems to play no other role in the calculations. To keep the theory reparameterization invariant, the regularised propagator must satisfy

$$\Delta_\epsilon(z, z) = \left( 2\sigma(z) - \log \epsilon^2 \right) + O(\epsilon^2),$$

$$\frac{\partial}{\partial z^a}\Delta_\epsilon(z, z') \big|_{z = z} = \partial_a\sigma(z) + O(\epsilon^2).$$

Finally, the symmetry that allows the absorption of $\hat{A}^\mu$ into the graviton in Eq. (12) can be made to persist at the quantum level (to $O(\lambda)$) if the regularised propagator satisfies the Leibnitz-like relation

$$D_a\left( [O_x O_y \Delta_\epsilon(z, z')]|_{x = y} \right) = [D_a\left( O_x O_y \Delta_\epsilon(z, z') \right)|_{x = y} + [O_a D_x\left( O_y \Delta_\epsilon(z, z') \right)]|_{x = y},$$

where $O_x$ is a linear function of $D_x$ and $O_y$ a linear function of $D_y$. This can be seen to be necessary by directly calculating the path integral to $O(\lambda)$ with $\hat{A}^\mu$ included. At the first massive level, the St"uckelberg fields may be absorbed if Eq. (21) holds for quadratic $O$ and $O'$.

Performing the weak-field expansion to $O(\lambda)$ and using the flat measure $d^D\xi$, the path integral yields

$$Z[J] = P[\sigma]\left( \frac{1}{V} \det^\frac{1}{4}\left( \frac{1}{4\pi} \right) \right) \left( \delta^D(p^\mu) - \frac{1}{4\pi} \int d^2 z e^{ipX} e^{-\frac{1}{2}p^2\Delta_\epsilon(z, z)} \right. \times \left[ \epsilon^{-2}\epsilon^{2\sigma}T(p) + \Box \sigma \phi(p) + \frac{1}{2} \left( \partial_a X^\mu \partial_a X^\nu + 2i p^\mu \partial_{z_1} \Delta_\epsilon(z, z_1) \partial_a X^\nu \right) \right|_{z_1 = z} e^D B_{\mu\nu}(p) + \frac{1}{2} \left( \partial_a X^\mu \partial_a X^\nu + 2i p^\mu \partial_{z_1} \Delta_\epsilon(z, z_1) \partial_a X^\nu + \eta^\mu_{\nu} \partial_{z_1} \partial_{z_2} \Delta_\epsilon(z_1, z_2) \right) \right. \left. - p^\mu p^\nu \left( \frac{\partial}{\partial z_1} \Delta_\epsilon(z, z_1) \right)^2 \right|_{z_1 = z_2 = z} h_{\mu\nu}(p) \right) \right).$$

In this formula $\det^\frac{1}{4}$ is the determinant without the zero mode, the tachyon has been scaled by $\epsilon^{-2}$ for convenience and

$$p^\mu + J_0^\mu \sqrt{V} - 2Q^\mu = 0.$$
Although the source is a worldsheet density, it must not vary under Weyl transformations. Thus $p^\mu$ is Weyl neutral and $\delta X^\mu = Q^\mu \delta \sigma$, as prescribed by Eq. (17).

The second derivatives of $\Delta_e$ are not entirely fixed by reparameterisation invariance\textsuperscript{17}. The most general form contains the 2 arbitrary numbers $\gamma_e$ and $\gamma_0$ and the symmetric traceless matrix\textsuperscript{b} $T_{ab}$ which only contains terms with two derivatives

$$\frac{\partial z}{\partial z_1} \frac{\partial z}{\partial z_2} \Delta_e(z_1, z_2) \bigg|_{z_1 = z_2 = z} = \gamma_e \delta_{ab} e^{-2 \epsilon^2 \sigma} + \frac{1}{2} \gamma_0 \delta_{ab} \square \sigma + T_{ab} + O(\epsilon^2). \quad (24)$$

On the superficial level this looks disastrous since the equations of motion may depend on the regularisation scheme used through the numbers $\gamma_e$ and $\gamma_0$ ($T_{ab}$ drops out of the calculation at this level of the string). In fact, it is clear that all regularisation dependence can be soaked-up by redefining the dilaton and the tachyon

$$\phi'(p) = \phi(p) + \frac{1}{2} (\gamma_0 - 1) \eta^\mu\nu h_{\mu\nu} \quad \text{and} \quad T'(p) = T(p) + \gamma_e \eta^\mu\nu h_{\mu\nu}. \quad (25)$$

It will soon become obvious that the factor of $\frac{1}{2} \eta^{\mu\nu} h_{\mu\nu}$ serves to covariantise the equations of motion. It is instructive to realise that these field redefinitions can be implemented by adding the local, worldsheet reparameterisation invariant, term

$$\frac{1}{8\pi} \int \eta^{\mu\nu} h_{\mu\nu} \square \Delta_e(z, z') \bigg|_{z' = z} , \quad (26)$$

to the action. Equivalently, the covariant measure of Eq. (10) can be used

$$[d^D\xi]_{\text{cov}} = \left[ d^D\xi \sqrt{\det \frac{1}{2} G_{\mu\nu}} \right]. \quad (27)$$

This measure has been previously considered by Andreev, Metsaev and Tseytlin\textsuperscript{18}. Regulating with the short-distance cutoff, leads to

$$[d^D\xi]_{\text{cov}} = [d^D\xi] \exp \left( -\frac{1}{8\pi} \int d^2 z \log \det G_{\mu\nu} \square \Delta_e(z, z') \bigg|_{z' = z} \right). \quad (28)$$

By performing a weak-field expansion of this new term and employing the relation Eq. (21)

$$\partial_a \left( \partial_b \Delta_e(z, z') \bigg|_{z = z'} \right) = \partial_a \partial_b \Delta_e(z, z') \bigg|_{z = z'} + \partial_b \partial'_a \Delta_e(z, z') \bigg|_{z = z'}, \quad (29)$$

all regularisation ambiguities disappear. So, either by field redefinitions\textsuperscript{c}, or by using the covariant measure, the generating functional can be cast into the form

$$Z[J] = P[\sigma] \left( \frac{1}{V} \sqrt{\det \square} \right)^{-\frac{1}{2} D} \left\{ \frac{1}{4} \int d^2 z e^\frac{i}{\hbar} \lambda_{\alpha} \epsilon^{(-p^2 + ip^0 Q)\sigma} |\epsilon|^2 \times \left[ e^{-2 \epsilon^2 \sigma} T(p) + \square \sigma (\phi(p) + \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu}) + \frac{1}{2} (\partial_a X^\mu \partial_b X^\nu + 2i p^\mu \partial_a \sigma \partial_b X^\nu) e^{ab} B_{\mu\nu}(p) \right. \right.$$  

$$+ \frac{1}{2} \left( \partial_a X^\mu \partial_a X^\nu + 2i p^\mu \partial_a \sigma \partial_a X^\nu - p^a \partial^a (\partial_a \sigma)^2 \right) h_{\mu\nu}(p) \bigg] \right\}. \quad (30)$$

\textsuperscript{b}Later we will argue that $T_{ab}$ is in fact not arbitrary, but is independent of the regularisation scheme.

\textsuperscript{c}in which case the tachyon and dilaton in Eq. (30) must be replaced by the redefined quantities $T'$ and $\Phi'$ given by Eq. (25).
Renormalisation at the linear level is trivial

\[ T_R(p) = \epsilon |p^2 - 2T(p) \quad \text{and} \quad (h_{R\mu}^\nu(p), B_{R\mu}^\nu(p), \phi_R(p)) = |\epsilon| |p^2 (h_{\mu\nu}(p), B_{\mu\nu}(p), \phi(p)) . \] (31)

This corresponds to a minimal subtraction scheme and can be clearly implemented by adding local counter-terms to the action. For notational simplicity the subscripts \( R \) will be dropped in what follows.

Finally, the limit \( \epsilon \to 0 \) can be taken and the generating functional can be varied with respect to \( \sigma \) to yield

\[
0 = \frac{\delta Z[J]}{\delta \sigma} = -P[\sigma] \left( \frac{1}{V} \text{det} \left( \square - \frac{1}{2} D \right) \right) \frac{1}{4\pi} e^{i\phi_0} e^{-p^2 + ip\cdot Q} \sigma \left\{ e^{2\sigma} \left( 2 - p^2 + ip\cdot Q \right) T \right. \\
- \left. \left[ \square \sigma + \partial_\mu \sigma \partial_\nu X^\lambda_i \phi^\lambda \right. \right] \right. \\
\times \left[ \left( \frac{D-26}{3} + (\eta_{\mu\nu} + h_{\mu\nu}) \right) Q^\mu Q^\nu - 2p^2 \phi + R + 2ip\cdot Q(\phi + \frac{1}{2}h) - 2ip^\mu Q^\nu h_{\mu\nu} \right] \\
+ \left( \frac{1}{2} e^{ab} \partial_\mu X^\mu_0 \partial_\nu X^\nu_0 + e^{ab} \partial_\mu \sigma \partial_\nu X^\mu_0 Q^\nu \right) \left( 3(-p^2 + iQ^\lambda) H_{\lambda\mu} \right. \\
+ \partial_\mu X^\mu_0 \partial_\nu X^\nu_0 \left( R_{\mu\nu} - p_\mu p_\nu + 2ip\cdot Q h_{\mu\nu} - iQ^\lambda h_{\mu\lambda} \right) \\
+ \square X^\mu_0 \left( ip^\mu (\phi + \frac{1}{2}h) - (ip^\nu + Q^\nu) h_{\mu\nu} \right) \right\} , \quad (32)
\]

where \( h = \eta^{\mu\nu} h_{\mu\nu} \) and

\[ 2R_{\mu\nu} = -p^2 h_{\mu\nu} - p_\mu p_\nu h + p_\mu h_{\nu\lambda} + p_\nu h_{\mu\lambda} , \]
\[ H_{\mu\nu} = \frac{1}{2} (p_\lambda B_{\mu\nu} + p_\mu B_{\nu\lambda} + p_\nu B_{\lambda\mu}) . \quad (33) \]

Identifying the coefficients of each linearly-independent term with zero yields the standard linearised field equations (now expressed in position coordinates)

\[ 0 = (\nabla^2 + Q \nabla + 2)T , \]
\[ 0 = \frac{D-26}{3} + (\nabla_\mu \Phi)^2 + 2\nabla^2 \Phi + R , \]
\[ 0 = (\nabla^\lambda + Q^\lambda) H_{\lambda\mu} , \]
\[ 0 = R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi , \quad (34) \]

and the gauge condition

\[ 0 = \partial_\mu (\phi + \frac{1}{2}h) - (\theta^\prime + Q^\prime) h_{\mu\nu} . \quad (35) \]

These equations are correct to first order in \( \lambda \) and \( \nabla_\mu \) is the covariant spacetime derivative.

A word can now be said about the connection of this work to the standard approach in which the beta-functions are calculated and set to zero. In Eq. (32) the beta functions are the coefficients of the linearly independent terms \( e^{2\sigma}, \square \sigma, \phi^\lambda, e^{ab} \partial_\mu X^\mu_0 \partial_\nu X^\nu_0 \) and \( \partial_\mu X^\mu_0 \partial_\nu X^\nu_0 \). However, there are also ‘beta-functions’ corresponding to terms that would have been non-local in the original action, \( \left( \partial \sigma \right)^2 \) and \( \partial_\mu \partial_\nu X^\mu_0 \). It is not just luck that setting the standard beta-functions to zero implies that these new ‘non-local beta-functions’ are zero too. This can be verified by writing the most general generating functional with two derivatives and varying it with respect to \( \sigma \). The ‘beta-functions’ corresponding to \( \left( \partial \sigma \right)^2 \) and \( \partial_\mu \partial_\nu X^\mu_0 \) are always derivatives of the dilaton and antisymmetric-tensor beta-functions. At the first massive level
this general argument no longer holds. There are operators in $\delta Z/\delta \sigma$, which correspond to non-local terms in the original action, whose coefficients are not necessarily derivatives of other beta-functions. However, although the general argument breaks down, in practice, setting the beta-functions and the gauge constraints equal to zero guarantees Weyl invariance of the theory.

2.2. Higher-order corrections to the tachyon field equation

In the final part of this section, the $T^2$ corrections to the generating functional of Eq. (30) will be discussed. The path integral is easily evaluated to give a term proportional to

$$\int dp_1 T(p_1)T(p_2) \int d^2 z_1 d^2 z_2 f_1(z_1) f_2(z_2) \exp \left( -p_1 \cdot p_2 \Delta_e(z_1, z_2) \right) ,$$

(36)

in which

$$f_i(z_i) = \exp \left[ \mp p_i X(z_i) + (2 - p_i^2)(\sigma(z_i) + \log |\epsilon|) \right] \text{ and } p_1^\mu + p_2^\mu + J_0^\mu \sqrt{V} - 2Q^\mu = 0 .$$

Up to derivatives on $\sigma$, the propagator can be written as

$$\Delta_e(z_+, z_-) = -\log \left( 4|z_-|^2 + \epsilon^2 e^{-2\sigma(z_+)} \right) ,$$

(37)

where $z_{\pm} = \frac{1}{2}(z_1 \pm z_2)$. Expanding the $f_i$ around $z_- = 0$, the integral over $z_-$ can now be performed. Of course this expansion is only valid for small $z_-$. It is assumed that the integral is finite because the worldsheet is compact and that the $f_i$ are well behaved. The integral is then dominated by small $z_-$. An aside can now be made explaining why the simple-minded method employed in this paper is not suited to finding all the field equations to quadratic order. The expansion of the $f_i$ and higher derivative terms in the regulated propagator generate terms with arbitrary powers of $\partial_\sigma \sigma$ and $\partial_\sigma X^\mu$. This means that, generically, $T^2$ terms will appear in every field equation. This is true for every other field too: The equation of motion and linear constraints $C_i$ for an arbitrary field $F$ look like

$$(\nabla^2 - M_F^2)F = O(\lambda^2) \text{ and } C_i(F) = O(\lambda^2) ,$$

(38)

to $O(\lambda^2)$. It would thus be impossible to calculate the infinite number of $O(\lambda^2)$ corrections. Instead, with a finite number of fields $F$ being $O(\lambda)$ and the rest $\bar{F}$ being $O(\lambda^2)$, it is assumed that the equations

$$(\nabla^2 - M_F^2)\bar{F} = O(F^2) \text{ and } C_i(\bar{F}) = O(F^2) ,$$

(39)

have a solution for $\bar{F}$. Then, to $O(\lambda^2)$, the only equations that need be considered are

$$(\nabla^2 - M_F^2)\bar{F} = O(F^2) \text{ and } C_i(F) = O(F^2) .$$

(40)

The $T^2$ contribution to the tachyon field equation is the easiest to calculate and will be of use later when 1-1 tachyon scattering is discussed. The contribution has no derivatives on $X^\mu$ and $\sigma$ and reads

$$Z_{TT} = -P[\sigma] \left( \frac{1}{V} \det t^{\square} \frac{1}{4\pi} \right)^{-\frac{1}{2}} \int d^2 \zeta e^{i p \cdot X} e^{(2 - p^2)(\sigma - \log |\epsilon|)} \frac{1}{32\pi} \int dp_1 T(p_1)T(p - p_1) ,$$

(41)

where $X^\mu$ is still given by Eq. (17) and $p^\mu + 3J_0^\mu \sqrt{V} = 0$. The renormalisation condition must be modified to read

$$T_R(p) = |\epsilon|^2 - 2T(p) + \frac{1}{8} \int dp_1 T(p_1)T(p - p_1) \left( 1 - |\epsilon|^{-2} - 2p_1(p - p_1) \right) .$$

(42)
Again, this can be implemented by adding local counterterms to the action. The linearised equation of motion for the tachyon, $T(p)$, implies that $-p^2 + iQp + 2 = 0$ to first order. Thus, to this order, it is correct to set $-p^2 + iQp_1 + 2 = 0$. Weyl invariance then gives
\[(\nabla^2 + Q \partial + 2)T - \frac{1}{4} T^2 = 0.\]

3. The Massive Fields

The analogous steps that were carried out at the massless level are now performed at the first massive level of the string. The most general action consistent with reparameterisation invariance, both on the worldsheet and in spacetime, is gauge-fixed by eliminating all Stückelberg degrees of freedom. Field redefinitions are employed in the path-integral to simplify the action further and to eliminate regularisation ambiguities. Renormalisation is performed and the linearised equations of motion are calculated. These are solved in two spacetime dimensions. A non-linear corrections to the tachyon field equation is then derived.

3.1. The linearised field equations for the first massive level

It is well known that the first massive level of the closed bosonic string consists of a field $E_{\mu\nu}\lambda\rho = E_{(\mu\nu)(\lambda\rho)}$ upon which the Virasoro constraints impose traceless inside pairs of indices and transversality\(^7\). These conditions have not yet been derived using the Wilson renormalisation group method\(^7\). Recently, Bardakci and Bernardo\(^2\) have presented an elegant procedure for deriving the field equations and constraints due to Weyl invariance in which they retain some degree of manifest covariance. Using their formalism, all the massive levels have been considered\(^2\), however, to date the calculations have only been performed in a flat empty background with $D \neq 26$.

The most general reparameterisation invariant action with four derivatives on a curved worldsheet has many gauge symmetries. A systematic study of these symmetries has been made by Buchbinder et. al\(^2\). Gauge invariance in string theory has also been extensively investigated by Evans et. al\(^2\). For each gauge symmetry there is an associated Stückelberg field\(^1\). Fixing each Stückelberg field to zero fixes its corresponding gauge. For instance, the action contains the three terms
\[
\int \sqrt{g} \left( W_{\mu\nu\lambda\rho}(\xi) \partial_\alpha \xi^\mu \partial_\beta \xi^\nu \partial_\gamma \xi^\lambda \partial_\delta \xi^\rho g^{ab} g^{cd} + A_{\mu\lambda}(\xi) D^2 \xi^\mu \partial_\alpha \xi^\nu \partial_\beta \xi^\rho g^{ab} + S_{\mu\nu\lambda}(\xi) D_\alpha \partial_\beta \xi^\nu \partial_\gamma \xi^\rho g^{ab} \right) .
\]

(44)

It is evident that this is invariant under
\[
\delta W_{\mu\nu}\lambda\rho = \frac{1}{2} \delta_{(\mu \Lambda_{\nu})(\lambda \rho)} + \frac{1}{2} \delta_{(\lambda \Lambda_{\rho})(\mu \nu)} , \\
\delta A_{\mu\nu\lambda} = A_{\mu(\nu\lambda)} , \\
\delta S_{\mu\nu\lambda} = A_{\nu(\mu\lambda)} + A_{\lambda(\mu\nu)} ,
\]

(45)
in which symmetrisation is indicated with round brackets, for example $A_{\mu(\nu\lambda)} = \frac{1}{2} A_{\mu\nu\lambda} + \frac{1}{2} A_{\mu\lambda\nu}$. In this case, the field $S_{\mu\nu\lambda}$ can be considered a Stückelberg field and eliminated by choosing an appropriate $A_{\mu\nu\lambda}$. After a complete gauge-fixing, the remaining terms can be written
\[
S_M(\xi^\mu, g_{ab}) = \int \sqrt{g} W_{\mu\nu\lambda\rho}(\xi) \partial_\alpha \xi^\mu \partial_\beta \xi^\nu \partial_\gamma \xi^\lambda \partial_\delta \xi^\rho g^{ab} g^{cd} + \sqrt{g} RW_{\mu\nu}(\xi) \partial_\alpha \xi^\mu \partial_\beta \xi^\nu g^{ab} + \sqrt{g} R^2 W(\xi)
\]
\[ + \int \tilde{W}_{\mu\lambda\rho}(\xi) \partial_\alpha \xi^\mu \partial_{\beta} \xi^\rho \partial_{\epsilon} \xi^\lambda \partial_{\delta} \xi^\delta \frac{g^{ab} i e^{cd}}{2} + R \tilde{W}_{\mu\lambda}(\xi) \partial_\alpha \xi^\mu \partial_{\beta} \xi^\rho \partial_{\epsilon} \xi^\lambda i e^{ab} .
\]
\[ + \int \sqrt{g} R D^2 \xi^\mu A_\mu(\xi) + \sqrt{g} D^2 \xi^\mu D^2 \xi^\nu A_{\mu\nu}(\xi) + \sqrt{g} A_{\mu\nu\lambda}(\xi) D^2 \xi^\mu \partial_\alpha \xi^\nu \partial_\beta \xi^\lambda g^{ab} .
\]
\[ + \int \bar{A}_{\mu\nu\lambda}(\xi) D^2 \xi^\mu \partial_\alpha \xi^\nu \partial_\beta \xi^\lambda i e^{ab} .
\] (46)

As in the massless case, the \( A \)-type fields can be shifted away by a change of variables in the path integral. Specifically, the analogue of the shift in Eq. (9) is
\[
\xi'^\mu = \xi^\mu - R A^\mu - A^\mu, D^2 \xi^\nu - R A^\mu, Q^\nu - A^\mu, \lambda \partial_\alpha \xi^\nu \partial_\beta \xi^\lambda g^{ab} - \bar{A}^\mu, \lambda \partial_\alpha \xi^\nu \partial_\beta \xi^\lambda i e^{ab} / \sqrt{g} ,
\] (47)
with covariant measure being
\[
\left[ d^D \xi \right]_{\text{cov}} = \left[ d^D \xi \det \frac{1}{2} \left( \frac{1}{2} G_{\mu} - R \nabla_{(\mu} A_{\nu)} - \nabla_{(\mu} (A_{\nu})_{\lambda} D^2 \xi^\lambda - R \nabla_{(\mu} (A_{\nu})_{\lambda} Q^\lambda \right)
\]
\[ - \nabla_{(\mu} (A_{\nu})_{\lambda} \partial_\alpha \xi^\nu \partial_\beta \xi^\rho - \nabla_{(\mu} (\bar{A}_{\nu})_{\lambda} \partial_\alpha \xi^\nu \partial_\beta \xi^\lambda i e^{ab} / \sqrt{g} \right) .
\] (48)

The \( W \)-type fields also need to be redefined just as the dilaton \( \phi \) was shifted to \( \phi + Q \bar{A} \) in Eq. (9).

The remaining terms in the action can be grouped together in a more compact form by using the 2-dimensional identity
\[
e^{ab} e^{cd} g_{ac} g_{bd} = 2 g .
\] (49)

Defining \( f^{ab} = \sqrt{g} g^{ab} + i e^{ab} \) this identity leads to the following symmetries of a product of two \( f^{ab} \) densities
\[
f^{ac} f^{bd} = f^{bc} f^{ad} \quad \text{and} \quad f^{ac} f^{bd} g_{ab} = 0 .
\] (50)

Then the action at the first massive level can be written in the standard fashion
\[
S_M(\xi^\mu, g_{ab}) = \int g^{-1/2} \left( E_{\mu\nu\lambda\rho}(\xi) \partial_\sigma \xi^\mu \partial_{\tau} \xi^\nu \partial_{\lambda} \xi^\rho \partial_{\delta} \xi^\sigma f^{ac} f^{bd} + \sqrt{g} RE_{\mu\nu}(\xi) \partial_\sigma \xi^\mu \partial_{\tau} \xi^\nu f^{ab} + (\sqrt{g} R)^2 E(\xi) \right) ,
\] (51)
in which
\[
2E_{\mu\nu\lambda\rho} & \equiv W_{\mu\nu\lambda\rho} + W_{\mu\nu\lambda\rho} + \tilde{W}_{\mu\nu\lambda\rho} + \tilde{W}_{\nu\mu\lambda\rho} , \\
E_{\mu\nu} & \equiv W_{\mu\nu} + \tilde{W}_{\mu\nu} , \\
E & \equiv W ,
\] (52)
The field \( E_{\mu\nu\lambda\rho} \) is symmetric in pairs of indices, \( E_{\mu\nu\lambda\rho} = E_{(\mu\nu)_{(\lambda\rho)}} \), which can be seen by using Eq. (50).

The partition function at the linear level is
\[
Z[J] = P[\sigma] \left( \frac{1}{\sqrt{4\pi}} \det \frac{1}{4\pi} \right)^{-\frac{1}{2} D} \left\{ \delta^D (p^\mu) - \frac{1}{4\pi} \int d^2 z e^{i p X} g^{-1/2} e^{-\frac{1}{2} p^2 \Delta(z,z)} e^2 \right. \\
\times \left[ E_{\mu\nu\lambda\rho} f^{ac} f^{bd} (\partial_\sigma \xi^\mu \partial_{\tau} \xi^\nu \partial_{\lambda} \xi^\rho \partial_{\delta} \xi^\sigma f^{ab} + \sqrt{g} RE_{\mu\nu}(\xi) \partial_\sigma \xi^\mu \partial_{\tau} \xi^\nu f^{ab} + (\sqrt{g} R)^2 E(\xi) \right) + (4\eta^\mu, \lambda \partial_\sigma \partial_{\tau} \Delta \partial_{\lambda} \partial_{\delta} X^\rho + 2 \eta^\mu, \nu \partial_\sigma \Delta X^\nu \partial_{\tau} \Delta X^\rho + 2 \eta^\mu, \nu \partial_\sigma \partial_{\tau} \Delta \partial_{\lambda} \partial_{\delta} X^\rho + (\eta^\mu, \nu \partial_\sigma \partial_{\tau} \Delta - \eta^\mu, \lambda \partial_\sigma \partial_{\tau} \Delta \partial_{\lambda} \partial_{\delta} \Delta) \partial_{\sigma} \Delta \partial_{\tau} \Delta \partial_{\lambda} \partial_{\delta} \Delta) \partial_{\lambda} \partial_{\delta} \Delta \partial_{\tau} \Delta \partial_{\sigma} \Delta \partial_{\lambda} \partial_{\delta} \Delta)
\]
of the regularised propagator, given by Eq. (28), contains the arbitrary numbers $\gamma_2$ and $\gamma_0$ and the symmetric traceless matrix $T_{ab}$. The situation is complicated further by the terms that look generically like $\partial_a \partial_b \Delta \partial_e \partial_e \Delta$ and $\partial_a \partial_b \Delta \partial_e \partial_e \Delta$. Recall that $\partial_a \partial_b \Delta$ is of order $\epsilon^{-2}$ so that in such terms, $O(\epsilon^2)$ corrections to $\partial_a \Delta$ and $\partial_a \partial_b \Delta$ must be considered. In fact, most ambiguities can be absorbed by following the same line of thought that lead to the covariant measure of Eq. (27). Consider first the terms

$$
\left[ E_{\mu \lambda \rho} f^{ac} f^{bd} 4 \eta^{\mu \lambda} \partial_a \partial_b \Delta \left( \partial_c X^\nu \partial_d X^\rho + i \partial_c \partial_b X^\nu \partial_d \Delta + \frac{1}{2} \eta^{\nu \rho} \partial_b \partial_b \Delta - \frac{1}{2} \eta^{\nu \rho} \partial_b \partial_b \Delta \right) + \sqrt{g} RE_{\mu \nu} f^{ab} i \eta^{\mu \nu} \partial_a \partial_b \Delta \right].
$$

By using the covariant measure

$$
\left[ d^D \xi \right]_{\text{cov}} = \left[ d^D \xi \sqrt{\det \left( \frac{1}{2} G_{\mu \nu} + RE_{\mu \nu} + 4 E_{\mu \lambda \rho \sigma} f^{ab} (\partial_a X^\lambda \partial_b X^\rho - \frac{1}{2} G^{\lambda \rho} \partial_b \Delta \partial_e) \right) / \sqrt{g} } \right],
$$

and employing the Leibnitz-like relation Eq. (21), the ambiguities are removed from Eq. (54). Upon regulating this measure as in Eq. (28), it is seen that the extra terms added to the action are both local and worldsheet reparameterisation invariant (and thus correspond to simple field redefinitions as in Eq. (25)).

However, there are more ambiguous terms in the partition function. These are

$$
\left[ E_{\mu \lambda \rho} f^{ac} f^{bd} \left\{ \eta^{\mu \nu} \partial_a \partial_b \Delta \left( \partial_c X^\lambda \partial_d X^\rho + i \partial_c \partial_b X^\lambda \partial_d \Delta + \frac{1}{2} \eta^{\lambda \rho} \partial_b \partial_b \Delta - \frac{1}{2} \eta^{\lambda \rho} \partial_b \partial_b \Delta \right) + (\mu \nu \sigma) \leftrightarrow (\lambda \rho \sigma) \right\} \right].
$$

Since $f^{ac} f^{bd}$ is symmetric and traceless in $ab$ (see Eq. (50)), the $\partial_a \partial_b \Delta$ becomes simply $T_{ab}$. To remove this ambiguity, the term

$$
g E_{\mu \lambda \rho} f^{ac} f^{bd} \left\{ \eta^{\mu \nu} T_{ab} \partial_c X^\lambda \partial_d X^\rho + (\mu \nu \sigma) \leftrightarrow (\lambda \rho \sigma) \right\},
$$

would have to be added to the Lagrangian. Unless $T_{ab}$ is zero then, this term is not a local counterterm since it will contain $\partial_a \partial_b \sigma$ or $\partial_a \sigma \partial_b \sigma$. However, one particular diffeomorphism covariant calculation yields a non-zero value for $T_{ab}$

$$
T_{ab} = \frac{1}{3} \left( \partial_a \partial_b \sigma + \partial_a \sigma \partial_b \sigma - \frac{1}{2} \delta_{ab} \square \sigma - \frac{1}{2} \delta_{ab} (\partial \sigma)^2 \right).
$$
Since a local counterterm to remove $T_{ab}$ cannot be found, we propose that its non-vanishing is independent of the regularisation scheme used.

It is worth emphasising this point given the discussion in Refs 7 and 22 where the traceless condition on $E_{\mu\nu}\lambda\rho$ was missed. It is now clear that tracelessness would be lost if there was a reparameterisation invariant regularisation scheme in which $T_{ab} = 0$, since Eq. (56) would vanish and there are no other terms in the partition function that depend on $\eta^{\mu\nu}$ and $\eta^{\lambda\rho}$. Notice that this can be seen in a vertex-operator calculation too — tracelessness would come from a self-contraction of $\partial X^\mu \partial X^\nu$, and if the regularisation was such that this was zero, no tracelessness condition would be found.

After performing minimal subtraction

$$\left(E_{\mu\nu}^{\lambda\rho}, E_{\mu\nu}, E_{\nu}\right) = |\epsilon|^{2+2} \left(E_{\mu\nu}^{\lambda\rho}, E_{\mu\nu}, E\right),$$

and taking the limit $\epsilon \to 0$, Weyl invariance can be imposed on the partition function and the equations of motion obtained

$$0 = (\nabla^2 + Q \nabla - 2)E_{\mu\nu}\lambda\rho, \quad 0 = \eta^{\mu\nu}E_{\mu\nu}\lambda\rho = \eta^{\lambda\rho}E_{\mu\nu}\lambda\rho, \quad 0 = (\nabla^\mu + Q^\mu)E_{\mu\nu}\lambda\rho = (\nabla^\lambda + Q^\lambda)E_{\mu\nu}\lambda\rho, \quad 0 = \eta^{\mu\nu}E_{\mu\nu} + E_{\nu}\rho, \quad 0 = \eta^{\mu\nu}E_{\mu\nu} + 4E.$$  \hspace{1cm} (60)

Evidently $E'_{\mu\nu}$ and $E'$ are just traces of $E_{\mu\nu}\lambda\rho$. $E_{\mu\nu}\lambda\rho$ is transverse and traceless inside the pairs of indices as expected.

In two spacetime dimensions, these equations can be solved to find a higher-mode version of the blackhole. The metric $\eta^{\mu\nu} = \text{diag}(\eta^{TT}, \eta^{XX}) = \text{diag}(-1, 1)$ is used. The most general solution of equations of motion is given by

$$E_{\mu\nu}\lambda\rho = \begin{cases} E_+ & \text{if } \mu + \nu + \lambda + \rho = \text{even} \\ E_- & \text{if } \mu + \nu + \lambda + \rho = \text{odd} \end{cases},$$

where

$$E_\pm = A e^{\pm \left(Q^X + \frac{2}{q\sqrt{x^2-q^2}}\right)X + \left(Q^T - \frac{2}{q\sqrt{x^2-q^2}}\right)T} \pm B e^{-\left(Q^X + \frac{2}{q\sqrt{x^2-q^2}}\right)X + \left(Q^T + \frac{2}{q\sqrt{x^2-q^2}}\right)T},$$

in which $A$ and $B$ are arbitrary constants of integration. In the non-generic case of $Q^X \pm Q^T = 0$ the term which contains the potential infinity must be dropped from the solution. Thus, for generic background charge, the background solution of $E_{\mu\nu}\lambda\rho$ is a time-dependent, two-parameter solution.

3.2. Massive-field corrections to the tachyon field equation

In a flat linear-dilaton background $0 = \phi_{\mu\nu} = B_{\mu\nu} = \phi'$, the linearised equations of motion simply consist of the equations for the massive field $E_{\mu\nu}\lambda\rho$ given by Eq. (60), the dilaton field equation $Q^2 = (26 - D)/3$ and the tachyon equation. This section considers higher-order corrections to the tachyon field equation in this background. It is sufficient to calculate the $T^2$ and $TE$ contributions in order to make a comparison with the proposal of DMW. The $T^2$ part has already been calculated in Sec. 2. There are two ways in
which a $TE$ term can be obtained. One comes from the covariant measure

$$
\frac{1}{16\pi^2} \int [d^D\xi] \exp \left( -\frac{1}{8\pi} \int \partial_\mu \xi^\mu \partial_\nu \xi^\nu \eta_{\mu\nu} + \frac{1}{4\pi} a_0 \cdot (J_0 - \frac{2Q}{\sqrt{\nu}}) \right) \int d^2 z_1 d^2 z_2 d^Dp_1 d^Dp_2
$$

\begin{align}
&\times e^{i(p_1(z_1 + x)(z_1 + z_2) + \frac{2Q}{\sqrt{\nu}}(z_1 - z_2))} T(p_1) \\
&\times 4\eta^{\mu\nu} E_{\mu\nu\lambda\rho}(p_2) f_{\mu\nu\lambda\rho}(z_2) \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) \\
&\times 2/\sqrt{\nu} \left( \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) \right) (-2 \gamma e^{-2\gamma} e^{2\gamma}) ,
\end{align}

(63)

while the other is simply

$$
\frac{1}{16\pi^2} \int [d^D\xi] \exp \left( -\frac{1}{8\pi} \int \partial_\mu \xi^\mu \partial_\nu \xi^\nu \eta_{\mu\nu} + \frac{1}{4\pi} a_0 \cdot (J_0 - \frac{2Q}{\sqrt{\nu}}) \right) \int d^2 z_1 d^2 z_2 d^Dp_1 d^Dp_2
$$

\begin{align}
&\times e^{i(p_1(z_1 + x)(z_1 + z_2) + \frac{2Q}{\sqrt{\nu}}(z_1 - z_2))} T(p_1) \\
&\times 4\eta^{\mu\nu} E_{\mu\nu\lambda\rho}(p_2) f_{\mu\nu\lambda\rho}(z_2) \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) \partial_\theta \xi^\rho(z_2) .
\end{align}

(64)

All other terms give derivatives on $\sigma$ or $X^\mu$ which will contribute to other field equations, as explained in Sec. 2. The result of performing these Gaussian integrals is quite simple because all terms containing derivatives on $\sigma$ or $X^\mu$ can be dropped. By construction, the parts from the covariant measure exactly cancel the regularisation dependent parts in Eq. (64), leading to a total contribution of

$$
\frac{1}{16\pi^2} \int d^2 z_1 d^2 z_2 d^Dp_1 e^{i(p_1 X_1 + p_2 X_2)} e^{2\gamma} T(p_1) E_{\mu\nu\lambda\rho}(p_2)
$$

\begin{align}
&\times p_{\mu}^{i} p_{\nu}^{j} p_{\lambda}^{k} p_{\rho}^{l}(\partial_\Delta^2(z_1, z_2))^{4} e^{-\frac{1}{2} p_2^2 (\Delta(z_1, z_2))} e^{-\frac{1}{2} p_1^2 (\Delta(z_1, z_2))} e^{-p_1 (\Delta(z_1, z_2))} ,
\end{align}

(65)

with momentum conservation $p_1 + p_2 + J_0 \sqrt{\nu} - 2Q^\mu = p_1 + p_2 + p = 0$. In Sec. 2, the $TT$ term was simplified by performing the integral over $z$. This integral can also be done here. Using

$$
(4\pi \partial_\Delta^2(z_1, z_2))^2 = \frac{4|z - z'|^2}{|z - z'|^2 + e^{-2\gamma}} ,
$$

(66)

which is true up to derivatives on $\sigma$, the $TE$ term finally reads

$$
Z_{TE} = P[\sigma] \left( \frac{1}{V} \det \left( \frac{\partial^D}{\partial \sigma} \right) - \frac{1}{2} \right) \int d^2 z e^{i\nu X} e^{(2-n)^2(\sigma - \log |1|)}
$$

\begin{align}
&\times 2 \pi \int dp_1 \frac{E_{\mu\nu\lambda\rho}(p - p_1) p_{\mu}^{i} p_{\nu}^{j} p_{\lambda}^{k} p_{\rho}^{l} T(p_1)}{(3 - p_1(p - p_1))(2 - p_1(p - p_1))(1 - p_1(p - p_1))} .
\end{align}

(67)

After renormalisation, the denominators of Eqs. (41) and (67) can be simplified by substituting in the mass-shell relations

$$
(-p^2 + iQp + 2) T(p) + O(\lambda^2) = 0 = (-p^2 + iQp + 2) E_{\mu\nu\lambda\rho}(p) + O(\lambda^2) .
$$

(68)

Imposing Weyl invariance and expanding the $TE$ contribution around $2 + iQp - p^2 = 0$, the field equation is finally obtained:

$$
(\nabla^2 + Q \nabla + 2) T - \frac{1}{4} T^2 + 8E_{\mu\nu\lambda\rho} \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\rho T = 0 .
$$

(69)

This equation is valid up to $O(E^2)$ with a flat linear-dilaton background. All higher derivative terms in the expansion of the $TE$ term around $2 + iQp - p^2 = 0$ have been left out. It will soon be explained why these terms have no bearing on the validity of DMW’s proposal.
4. Tachyon Scattering in the Matrix Model

As mentioned in the introduction, the double-scaling limit of the matrix model defines the nonperturbative physics of 2D string theory. After taking the limit, the matrix model consists of nonrelativistic, non-interacting fermions living in an inverted harmonic potential. The ground state consists of a flat sea filled up to the Fermi-surface which lies \( O(1/g_{str}) \) below the top of the potential. The tachyons in spacetime have been found to be the bosonised fluctuations of the Fermi surface. To reproduce the perturbative string theory results only very small bumps on top of the Fermi sea need be considered. Tunnelling through the potential barrier and fluid washing over the wall are both non-perturbative effects. So, when working in the semi-classical limit with very small pulses, it seems safe to work with only one side of the potential. However, DMW argued that the spacetime metric must couple to the entire energy-momentum tensor of the theory. The Hamiltonian of the string theory is identical to that of the matrix model (unless the potential is modified by hand from the beginning). A generic perturbation of the Fermi fluid has total energy coming from both sides of the potential, so the metric must couple to this total energy. Allowing excitations of both halves of the sea introduced another degree of freedom into the model that had not yet been utilised. There were now two scalar fields — the average and the difference of the bosonised fluctuations on each side of the barrier. Comparing with the tachyon-graviton effective theory, they found that the tachyon, \( T \), is the leg-pole transform of the average of the fluid fluctuations, while the total energy of the difference variable, \( \Delta \), could be identified with the mass of the blackhole.

Specifically, after scaling \( T = e^{-2x} S \), the massless field \( S \) satisfied the equation of motion (see DMW Eq. (5.4))

\[
\partial_+ \partial_- X = M e^{-4x} O_{x,t} S + e^{-2x} S_0 S , \tag{70}
\]

where \( O_{x,t} \) is some second-order differential operator whose details are not important, \( S_0 \) is the tachyon background and \( x_{\pm} = t \pm x \). The first term originates from the \( Th \) term while the second comes from the \( TT \) term. The factor of \( Me^{-4x} \) comes from the blackhole solution to the graviton equation of motion, \( h_{\mu\nu} = Me^{-4x} \) in which \( M \) is a constant and called the ‘mass’ of the blackhole. Integrating the equation of motion to first order in the background \( S_0 \) with the boundary condition \( S(x,t) \big|_{x=\pm\infty} = S_0(x^\pm) \) they obtained

\[
S_{\text{out}}(x^-) = e^{2x^-} \int_{-\infty}^{x^-} dx^+ e^{-x^+} \int_{-\infty}^{x^-} du^- e^{u^-} S_0(u^-, x^+) S_0(x^+) - \frac{1}{2} M e^{2x^-} \int_{-\infty}^{\infty} e^{-2x^+} S_0(x^+) . \tag{71}
\]

The first term describes tachyon scattering in the presence of the background \( S_0 \) and the second is due to scattering off the blackhole background. Mathematically the second term comes about by going by parts with the operator \( O_{x,t} \) under the integrals \( \int_{-\infty}^{x^-} du^- \int_{-\infty}^{\infty} dx^+ \). The matrix model predicts that the term describing tachyon scattering off a blackhole background is proportional to

\[
\int_{-\infty}^{\infty} d\tau \Delta^2(\tau) e^{2x^-} \int_{-\infty}^{\infty} dx^+ e^{-2x^+} S_0(x^+) , \tag{72}
\]

This lead DMW to conclude that the mass of the blackhole was equal to the energy contained in the difference variable \( \int_{-\infty}^{\infty} d\tau \Delta^2(\tau) \).
Having found the effective theory describing tachyon-\( E_{\mu\nu}\lambda\) scattering, the same steps can be followed to find an equivalent to Eq. (71). Taking the dilaton to lie purely in the \( X^1 = X \) direction means \( Q^X = 2\sqrt{2} \). DMW’s conventions (indicated by lower-case letters) differ slightly from those used in this paper, the connection is \( x^\mu = X^\mu / \sqrt{2} \). The massless tachyon \( S \) is related to \( T \) by \( T = e^{-Q^X / 2} S = e^{-2x} S \). Inserting this into the equation of motion for the tachyon yields the equivalent of Eq. (70)

\[
\partial_+ \partial_- S = O_{x,t} E_{\mu\nu}\lambda\rho S_{x,t} + \sqrt{2} e^{-2x} S_0 S.
\]  

(73)

The \( \sqrt{2} \) in front of the \( S_0 S \) term is arbitrary since it can be tuned by rescaling \( T \), and the differential operators \( O_{x,t} \) and \( O_{\mu\nu}\lambda\rho \) occur through expanding the denominators of the \( TE \) term in small \( 2 + ipQ - p^2 \).

Substituting the background solution for the \( E \)-field in Eq. (62) and integrating this equation with the same boundary condition leads to

\[
S_{\text{out}}(x^-) = e^{2x_-} \int_{-\infty}^{\infty} dx^+ e^{-x^+} \int_{-\infty}^{x^-} e^{u^-} S_0(u^-, x^+) S_{\text{in}}(x^+) \\
+c_1 A e^{2x_-} \int_{-\infty}^{\infty} e^{-3x^+} S_{\text{in}}(x^+) + c_2 B e^{3x_-} \int_{-\infty}^{\infty} e^{-2x^+} S_{\text{in}}(x^+) .
\]

(74)

The numbers \( c_i \) come from going by parts with the operator \( O_{\mu\nu}\lambda\rho \) and by applying \( O_{x,t} \) to the background solution for \( E_{\mu\nu}\lambda\rho \). DMW’s representation of the matrix model predicts that the terms describing tachyon scattering off a discrete-mode background are proportional to

\[
\int_{-\infty}^{\infty} d\tau \Delta^2(\tau) e^{(m-n)\tau} e^{(n+1)x^-} \int_{-\infty}^{\infty} dx^+ e^{-(m+1)x^+} S_m(x^+) ,
\]

where \( m, n > 0 \). The \( m = n = 1 \) term is the blackhole background. By comparing this equation with Eq. (74) it is clear that the \((m,n) = (2,1)\) and \((m,n) = (1,2)\) can be identified with the \( A \) and \( B \) terms of the first massive level of the string. The \( c_i \) (and thus the extra terms from the expansion in small \( 2 + ipQ - p^2 \)) just change the constants of proportionality in the identification

\[
A \propto \int_{-\infty}^{\infty} d\tau \Delta^2(\tau) e^\tau \quad \text{and} \quad B \propto \int_{-\infty}^{\infty} d\tau \Delta^2(\tau) e^{-\tau} .
\]

(76)

Thus, DMW’s proposal has been checked to the first massive level in the string spectrum.

5. Conclusion

By imposing Weyl invariance on the generating functional of closed bosonic string theory, the linearised equation of motion and constraints for the first massive level, in a flat linear-dilaton background, have been derived. This is the first time that the correct equations have been obtained using a variant of the Wilson renormalisation group method.

In two spacetime dimensions, these constraints were solved to find a two-parameter time-dependent solution. One-to-one tachyon scattering in this discrete-state background was studied and the results agree with the prediction made by Dhar, Mandal and Wadia’s\cite{13,14} representation of the matrix model.

It would be a thankless task to derive field equations for the higher states using the approach of this paper. It is interesting, however, that according to DMW’s formula Eq. (75), the charges (such as \( M \), \( A \) and \( B \)) at fixed \( n - m \) are related. Thus, for example, the charge of any state with \( n = m \) should be
proportional to the mass of the blackhole at \( n = 1 = m \). Presumably this is a consequence of the \( W_\infty \) symmetry in the string theory, but further investigation is beyond the scope of this paper.

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**References**